QM HW 2a Tues.10 April 2007 - Moore SP4.1, Shankar ~~4.2.1~~ (p.135, ed.1), 4.2.2 (p.146), 4.2.3

Instead of 4.2.1, do Moore's very similar and equally important (but less tedious) problem:

Moore SP4.1: The following operators are the Hermitian operators corresponding to the x,y,z components of a fermion's spin angular momentum respectively (as we will see in a later chapter):

$$S_x \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ S_y \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ S_z \leftrightarrow \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ we can read the S_z eigenvalues off the diagonal

Note that we have expressed these matrices in the basis of the eigenstates of S_z.

a. If we measure the z-component of the particle's spin, what are the possible results that we could get? $\hbar/2$ and $-\hbar/2$: spin up and spin down
b. Take the state where $s_z = \hbar/2$ (i.e. the eigenstate of S_z with eigenvalue $s_z = \hbar/2$).

   In this state, what are $\langle S_x \rangle, \langle S_x^2 \rangle$, *and* ΔS_x?

$$\langle S_x \rangle = [1, 0]\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \langle S_z | S_x | S_z \rangle$$

$$= \frac{\hbar}{2}[1, 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2}[0] = 0$$

equally likely to get S_x up or down

$$S_x^2 = \frac{\hbar^2}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\frac{\hbar^2}{4}$$

$$\langle S_x^2 \rangle = \langle S_z | S_x^2 | S_z \rangle = [1, 0]\frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\hbar^2}{4}[1, 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hbar^2/4$$

Recall that $\Delta S^2 = \langle S^2 \rangle - \langle S \rangle^2$

So $\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\hbar^2/4 - 0} = \hbar/2$

c. Find the eigenvalues and normalized eigenstates of S_x in the S_z basis (i.e. the basis currently being used).

$\det(S_x - I\omega) = 0 \rightarrow \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{vmatrix} \omega & 0 \\ 0 & \omega \end{vmatrix} = \begin{vmatrix} -\omega & \hbar/2 \\ \hbar/2 & -\omega \end{vmatrix}$

$\omega^2 - \frac{\hbar^2}{4} = 0 \rightarrow \omega = \pm\frac{\hbar}{2}$: Spin in the x-direction is up or down $\hbar/2$ also - same as for z.

The eigenvectors $|S_x\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ satisfy $[S_x - I\omega]|S_x\rangle = 0$

$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega & \hbar/2 \\ \hbar/2 & -\omega \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

For $\omega = +\frac{\hbar}{2}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \hbar/2\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow a = b$

$|S_x = \hbar/2\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}/\sqrt{2}$

For $\omega = -\frac{\hbar}{2}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\hbar/2 & -\hbar/2 \\ -\hbar/2 & -\hbar/2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -\hbar/2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow a = -b$

$|S_x = -\hbar/2\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}/\sqrt{2}$

d. If the particle is in the state with $s_z = -\hbar/2$ and s_x is measured, what are the possible outcomes and their probabilities?

$|S_z = -\hbar/2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ from the S_z matrix.

If we measure S_x, the possible outcomes are the S_x eigenvalues, $\pm\hbar/2$.

(d) continued... The probability of measuring S_x if we start in state $|S_z = -\hbar/2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $P(S_x) = |\langle S_x | S_z = -\hbar/2\rangle|^2$

$$P(S_x = +\hbar/2) = |\langle S_x = +\hbar/2 | S_z = -\hbar/2\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}[1, 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right|^2$$
$$= \left(\tfrac{1}{\sqrt{2}} \cdot 1\right)^2 = \tfrac{1}{2}$$

$$P(S_x = -\hbar/2) = |\langle S_x = -\hbar/2 | S_z = -\hbar/2\rangle|^2 = \left|\tfrac{1}{\sqrt{2}}[-1, 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right|^2$$
$$= \left(\tfrac{1}{\sqrt{2}} \cdot 1\right)^2 = \tfrac{1}{2}$$

No matter which S_z eigenstate the system starts in, we have an even chance of measuring $S_x = \pm\hbar/2$.

e. Consider the state $|\psi\rangle \leftrightarrow \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$, which we have expressed in the same basis we have been using all along, the basis of eigenstates of S_z. If s_z^2 is measured and $\hbar^2/4$ is the result obtained, what is the state after the measurement?
How probable was this result?
If s_z is then measured, what are the possible outcomes and their respective prababilities?

$$S_z^2 = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\hbar^2}{4} I$$

Since this is a multiple of the identity matrix, All vectors are eigenstates of S_z^2, with eigenvalue $\frac{\hbar^2}{4}$: $S_z^2|\psi\rangle = \frac{\hbar^2}{4}|\psi\rangle$ for all $|\psi\rangle$. So $|\psi\rangle = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ is an eigenstate of S_z^2 and a measurement of S_z^2 on $|\psi\rangle$ will yield $\hbar^2/4$ with 100% probability.

