Physical Systems – spring 2007

QM HW 6a, L and Spherical coordinates, Tues.8 May 2007

Shankar 12.3.1, 12.5.3 (i-iii), 12.5.12, 12.5.13, 12.6.1, 12.6.2

The fact that complex eigenvalues enter the answer, signals that we are overlooking the Hermiticity constraint. Let us impose it. The condition

$$\langle \psi_1 | L_z | \psi_2 \rangle = \langle \psi_2 | L_z | \psi_1 \rangle^* \tag{12.3.4}$$

becomes in the coordinate basis $L|\psi\rangle = -i\hbar \frac{\partial}{\partial\phi}|\psi\rangle$

$$\int_0^\infty \int_0^{2\pi} \psi_1^*\left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi_2 \varrho \, d\varrho \, d\phi = \left[\int_0^\infty \int_0^{2\pi} \psi_2^*\left(-i\hbar\frac{\partial}{\partial\phi}\right)\psi_1 \varrho \, d\varrho \, d\phi\right]^*$$

$$= \int\int \psi_2 \left(i\hbar \frac{\partial}{\partial\phi}\right)\psi^* \rho \, d\rho \, d\phi \tag{12.3.5}$$

If this requirement is to be satisfied for all ψ_1 and ψ_2, one can show (upon integrating by parts) that each ψ must obey

Hermiticity → periodicity : $$\psi(\varrho, 0) = \psi(\varrho, 2\pi) \tag{12.3.6}$$

If we impose this constraint on the L_z eigenfunctions, Eq. (12.3.3), we find

$$e^0 = 1 = e^{2\pi i l_z/\hbar} \tag{12.3.7}$$

This forces l_z not merely to be real, but also an integral multiple of $\hbar$:

QUANTIZATION! $$\boxed{l_z = m\hbar,} \quad m = 0, \pm1, \pm2, \ldots \tag{12.3.8}$$

One calls m the ***magnetic quantum number***. Notice that $l_z = m\hbar$ implies that ψ is a single-valued function of ϕ.

#1 *Exercise 12.3.1.* Provide the steps linking Eq. (12.3.5) to Eq. (12.3.6).

Since L_z is Hermitian,

$$\int_0^\infty\int_0^{2\pi} \psi_1^*\left(-i\hbar\frac{\partial}{\partial\varphi}\right)\psi_2 \rho \, d\rho \, d\varphi = \left[\int_0^\infty\int_0^{2\pi} \psi_2^*\left(-i\hbar\frac{\partial}{\partial\varphi}\right)\psi_1 \rho \, d\rho \, d\varphi\right]^*$$

$$= \int_0^\infty\int_0^{2\pi} \psi_2\left(i\hbar\frac{\partial}{\partial\varphi}\right)\psi_1^* \rho \, d\rho \, d\varphi = \int u \, dv$$

$$\int v \, du = \int_0^\infty\int_0^{2\pi} \psi_1^*\left(i\hbar\frac{\partial}{\partial\varphi}\right)\psi_2 \rho \, d\rho \, d\varphi$$

$$d(uv) = u\,dv + u\,dv \to uv\Big| = \int u\,dv + \int v\,du$$

Integrating by parts with respect to φ,

$$uv\Big| = i\hbar\int_0^\infty \psi_2\psi_1^* \rho \, d\rho \Big|_0^{2\pi} = \int_0^\infty\int_0^{2\pi} \psi_2\left(i\hbar\frac{\partial}{\partial\varphi}\right)\psi_1^* \rho\,d\rho\,d\varphi + \int_0^\infty\int_0^{2\pi} \psi_1^*\left(i\hbar\frac{\partial}{\partial\varphi}\right)\psi_2 \rho\,d\rho\,d\varphi$$

$= 0$ since $\int_0^\infty\int_0^{2\pi}$ integrals cancel.

$$0 = \int_0^\infty \psi_2(\rho,\varphi)\,\psi_1^*(\rho,\varphi)\,\rho\,d\rho\,\Big|_0^{2\pi}$$

$$= \int_0^\infty \psi_2(\rho,2\pi)\,\psi_1^*(\rho,2\pi)\,\rho\,d\rho - \int_0^\infty \psi_2(\rho,0)\,\psi_1^*(\rho,0)\,\rho\,d\rho$$

$$0 = \int_0^\infty \left[\psi_2(\rho,2\pi)\,\psi_1^*(\rho,2\pi) - \psi_2(\rho,0)\,\psi_1^*(\rho,0)\right]\rho\,d\rho$$

$$0 = \psi_2(\rho,2\pi)\,\psi_1^*(\rho,2\pi) - \psi_2(\rho,0)\,\psi_1^*(\rho,0)$$

$$\psi_2(\rho,2\pi)\,\psi_1^*(\rho,2\pi) = \psi_2(\rho,0)\,\psi_1^*(\rho,0)$$

Need $\psi_2(\rho,2\pi) = \psi_2(\rho,0)$ and $\psi_1^*(\rho,2\pi) = \psi_1^*(\rho,0)$

for L_z to be Hermitian.

