Quantum mechanics HW 1a, Tuesday 3 April 2007

1.1.4, 1.6.1, 1.6.2, 1.6.3

*Exercise 1.1.4.** (a) Consider the vector space discussed in Exercise 1.1.1. Show that the elements (1, 1, 0), (1, 0, 1), and (3, 2, 1) are linearly dependent. [Assume that one of them is a linear combination of the other two, and find the (nontrivial) coefficients of the expansion.] (b) Show likewise that (1, 1, 0), (1, 0, 1), and (0, 1, 1) are LI.

(a) $\alpha\begin{bmatrix}1\\1\\0\end{bmatrix} + \beta\begin{bmatrix}1\\0\\1\end{bmatrix} + \gamma\begin{bmatrix}3\\2\\1\end{bmatrix} = 0 \rightarrow \alpha + \beta + 3\gamma = 0$

$\alpha + 0\beta + 2\gamma = 0 \rightarrow \alpha = -2\gamma$

$0\alpha + \beta + \gamma = 0 \rightarrow \beta = -\gamma$

Björn: Use row reduction (same result):

$$\begin{bmatrix}1 & 1 & 3\\1 & 0 & 2\\0 & 1 & 1\end{bmatrix} \rightarrow \begin{pmatrix}-1 & -1 & -3\\+1 & 0 & 2\end{pmatrix} \rightarrow \begin{bmatrix}1 & 1 & 3\\0 & -1 & -1\\0 & 1 & 1\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

$0 \quad -1 \quad -1$

$\beta = -\gamma, \quad \alpha = -2\cancel{\gamma} = 2\beta$

Ex: $\beta = 1, \gamma = -1, \alpha = +2$

We can add a combination of these vectors to zero, so they are linearly dependent.

(b) $a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\\1\end{bmatrix} + c\begin{bmatrix}0\\1\\1\end{bmatrix} \overset{?}{=} 0$ If so, then LD.

$a + b + 0c = 0 \rightarrow a = -b$ $\quad b = 0$

$a + 0b + c = 0 \rightarrow a = -c$ $\quad a = 0$

$0a + 0b + c = 0 \rightarrow c = 0$

The only way to add a combination of these three vectors to zero is to multiply them all by zero: TRIVIAL solution.

Therefore, these are linearly INDEPENDENT: LI.

$\Omega|1\rangle = |2\rangle$ $\Omega|2\rangle = |3\rangle$ $\Omega|3\rangle = |1\rangle$

Exercise 1.6.1. An operator Ω is given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

What is its action? What does it do? Rotation of 120° around $\begin{bmatrix}1\\1\\1\end{bmatrix}$ axis
Check by operating on $\begin{bmatrix}1\\1\\1\end{bmatrix}$ - should produce NO CHANGE.

$\Omega|1\rangle = |2\rangle$: Ω operates on $\hat{x}$ and rotates it into $\hat{y}$

$\Omega|2\rangle = |3\rangle$: Ω rotates $\hat{y}$ into $\hat{z}$

$\Omega|3\rangle = |1\rangle$: Ω rotates $\hat{z}$ into $\hat{x}$

Check: $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ ✓ Ω doesn't change $\begin{bmatrix}1\\1\\1\end{bmatrix}$ axis.

*Exercise 1.6.2.** Given Ω and Λ are Hermitian what can you say about (i) $\Omega\Lambda$; (ii) $\Omega\Lambda + \Lambda\Omega$; (iii) $[\Omega, \Lambda]$; and (iv) $i[\Omega, \Lambda]$?

*Exercise 1.6.3.** Show that a product of unitary operators is unitary.

$\Omega^\dagger$ is the transpose conjugate of $\Omega = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\Omega^\dagger = \begin{vmatrix} a^* & d^* & g^* \\ b^* & e^* & h^* \\ c^* & f^* & i^* \end{vmatrix}$. If $\Omega = \Omega^\dagger$ then Ω is HERMITIAN.

1.6.2 $\Omega = \Omega^\dagger$ and $\Lambda = \Lambda^\dagger$. (i) $(\Omega\Lambda)^\dagger = \Lambda^\dagger\Omega^\dagger$ (see eqn 1.6.16)
$= \Lambda\Omega$

So $(\Omega\Lambda)$ is NOT Hermitian unless $(\Omega\Lambda) = (\Lambda\Omega)$, that is, unless Ω and Λ commute.