(e) continued... If S_z is measured on state $|\psi\rangle = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$ the possible outcomes are $\pm \hbar/2$ (as usual) with probability $P(S_z) = |\langle S_z | \psi \rangle|^2$

$$P(S_z = \hbar/2) = |\langle S_z = \hbar/2 | \psi \rangle|^2 = \left| [1 \ 0] \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \right|^2 = (\sqrt{3}/2)^2 = \frac{3}{4}$$

$$P(S_z = -\hbar/2) = |\langle S_z = -\hbar/2 | \psi \rangle|^2 = \left| [0 \ 1] \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \right|^2 = (1/2)^2 = \frac{1}{4}$$

We are more likely to measure S_z spin up since ψ, which is a mixture of states, has a bigger spin-up component in the S_z basis.

f. A particle is in a state with probabilities $P(s_z = \hbar/2) = 2/5$ and $P(s_z = -\hbar/2) = 3/5$.
Convince yourself that the most general normalized state with this property is:

$$|\psi\rangle = \tfrac{2}{5} e^{i\delta_1} \left| s_z = \tfrac{\hbar}{2} \right\rangle + \tfrac{3}{5} e^{i\delta_2} \left| s_z = \tfrac{-\hbar}{2} \right\rangle.$$

If $|\psi\rangle$ is a normalized state, then the state $e^{i\theta}|\psi\rangle$ is a *physically equivalent* normalized state. So the overall phase θ of the state above is physically unimportant. What about the relative phase $e^{i(\delta_1 - \delta_2)}$ between the two parts of the state?
[Calculate, for example, $P(s_x = \hbar/2)$ for the state above.]

We can use $|S_z = \hbar/2\rangle$ and $|S_z = -\hbar/2\rangle$ as basis vectors. Any arbitrary state $|\psi\rangle$ can be written as a linear combination of these vectors:

$$|\psi\rangle = a|S_z = \hbar/2\rangle + b|S_z = -\hbar/2\rangle$$

Simpler spin-up/down notation: $|\psi\rangle = a|+\rangle + b|-\rangle$

$$\text{Probability } P|+\rangle = \frac{2}{5} = |\langle + | \psi \rangle|^2 = |\langle + | (a|+\rangle + b|-\rangle)|^2$$
$$= |a\langle + | + \rangle + b\langle + | - \rangle|^2$$

Orthonormal: so $\frac{2}{5} = a^2$

$\langle + | + \rangle = 1, \ \langle + | - \rangle = 0$

(f) continued ...

We are also given the probability of finding spin down:

$$P|-\rangle = \frac{3}{5} = |\langle -|\psi\rangle|^2 = |\langle -|(a|+\rangle + b|-\rangle)|^2$$

$$= |a\langle -|+\rangle + b\langle -|-\rangle|^2$$

$$\frac{3}{5} = b^2 \qquad\qquad 0 \qquad\qquad 1$$

The most general a^2 that satisfies $a^2 = \frac{2}{5}$ is $a = \sqrt{\frac{2}{5}}\, e^{i\delta_1}$ where δ_1 is some real number - the phase of an oscillating function with amplitude $\sqrt{\frac{2}{5}}$.

Similarly, the most general b^2 that satisfies $b^2 = \frac{3}{5}$ is $b = \sqrt{\frac{3}{5}}\, e^{i\delta_2}$

Therefore $|\psi\rangle = a|+\rangle + b|-\rangle$

$$= \sqrt{\tfrac{2}{5}}\, e^{i\delta_1} |S_z = \tfrac{\hbar}{2}\rangle + \sqrt{\tfrac{3}{5}}\, e^{i\delta_2} |S_z = -\tfrac{\hbar}{2}\rangle$$

What about the relative phase $(\delta_1 - \delta_2)$? Calculate

$$P(S_x = \hbar/2) = |\langle S_x = \hbar/2 | \psi\rangle|^2$$

$$= \left| \tfrac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\left(\sqrt{\tfrac{2}{5}}\, e^{i\delta_1}\begin{bmatrix}1\\0\end{bmatrix} + \sqrt{\tfrac{3}{5}}\, e^{i\delta_2}\begin{bmatrix}0\\1\end{bmatrix}\right)\right|^2$$

$$= \left|\sqrt{\tfrac{2}{10}}\, e^{i\delta_1} + \sqrt{\tfrac{3}{10}}\, e^{i\delta_2}\right|^2$$

