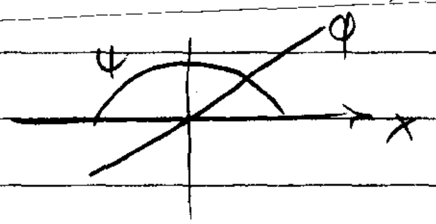
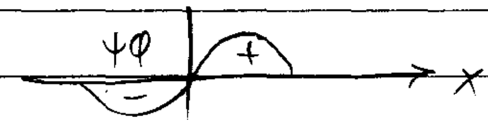
Physical Systems – spring 2007

QM HW 3a Tues. 17 April 2007 - Shankar 7.3.3, 7.3.5, 7.4.1, 7.4.2, 7.5.2 Quantum Harmonic Oscillator

Exercise 7.3.3. **If $\psi(x)$ is even and $\phi(x)$ is odd under $x \to -x$, show that**

$$\int_{-\infty}^{\infty} \psi(x)\phi(x)\,dx = 0$$





$|\psi\phi| = \psi\phi$: Same magnitude, opposite sign of product for $x>0$ or $x<0$. $\therefore$ Sum $= 0$.

Exercise 7.3.5.* **Using the symmetry arguments from Exercise 7.3.3 show that $\langle n | X | n\rangle = \langle n | P | n\rangle = 0$ and thus that $\langle X^2\rangle = (\Delta X)^2$ and $\langle P^2\rangle = (\Delta P)^2$ in these states. Show that $\langle 1 | X^2 | 1\rangle = 3\hbar/2m\omega$ and $\langle 1 | P^2 | 1\rangle = \frac{3}{2}m\omega\hbar$. Show that $\psi_0(x)$ saturates the uncertainty bound $\Delta X \cdot \Delta P > \hbar/2$.**

$|n\rangle$ = eigenstate of QHO.

$$\langle n|X|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a+a^+|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n|a|n\rangle + \langle n|a^+|n\rangle\right]$$

$$\langle n|a|n\rangle = \sqrt{n}\,\langle n|n-1\rangle \xrightarrow{} 0 = 0$$ since $|n\rangle$ and $|n-1\rangle$ are orthogonal (even & odd)

$$\langle n|a^+|n\rangle = \sqrt{n+1}\,\langle n|n+1\rangle = 0$$ ″

$$\langle X\rangle = 0$$

$$\langle n|P|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle n|a^+-a|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left[\langle n|a^+|n\rangle - \langle n|a|n\rangle\right]$$

$$= i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n+1}\,\langle n|n+1\rangle - \sqrt{n}\,\langle n|n-1\rangle\right] = 0$$

To find the expectation value for X^2 in QHO state $|n\rangle$

$\langle n|X^2|n\rangle$ use $X = \left(\frac{\hbar}{2m\omega}\right)^{\frac{1}{2}}(a+a^+)$

$$X^2 = \left(\frac{\hbar}{2m\omega}\right)(a+a^+)(a+a^+)$$

$$= \left(\frac{\hbar}{2m\omega}\right)(a^2 + a^+a + aa^+ + a^+a^+)$$

$$\langle n|X^2|n\rangle = \left(\frac{\hbar}{2m\omega}\right)\left[\langle n|a^2|n\rangle + \langle n|a^+a|n\rangle + \langle n|aa^+|n\rangle + \langle n|a^+a^+|n\rangle\right]$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \quad \text{and} \quad a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\langle n|a^2|n\rangle = \sqrt{n}\langle n|a|n-1\rangle = \sqrt{n}\sqrt{n-1}\langle n|n-2\rangle$$

$$= \sqrt{n}\sqrt{n-1}\;\delta_{n,n-2} = 0$$

Similarly $\langle n|a^+a^+|n\rangle \rightarrow \langle n|n+2\rangle = 0$

$$\langle n|a^+a|n\rangle = \sqrt{n}\langle n|a^+|n-1\rangle = \sqrt{n}\sqrt{n}\langle n|n\rangle = n\cdot 1$$

$$\langle n|aa^+|n\rangle = \sqrt{n+1}\langle n|a|n+1\rangle = \sqrt{n+1}\sqrt{n+1}\langle n|n\rangle = n+1$$

$$\langle n|X^2|n\rangle = \frac{\hbar}{2m\omega}(n+n+1) = \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle 1|X^2|1\rangle = \frac{\hbar}{2m\omega}(2+1) = 3\hbar/2m\omega \quad = \langle X^2\rangle_1$$

for $|n=1\rangle$, $\Delta X^2 = \langle X^2\rangle - \langle X\rangle^2$

$$= \frac{3\hbar}{2m\omega} - 0$$

Therefore $\Delta X = \sqrt{\frac{3\hbar}{2m\omega}}$

To find the expectation value for P^2 in QHO state $|n\rangle$

$\langle n|P^2|n\rangle$ use $P = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(a^+ - a)$

$$P^2 = -\left(\frac{m\omega\hbar}{2}\right)(a^+a^+ - a^+a - aa^+ + a^2)$$

$$\langle n|P^2|n\rangle = -\left(\frac{m\omega\hbar}{2}\right)\left[\langle n|a^+a^+|n\rangle - \langle n|a^+a|n\rangle - \langle n|aa^+|n\rangle + \langle n|a^2|n\rangle\right]$$

$$\langle n|a^+a^+|n\rangle \rightarrow \langle n|n+2\rangle = 0, \quad \langle n|a^2|n\rangle \rightarrow \langle n|n-2\rangle = 0$$