2 *Exercise 12.5.3.** (i) Show that $\langle J_x \rangle = \langle J_y \rangle = 0$ in a state $|jm\rangle$.
(ii) Show that in these states

$$\langle J_x^2 \rangle = \langle J_y^2 \rangle = \tfrac{1}{2}\hbar^2[j(j+1) - m^2]$$

(use symmetry arguments to relate $\langle J_x^2 \rangle$ to $\langle J_y^2 \rangle$).
(iii) Check that $\Delta J_x \cdot \Delta J_y$ from part (ii) satisfies the inequality imposed by the uncertainty principle [Eq. (9.2.9)]. $(\Delta J_x)^2(\Delta J_y)^2 \geq |\langle \psi | J_x J_y | \psi \rangle|^2$

(i) (12.5.21.a) $\langle jm | J_x | jm \rangle = \frac{\hbar}{2}\{\delta_{jj}\delta_{m,m+1}\sqrt{(j-m)(j+m+1)} + \delta_{jj}\delta_{m,m-1}\sqrt{(j+m)(j-m+1)}\}$

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Since $\delta_{m,m+1}$ and $\delta_{m,m-1} = 0$, $\langle J_x \rangle = 0$ in state $|jm\rangle$
Similarly, $\langle jm | J_y | jm \rangle = 0$.

(ii) $J^2 = J_x^2 + J_y^2 + J_z^2$ so $\langle J^2 \rangle = \langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle$

(12.5.17.a) $J^2 |jm\rangle = j(j+1)\hbar^2 |jm\rangle$
p.335 (12.5.17.b) $J_z |jm\rangle = m\hbar |jm\rangle$

In the state $|jm\rangle$, $\langle J^2 \rangle = \langle jm | J^2 | jm \rangle = \langle jm | j(j+1)\hbar^2 | jm \rangle$
$\langle J^2 \rangle = j(j+1)\hbar^2$

and $\langle J_z^2 \rangle = \langle jm | m^2\hbar^2 | jm \rangle = m^2\hbar^2$

$\langle J_x^2 \rangle + \langle J_y^2 \rangle = \langle J^2 \rangle - \langle J_z^2 \rangle$
$= j(j+1)\hbar^2 - m^2\hbar^2$

There is no reason to prefer x or y direction (in the absence of a symmetry-breaker like a magnetic field), so $\langle J_x^2 \rangle = \langle J_y^2 \rangle$
and $\langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{\hbar^2}{2}[j(j+1) - m^2]$

12.5.3 (iii) $\Delta J_x^2 = \langle J_x^2 \rangle - \langle J_x \rangle^2$ as usual

$$= \frac{\hbar^2}{2}[j(j+1)-m^2]-0 = \Delta J_y^2$$

$$\Delta J_x \Delta J_y = \Delta J_x \Delta J_x = \frac{\hbar^2}{2}[j(j+1)-m^2]$$

① or $\Delta J_x^2 \Delta J_y^2 = \frac{\hbar^4}{4}[j(j+1)-m^2]^2$

Uncertainty relation from $\underset{p.246}{(9.2.9)}$ is $(\Delta J_x)^2(\Delta J_y)^2 = |\langle jm|\hat{J}_x \hat{J}_y|jm\rangle|$

$$(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}\langle jm|[J_x, J_y]|jm\rangle^2 + \frac{\hbar^2}{4} \quad \underset{p.247}{(9.2.13)}$$

$$[J_x, J_y] = i\hbar J_z \quad \underset{p.327}{(12.4.4a)}$$

② $(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4}\langle jm|i\hbar J_z|jm\rangle^2 = \frac{|i\hbar|^2}{4}(m\hbar)^2 = \frac{m^2\hbar^4}{4}$

Is it true that ① ≥ ②?

$$\frac{\hbar^4}{4}[j(j+1)-m^2]^2 \overset{?}{\geq} \frac{m^2\hbar^4}{4}$$ The smallest $j = m$:

① $\frac{\hbar^4}{4}[m(m+1)-m^2]^2 = \frac{\hbar^4}{4}[m^2-m^2+m^2] = \frac{\hbar^4}{4}m^2$ = ②

just barely.

*Exercise 12.5.13.** Consider a particle in a state described by

$$\psi = N(x + y + 2z)e^{-\alpha r}$$ Ty 12.5.10 (ii)?

where N is a normalization factor.