$$= \left(\sqrt{\tfrac{2}{10}}\, e^{-i\delta_1} + \sqrt{\tfrac{3}{10}}\, e^{-i\delta_2}\right)\left(\sqrt{\tfrac{2}{10}}\, e^{+i\delta_1} + \sqrt{\tfrac{3}{10}}\, e^{+i\delta_2}\right)$$

$$= \tfrac{2}{10} e^0 + \tfrac{3}{10} e^0 + \tfrac{\sqrt{6}}{10}\left[e^{i(\delta_1-\delta_2)} + e^{i(\delta_2-\delta_1)}\right]$$

$$= \tfrac{5}{10} + \tfrac{\sqrt{6}}{10}\left[e^{i(\delta_1-\delta_2)} + e^{-i(\delta_1-\delta_2)}\right]$$

$$= \tfrac{1}{2} + \tfrac{\sqrt{6}}{10}\, 2\cos(\delta_1 - \delta_2)$$

$$= \tfrac{1}{2} + \tfrac{\sqrt{6}}{5}\cos(\delta_1 - \delta_2)$$

So a phase difference between terms affects the outcome of measurements.

$\psi^*(x)$

*Exercise 4.2.2.** Show that for a real wave function $\psi(x)$, $= \psi^*(x)$ the expectation value of momentum $\langle P\rangle = 0$. (Hint: Show that the probabilities for the momenta $\pm p$ are equal). Generalize this result to the case $\psi = c\psi_r$, where ψ_r is real and c an arbitrary (real or complex) constant. (Recall that $|\psi\rangle$ and $\alpha|\psi\rangle$ are physically equivalent.)

p. 144 $$\langle p|\psi\rangle = \int_{-\infty}^{\infty} \langle p|x\rangle\langle x|\psi\rangle\,dx = \int_{-\infty}^{\infty} \psi_p^*(x)\,\psi(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\psi(x)\,dx$$

$$\langle\psi|p\rangle = \int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|p\rangle\,dx = \int_{-\infty}^{\infty} \psi^*(x)\,\psi_p(x)\,dx$$

$$= \int_{-\infty}^{\infty} \underset{\text{real}}{\psi(x)}\,\frac{e^{+ipx/\hbar}}{\sqrt{2\pi\hbar}}\,dx = \int_{-\infty}^{\infty} \frac{e^{-i(-p)x/\hbar}}{\sqrt{2\pi\hbar}}\,\psi(x)\,dx$$

$$= \langle -p|\psi\rangle$$

So the probability of $(+p)$ is the same as for $(-p)$,

$\langle p|\psi\rangle = \langle -p|\psi\rangle = \langle\psi|p\rangle$. Conversely,

$\langle -p|\psi\rangle^* = \langle -(-p)|\psi\rangle = \langle\psi|-p\rangle$ (Since real $\psi = \psi^*$)

So $|\langle p|\psi\rangle|^2 = \langle\psi|p\rangle\langle p|\psi\rangle = \langle -p|\psi\rangle\langle\psi|-p\rangle = |\langle -p|\psi\rangle|^2$

Since $|\langle p|\psi\rangle|^2$ is an EVEN function of p,

$$\langle P\rangle = \langle\psi|P|\psi\rangle = \int_{-\infty}^{\infty} \langle\psi|p\rangle\,p\,\langle p|\psi\rangle\,dp = \int_{-\infty}^{\infty} \underbrace{|\langle p|\psi\rangle|^2 p}_{\text{ODD integrand}}\,dp = 0$$

*Exercise 4.2.3.** Show that if $\psi(x)$ has mean momentum $\langle P\rangle$, $e^{ip_0x/\hbar}\psi(x) = \psi_0$ has mean momentum $\langle P\rangle + p_0$.

The mean momentum in the state ψ_0 is

$$\langle P\rangle_0 = \langle \psi_0|P|\psi_0\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi_0(x)\,dx$$

$$= \int \psi^*(x)e^{-ip_0x/\hbar}(-i\hbar)\frac{\partial}{\partial x}\psi(x)e^{ip_0x/\hbar}$$

$$= \int \psi^*(x)e^{-ip_0x/\hbar}(-i\hbar)\left[e^{ip_0x/\hbar}\left(\frac{\partial\psi(x)}{\partial x} + \frac{ip_0}{\hbar}\psi(x)\right)\right]dx$$

$$= \int \psi^*(x)\,(-i\hbar)\left(\frac{\partial\psi(x)}{\partial x} + \frac{ip_0}{\hbar}\psi(x)\right)dx$$

$$= \int \psi^*(x)\,p_0\,\psi(x)\,dx + \int \psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)\,dx$$

$$= p_0\langle\psi|\psi\rangle + \langle\psi|P|\psi\rangle$$

$$= p_0 + \langle P\rangle$$ ✓