($\Omega = \Omega^\dagger$ and $\Lambda = \Lambda^\dagger$ in this problem: Hermitian operators)

1.6.2 (ii) $\underline{(\Omega\Lambda + \Lambda\Omega)^\dagger} = (\Omega\Lambda)^\dagger + (\Lambda\Omega)^\dagger$

$$= \Lambda^\dagger\Omega^\dagger + \Omega^\dagger\Lambda^\dagger$$
$$= \Lambda\Omega + \Omega\Lambda = \underline{\Omega\Lambda + \Lambda\Omega}$$

Therefore $(\Omega\Lambda + \Lambda\Omega)$ is Hermitian.

(iii) $[\Omega, \Lambda]$ is defined as $\Omega\Lambda - \Lambda\Omega$ = COMMUTATOR of Ω & Λ (pp 22-23)

$$([\Omega,\Lambda])^\dagger = \underline{(\Omega\Lambda - \Lambda\Omega)^\dagger} = (\Omega\Lambda)^\dagger - (\Lambda\Omega)^\dagger$$
$$= \Lambda^\dagger\Omega^\dagger - \Omega^\dagger\Lambda^\dagger$$
$$= \Lambda\Omega - \Omega\Lambda = -\underline{(\Omega\Lambda - \Lambda\Omega)}$$
$$= -[\Omega, \Lambda]$$

Therefore $[\Omega, \Lambda]$ is ANTIHERMITIAN.

(iv) $\underline{i[\Omega,\Lambda]} = i(\Omega\Lambda - \Lambda\Omega) = i(\Omega^\dagger\Lambda^\dagger - \Lambda^\dagger\Omega^\dagger)$ (not necessary...)

$$\underline{(i[\Omega,\Lambda])^\dagger} = -i[\Omega,\Lambda]^\dagger = -i(-[\Omega,\Lambda]) = \underline{+i[\Omega,\Lambda]}$$

Therefore, $i[\Omega, \Lambda]$ is HERMITIAN.

1.6.3 Let U and V be unitary: $U^\dagger U = I$, $V^\dagger V = I$.

Then $(UV)^\dagger(UV) = (V^\dagger U^\dagger)(UV)$ (eq 1.6.16)

$$= V^\dagger(U^\dagger U)V$$
$$= V^\dagger I V = V^\dagger V = I$$

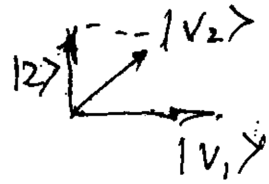
Therefore, the PRODUCT of unitary operators is also UNITARY.

Theorem 5 (Gram–Schmidt Theorem). Given n vectors $|V_1\rangle, |V_2\rangle, \ldots, |V_n\rangle$ that are LI, we can get, by forming linear combinations, n orthonormal vectors, $|1\rangle, |2\rangle, \ldots, |i\rangle, \ldots, |n\rangle$.

Proof. Let us first construct n mutually *orthogonal* vectors. Let

$$|1'\rangle = |V_1\rangle$$

$$|2'\rangle = |V_2\rangle - \frac{|1'\rangle\langle 1'|V_2\rangle}{\langle 1'|1'\rangle}$$



Exercise 1.3.2. Consider the vectors

$$|V_1\rangle \leftrightarrow \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \qquad |V_2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \qquad |V_3\rangle \leftrightarrow \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$

Use the Gram–Schmidt procedure to get the following orthonormal basis:

$$|1\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad |2\rangle \leftrightarrow \begin{bmatrix} 0 \\ 1/5^{1/2} \\ 2/5^{1/2} \end{bmatrix}, \qquad |3\rangle \leftrightarrow \begin{bmatrix} 0 \\ -2/5^{1/2} \\ 1/5^{1/2} \end{bmatrix}$$

Is this the only orthonormal basis you can get in this case? (What if you change the sign of the components of $|1\rangle$?)

$$|1\rangle = \frac{|V_1\rangle}{|V_1|} = \frac{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}}{3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is normal.