(i) Show, by rewriting the $Y_1^{\pm 1,0}$ functions in terms of x, y, z, and r, that

$$Y_1^{\pm 1} = \mp\left(\frac{3}{4\pi}\right)^{1/2} \frac{x \pm iy}{2^{1/2}r}$$
$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \frac{z}{r} \tag{12.5.42}$$

(ii) Using this result, show that for a particle described by ψ above, $P(l_z = 0) = 2/3$; $P(l_z = +\hbar) = 1/6 = P(l_z = -\hbar)$. Hint: Expand $\psi(r)$ in terms of the Y_1^m's.

Here are the first few Y_l^m functions:

$$\begin{aligned}
l=0 - \quad & Y_0^0 = (4\pi)^{-1/2} \\
& Y_1^{\pm 1} = \mp(3/8\pi)^{1/2} \sin\theta \, [e^{\pm i\phi} = \cos\varphi \pm i \sin\varphi] \\
l=1 - \quad & Y_1^0 = (3/4\pi)^{1/2} \cos\theta \\
& Y_2^{\pm 2} = (15/32\pi)^{1/2} \sin^2\theta \, e^{\pm 2i\phi} \\
& Y_2^{\pm 1} = \mp(15/8\pi)^{1/2} \sin\theta \cos\theta \, e^{\pm i\phi} \\
l=2 - \quad & Y_2^0 = (5/16\pi)^{1/2}(3\cos^2\theta - 1)
\end{aligned} \tag{12.5.39}$$

Note that

$$Y_l^{-m} = (-1)^m (Y_l^m)^* \tag{12.5.40}$$

In spherical coordinates, $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta[\cos\varphi \pm i\sin\varphi] = \mp\sqrt{\frac{3}{8\pi}}[\sin\theta\cos\varphi \pm i\sin\theta\sin\varphi]$$
$$= \mp\sqrt{\frac{3}{8\pi}}\left[\frac{x}{r} \pm i\, y/r\right]$$

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r}$$

(b) $\psi = N(x+y+2z)e^{-\alpha r}$ To find the probability of each l_z, I need to solve each state for x, y, z.

$l_z = 0, \pm 1$

Oh boy...

$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \qquad Y_1^0 = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}, \qquad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\left(\frac{x}{r} \pm \frac{iy}{r}\right)$

To get x and y, add and subtract $Y_1^{\pm 1}$:

$$Y_1^{+1} + Y_1^{-1} = -\sqrt{\frac{3}{8\pi}}\left(\frac{x}{r} + \frac{iy}{r}\right) + \sqrt{\frac{3}{8\pi}}\left(\frac{x}{r} - \frac{iy}{r}\right) = \sqrt{\frac{3}{8\pi}}\left(-\frac{2iy}{r}\right)$$

$$\frac{y}{r} = \frac{i}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^{+1} + Y_1^{-1}\right) = i\sqrt{\frac{2\pi}{3}}\left(Y_1^{+1} + Y_1^{-1}\right)$$

$$Y_1^{+1} - Y_1^{-1} = -\sqrt{\frac{3}{8\pi}}\left(\frac{x}{r} + \frac{iy}{r}\right) - \sqrt{\frac{3}{8\pi}}\left(\frac{x}{r} - \frac{iy}{r}\right) = -\sqrt{\frac{3}{8\pi}}\,\frac{2x}{r}$$

$$\frac{x}{r} = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\left(Y_1^{-1} - Y_1^{+1}\right) = \sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{+1}\right)$$

And $\frac{z}{r} = \sqrt{\frac{4\pi}{3}}\,Y_1^0 = 2\sqrt{\frac{\pi}{3}}\,Y_1^0$

$$\psi = N(x+y+2z)e^{-\alpha r} = Nre^{-\alpha r}\left[\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{+1} + iY_1^{+1} + iY_1^{-1}\right) + 4\sqrt{\frac{\pi}{3}}\,Y_1^0\right]$$
$$= Nre^{-\alpha r}\sqrt{\frac{\pi}{3}}\left[\sqrt{2}\left(Y_1^{-1} - Y_1^{+1} + iY_1^{+1} + iY_1^{-1}\right) + 4Y_1^0\right]$$

Now we can find the probabilities of each l_z - they will be proportional to the square of the coefficient of each corresponding $Y_1^{l_z = m\hbar}$. Let's call the unknown proportionality constant k for now.

$P(l_z = 0) = k \cdot 4^2 = k \cdot 16$
$P(l_z = +\hbar) = k\,|\sqrt{2}(i-1)|^2 = k\,2\sqrt{1+1}^{\,2} = k \cdot 4$
$P(l_z = -\hbar) = k\,|\sqrt{2}(1+i)|^2 = k \cdot 4$

$\Sigma P = 1 = 16k + 4k + 4k = 24k \Rightarrow k = \frac{1}{24}$. Now we can normalize:

$P(l_z = 0) = \frac{16}{24} = \frac{2}{3}, \quad P(l_z = \hbar) = P(l_z = -\hbar) = \frac{4}{24} = \frac{1}{6}$ ✓