Orthogonal (not necessarily normal) $|2'\rangle$ can be constructed by subtracting $\frac{\langle 1|V_2\rangle |1\rangle}{\langle 1|1\rangle}$. The part of $|V_2\rangle$ parallel to $|1\rangle$ is

dot product $\langle 1|V_2\rangle = [1\ 0\ 0]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 0$. Great, $|V_2\rangle$ is already perpendicular to $|V_1\rangle$

So $|2\rangle = \frac{|V_2\rangle}{|V_2|} = \frac{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}{\sqrt{0+1+2^2}} = \frac{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

To construct $|3'\rangle$ orthogonal to $|1\rangle$ and $|2\rangle$
(or $|V_1\rangle$ and $|V_2\rangle$ in this case)

we need to subtract out the parallel components
$\langle 1|V_3\rangle$ and $\langle V_2|V_3\rangle$ (I'll use $|1\rangle$ since it's simpler than $|V_1\rangle$ and $|V_2\rangle$ since its simpler than $|2\rangle$)

$$\langle 1|V_3\rangle = [1\ 0\ 0]\begin{bmatrix}0\\2\\5\end{bmatrix} = 0$$

$|V_3\rangle$ is already orthogonal to $|V_1\rangle$ - great.

$$\langle V_2|V_3\rangle = [0\ 1\ 2]\begin{bmatrix}0\\2\\5\end{bmatrix} = 0 + 2 + 10 = 12$$

$$|3'\rangle = |V_3\rangle - \frac{|2'\rangle\langle 2'|V_3\rangle}{\langle 2'|2'\rangle} \quad \text{where } |2'\rangle = |V_2\rangle$$
in this case

$$\frac{|2'\rangle\langle 2'|V_3\rangle}{\langle 2'|2'\rangle} = \frac{|V_2\rangle\langle V_2|V_3\rangle}{|V_2|^2} = \frac{12\,|V_2\rangle}{5}$$

$$|3'\rangle = \begin{bmatrix}0\\2\\5\end{bmatrix} - \frac{12}{5}\begin{bmatrix}0\\1\\2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}0\\10\\25\end{bmatrix} - \frac{1}{5}\begin{bmatrix}0\\12\\24\end{bmatrix} = \frac{1}{5}\begin{bmatrix}0\\-2\\1\end{bmatrix}$$

$$|3'| = \sqrt{\langle 3'|3'\rangle} = \sqrt{0+2^2+1}\,/\sqrt{5^2} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$$

$$|3\rangle = \frac{|3'\rangle}{|3'|} = \begin{bmatrix}0\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$$

QM 1b Physical Systems HW Thus.5.April 2007
Math review II: Ch.1.7 – 1.10
Do 1.8.1 (p.45), 1.8.3, 1.8.5, 1.9.2 (p.60), 1.10.1, 1.10.2

Exercise 1.8.1. **(a) Find the eigenvalues and normalized eigenvectors of the matrix**

Find roots of $\det(\Omega - \omega I) = 0$

$$\Omega = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{bmatrix} \qquad (\Omega - \omega I) = \begin{bmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{bmatrix}$$

(b) Is the matrix Hermitian? Are the eigenvectors orthogonal?

$$\det(\Omega - \omega I) = (1-\omega)\left[(2-\omega)(4-\omega) - 0\right] - 3(0-0) + 1(0-0)$$
$$= (1-\omega)(2-\omega)(4-\omega)$$

Eigenvalues are $\omega = 1, 2, 4$. Now find the eigenvectors $|\omega\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ corresponding to each.

$$(\Omega - \omega I)|\omega\rangle = 0 \rightarrow \begin{bmatrix} 1-\omega & 3 & 1 \\ 0 & 2-\omega & 0 \\ 0 & 1 & 4-\omega \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\omega = 1$: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{matrix} 3b + c = 0 \\ 2b = 0 \\ b + 3c = 0 \end{matrix} \rightarrow \begin{matrix} b = c = 0 \\ a = \text{anything} \end{matrix}$

So the normalized eigenvector for $\omega = 1$ is $|\omega = 1\rangle \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\omega = 2$: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{matrix} -a + 3b + c = 0 \rightarrow a = -5c \\ b + 2c = 0 \rightarrow b = -2c \end{matrix}$

Let's choose $c = 1$. Then one form of the $|\omega = 2\rangle$ eigenvector is $\begin{bmatrix} -5c \\ -2c \\ c \end{bmatrix} \rightarrow \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$ Norm $= \sqrt{5^2 + 2^2 + 1}$
$= \sqrt{25 + 4 + 1} = \sqrt{30}$

So $|\omega = 2\rangle \leftrightarrow \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} / \sqrt{30}$ (normalized)

$$\omega=4: \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}-3 & 3 & 1\\ 0 & -2 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} \rightarrow \begin{aligned}-3a + b + c &= 0\\ -2b &= 0\\ b &= 0\end{aligned} \quad \begin{aligned}3a &= c\\ \\ \rightarrow b &= 0\end{aligned}$$

Let's choose $a=1$. Then one form of the $|\omega=4\rangle$ eigen vector is $\begin{bmatrix}1\\0\\3\end{bmatrix}$. Norm $= \sqrt{1^2+0^2+3^2}$
$= \sqrt{1+9} = \sqrt{10}$

So $|\omega=4\rangle \leftrightarrow \begin{bmatrix}1\\0\\3\end{bmatrix} / \sqrt{10}$ (normalized)

(b) Is Ω Hermitian? Is $\Omega^\dagger = \Omega$?

$$\Omega = \begin{bmatrix}1 & 3 & 1\\ 0 & 2 & 0\\ 0 & 1 & 4\end{bmatrix} \qquad \Omega^\dagger = \begin{bmatrix}1 & 0 & 0\\ 3 & 2 & 1\\ 1 & 0 & 4\end{bmatrix}^* \neq \Omega$$

NOT HERMITIAN.

Thm 10: If Hermitian $\rightarrow$ then $\exists$ an orthogonal basis set of vectors.

Are the eigenvectors orthogonal? Since Ω is not Hermitian, we have to check.
$\langle V_i | V_j \rangle = \delta_{ij}$ for orthonormal vectors

$$\langle \omega=1 | \omega=4 \rangle = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\3\end{bmatrix}\frac{1}{\sqrt{10}} = \frac{1}{\sqrt{10}} \neq 0$$

NOT ORTHOGONAL

*Exercise 1.8.2.** **Consider the matrix**

$$\Omega = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \Omega^\dagger = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}^* = \Omega$$

(a) Is it Hermitian? YES; $\Omega = \Omega^\dagger$

(b) Find its eigenvalues and eigenvectors.

(c) Verify that $U^\dagger \Omega U$ is diagonal, U being the matrix of eigenvectors of Ω.

(b) Eigenvalues ω solve $\det(\Omega - I\omega) = 0$

$$\det\begin{pmatrix} 0-\omega & 0 & 1 \\ 0 & 0-\omega & 0 \\ 1 & 0 & 0-\omega \end{pmatrix} = \begin{vmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{vmatrix} = \begin{matrix} -\omega(\omega^2 - 0) \\ -0 \\ +1(0+\omega) \end{matrix}$$

$$0 \doteq -\omega^3 + \omega = -\omega(\omega^2 - 1) = -\omega(\omega+1)(\omega-1)$$

Eigen values $\omega = 0, 1, -1$

Eigenvectors $|\omega\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfy $(\Omega - \omega I)|\omega\rangle = 0$

$$\begin{bmatrix} -\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & -\omega \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\omega = 0$: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c = 0 \\ \\ a = 0 \end{matrix} \qquad b = \text{anything}$

$|\omega = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ✓

$\omega = 1$: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} a = c \\ b = 0 \\ a = 0 \end{matrix} \rightarrow |\omega = 2\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} / \sqrt{2}$

*Exercise 1.8.3.** Consider the Hermitian matrix

$$\Omega = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

(a) Show that $\omega_1 = \omega_2 = 1$; $\omega_3 = 2$.
(b) Show that $|\omega = 2\rangle$ is any vector of the form

$$\frac{1}{(2a^2)^{1/2}}\begin{bmatrix} 0 \\ a \\ -a \end{bmatrix}$$

(c) Show that the $\omega = 1$ eigenspace contains all vectors of the form

$$\frac{1}{(b^2 + 2c^2)^{1/2}}\begin{bmatrix} b \\ c \\ c \end{bmatrix}$$

either by feeding $\omega = 1$ into the equations or by requiring that the $\omega = 1$ eigenspace be orthogonal to $|\omega = 2\rangle$.

(a) Eigenvalues solve $\det(\Omega - I\omega) = 0 = \begin{vmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2}-\omega \end{vmatrix}$

$0 = (1-\omega)\left[\left(\frac{3}{2}-\omega\right)\left(\frac{3}{2}-\omega\right) - \frac{1}{4}\right]$

$= (1-\omega)\left[\frac{9}{4} - \frac{1}{4} - 3\omega + \omega^2\right] = (1-\omega)(\omega^2 - 3\omega + 2)$

$0 = (1-\omega)(\omega-1)(\omega-2)$

$\rightarrow \omega = 1, 1, 2$ (degenerate)

(b) Eigenvectors $|\omega\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfy $(\Omega - \omega I)|\omega\rangle = 0$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\omega & 0 & 0 \\ 0 & \frac{3}{2}-\omega & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2}-\omega \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\omega = 2$. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{matrix} a = 0 \\ b = -c \end{matrix} \rightarrow |\omega = 2\rangle = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} / \sqrt{2}$

or any vector of the form $|\omega = 2\rangle = \begin{bmatrix} 0 \\ a \\ -a \end{bmatrix} / \sqrt{2a^2}$

(c)

$$\omega=1: \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\ 0 & \frac{1}{2} & -\frac{1}{2}\\ 0 & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}b\\c\\d\end{bmatrix} \rightarrow \begin{array}{l} b = \text{anything} \\ c = d \end{array}$$

$$|\omega=1\rangle = \begin{bmatrix}b\\c\\c\end{bmatrix} \Big/ \sqrt{b^2+c^2+c^2} \checkmark$$

*Exercise 1.9.2.** If H is a Hermitian operator, show that $U = e^{iH}$ is unitary. (Notice the analogy with c numbers: if θ is real, $u = e^{i\theta}$ is a number of unit modulus.)

$$U = e^{iH} \equiv \sum_{n=0}^{\infty} \frac{(iH)^n}{n!} \qquad (\text{where } (iH)^0 = I)$$

$$H^+ = H$$

$$U^+ = \left(\sum_{n=0}^{\infty} \frac{(iH)^n}{n!}\right)^+ = \sum\left[\frac{(iH)^n}{n!}\right]^+ = \sum \frac{(-iH^+)^n}{n!} = \sum \frac{(-iH)^n}{n!} = e^{-iH}$$

So $UU^+ = e^{iH} e^{-iH} = e^0 = I$ ✓

*Exercise 1.10.1.** Show that $\delta(ax) = \delta(x)/|a|$. [Consider $\int \delta(ax)\, d(ax)$. Remember that $\delta(x) = \delta(-x)$.]

$$\int_{-\infty}^{\infty} \delta(ax) f(x) dx = \int_{-\infty}^{\infty} \delta(|a|x) f(x)\, dx \qquad \text{Let } y = |a|x, \quad dy = |a|\,dx$$

$$\text{Then } \int_{-\infty}^{\infty} \delta(ax) f(x)\, dx = \int_{-\infty}^{\infty} \delta(y) f\left(\frac{y}{|a|}\right) \frac{dy}{|a|} = \frac{1}{|a|} f\left(\frac{y}{|a|}\right)\bigg|_{y=0} = \frac{1}{|a|} f(0)$$

$$= \frac{1}{|a|} \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{\infty} \left[\frac{1}{|a|}\delta(x)\right] f(x)\, dx \rightarrow \delta(ax) = \frac{1}{|a|}\delta(x) \ ✓$$

*Exercise 1.10.2.** Show that

$$\delta(f(x)) = \sum_i \frac{\delta(x_i - x)}{|df/dx_i|}$$

is nonzero only in the infinitesimal region around each x_i.

where x_i are the zeros of $f(x)$. Hint: Where does $\delta(f(x))$ blow up? Expand $f(x)$ near such points in a Taylor series, keeping the first nonzero term.

Near x_i, $f(x) \approx f(x_i) + \frac{df}{dx}\Big|_{x_i}(x - x_i) = 0 + \frac{df}{dx}\Big|_{x_i}(x - x_i) = \frac{df}{dx_i}$

Use 1.10.1:

near x_i, $\delta(f(x)) = \dfrac{\delta(x - x_i)}{|df/dx_i|}$. The total $\delta(f(x)) = \sum_i \dfrac{\delta(x - x_i)}{|df/dx_i|}$

Recall $\delta(x - x_i) = 0$ if $x_i \neq x$. So the ith term won't interfere with the spike at any other x_j.

1.8.5 Consider the matrix
p.46

$$\Omega = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \frac{1}{2}(e^{i\theta}+e^{-i\theta}) & \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ -\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) & \frac{1}{2}(e^{i\theta}+e^{-i\theta}) \end{bmatrix}$$

(a) Show that Ω is unitary. $U^\dagger U = I$

$$\Omega^\dagger \Omega = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\sin - \sin\cos \\ \sin\cos - \cos\sin & \sin^2 + \cos^2 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ ✓

(b) Show that its eigenvalues are $e^{i\theta}$ and $e^{-i\theta}$

Eigenvalues ω solve $\det(\Omega - I\omega) = 0$. $\begin{vmatrix} \frac{1}{2}(e^{i\theta}+e^{-i\theta})-\omega & \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{-i\theta}-e^{i\theta}) & \frac{1}{2}(e^{i\theta}+e^{-i\theta})-\omega \end{vmatrix}$

$$0 = \left[\frac{1}{2}(e^{i\theta}+e^{-i\theta})-\omega\right]\left[\frac{1}{2}(e^{i\theta}+e^{-i\theta})-\omega\right] - \left(\frac{1}{2i}(e^{i\theta}-e^{-i\theta})\frac{1}{2i}(e^{-i\theta}-e^{i\theta})\right]$$

$$= \cos^2\theta + \omega^2 - (e^{i\theta}+e^{-i\theta})\omega + \sin^2\theta$$

$$= \omega^2 - (e^{i\theta}+e^{-i\theta})\omega + 1$$

$$= (\omega - e^{i\theta})(\omega - e^{-i\theta}) \quad \text{(check)}$$

$$= \omega^2 - \omega(e^{i\theta}+e^{-i\theta}) + e^0$$ ✓

$\rightarrow \omega = e^{i\theta}, e^{-i\theta}$ ✓

(c) Find the corresponding eigenvectors. $|\omega\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ satisfy

$0 = (\Omega - \omega I)|\omega\rangle$

$\omega = e^{i\theta}$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^{i\theta}+e^{-i\theta})-e^{i\theta} & \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{-i\theta}-e^{i\theta}) & \frac{1}{2}(e^{i\theta}+e^{-i\theta})-e^{i\theta} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^{-i\theta}-e^{i\theta}) & \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{-i\theta}-e^{i\theta}) & \frac{1}{2}(e^{i\theta}-e^{-i\theta}) \end{bmatrix}$$

$\rightarrow (e^{-i\theta}-e^{i\theta})a = i(e^{i\theta}-e^{-i\theta})b \qquad |\omega = e^{i\theta}\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}/\sqrt{2}$

$-a = ib$

$b = ia$

(c) Is it easier to find eigenvectors with original form of $\mathcal{R}$:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta - \omega & \sin\theta \\ -\sin\theta & \cos\theta - \omega \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

For $\omega = e^{i\theta} = \cos\theta + i\sin\theta$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta - (\cos\theta + i\sin\theta) & \sin\theta \\ -\sin\theta & \cos\theta - (\cos\theta + i\sin\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{bmatrix} \begin{matrix} a \\ b \end{matrix}$$

$\rightarrow -i\sin\theta\, a + b\sin\theta = 0 \rightarrow ia = b$ Same, and easier.

For $\omega = e^{-i\theta} = \cos\theta - i\sin\theta$, $\cos\theta - \omega = \cos\theta - (\cos\theta - i\sin\theta) = +i\sin\theta$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{matrix} -ia\sin\theta = b\sin\theta \\ -ia = b \end{matrix}$$

$$|\omega = e^{-i\theta}\rangle = \begin{bmatrix} 1 \\ -i \end{bmatrix} / \sqrt{2}$$

Show that these eigenvectors are orthogonal:

$$\langle \omega = e^{i\theta} | \omega = e^{-i\theta} \rangle = \underbrace{\tfrac{1}{2}\begin{bmatrix} 1 & -i \end{bmatrix}}_{\text{take complex conjugate}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \tfrac{1}{2}(1 + i^2) = \tfrac{1}{2}(1-1) =$$

YES - orthogonal

(d) Verify that $U^\dagger \mathcal{R} U =$ (diagonal matrix) where

$U =$ matrix of eigenvectors : $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, $U^\dagger = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$

$$\mathcal{R}U = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos\theta + i\sin\theta & \cos\theta - i\sin\theta \\ -\sin\theta + i\cos\theta & -\sin\theta - i\cos\theta \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\theta} & e^{-i\theta} \\ ie^{i\theta} & -ie^{-i\theta} \end{bmatrix}$$

$$U^\dagger \mathcal{R} U = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} e^{i\theta} & e^{-i\theta} \\ ie^{i\theta} & -ie^{-i\theta} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{i\theta} + e^{i\theta} & e^{-i\theta} - e^{-i\theta} \\ e^{i\theta} - e^{i\theta} & e^{-i\theta} + e^{-i\theta} \end{bmatrix} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

Sure enough, this diagonalizes $\mathcal{R}$ with ω on diagonal.