$\langle n|a^+a|n\rangle = n$, $\langle n|aa^+|n\rangle = n+1$ from previous page

$$\langle n|P^2|n\rangle = -\left(\frac{m\omega\hbar}{2}\right)\left[-n-(n+1)\right] = \frac{m\omega\hbar}{2}(2n+1)$$

$$\langle 1|P^2|1\rangle = \frac{m\omega\hbar}{2}(2+1) = \frac{3}{2}m\omega\hbar$$

$$\Delta P^2 = \langle P^2\rangle - \langle P\rangle^2 = \langle P^2\rangle$$

For $|n=1\rangle$, $\Delta P^2 = \frac{3}{2}m\omega\hbar$ so $\Delta P = \sqrt{\frac{3}{2}m\omega\hbar}$

Uncertainty relation for $|n=1\rangle$

$$\Delta X\,\Delta P = \sqrt{\frac{3\hbar}{2m\omega}}\sqrt{\frac{3m\omega\hbar}{2}} = \frac{3}{2}\hbar > \frac{\hbar}{2} \checkmark$$

Exercise 7.4.1.* **Compute the matrix elements of X and P in the $|n\rangle$ basis and compare with the result from Exercise 7.3.4.**

$$\langle n'|X|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n'|a+a^{+}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\langle n'|a|n\rangle + \langle n'|a^{+}|n\rangle\right]$$

$$\langle n'|a|n\rangle = \sqrt{n}\,\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n',n-1}$$

$$\langle n'|a^{+}|n\rangle = \sqrt{n+1}\,\langle n'|n+1\rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

$$\langle n'|X|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right]$$

$$\langle n'|P|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\langle n'|a^{+}-a|n\rangle =$$

$$= i\sqrt{\frac{m\omega\hbar}{2}}\left[\langle n'|a^{+}|n\rangle - \langle n'|a|n\rangle\right]$$

$$\langle n'|P|n\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\left[\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right]$$

Exercise 7.3.4.* **Using Eqs. (7.3.23)–(7.3.25), show that**

$$\langle n'|X|n\rangle = \left(\frac{\hbar}{2m\omega}\right)^{1/2}[\delta_{n',n+1}(n+1)^{1/2} + \delta_{n',n-1}n^{1/2}]$$

$$\langle n'|P|n\rangle = \left(\frac{m\omega\hbar}{2}\right)^{1/2} i[\delta_{n',n+1}(n+1)^{1/2} - \delta_{n',n-1}n^{1/2}]$$

Matches ✓

*Exercise 7.4.2.** Find $\langle X \rangle$, $\langle P \rangle$, $\langle X^2 \rangle$, $\langle P^2 \rangle$, $\Delta X \cdot \Delta P$ in the state $|n\rangle$.

I showed in 7.3.5 that $\langle X \rangle = 0$, $\langle P \rangle = 0$

$\langle X^2 \rangle = \frac{\hbar}{2m\omega}(2n+1)$, $\langle P^2 \rangle = \frac{m\omega\hbar}{2}(2n+1)$

$\Delta X \Delta P = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{2n+1}\sqrt{\frac{m\omega\hbar}{2}}\sqrt{2n+1} = (2n+1)\frac{\hbar}{2}$

Exercise 7.5.2. Project the relation

$$(1) \qquad a\,|n\rangle = n^{1/2}\,|n-1\rangle$$

on the X basis and derive the recursion relation

$$H_n'(y) = 2nH_{n-1}(y)$$

using Eq. (7.3.22).

$$\psi_E(x) = \psi_{(n+1/2)\hbar\omega}(x) = \psi_n(x)$$
$$= \left(\frac{m\omega}{\pi\hbar 2^{2n}(n!)^2}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left[\left(\frac{m\omega}{\hbar}\right)^{1/2} x\right] \qquad (7.3.22)$$

In the x-basis, (1) becomes

$$a|n\rangle \to \int_{-\infty}^{\infty} \langle x|a|x'\rangle \overbrace{\langle x'|n\rangle}^{\psi_n(x)} dx' = \sqrt{n}\,\overbrace{\langle x|n-1\rangle}^{\psi_{n-1}(x)}. \text{ Use } a = \sqrt{\tfrac{m\omega}{2\hbar}}\,X + i\sqrt{\tfrac{1}{2m\omega\hbar}}\,P$$

Matrix elements of a in the x-basis are

$$\langle x|a|x'\rangle = \sqrt{\frac{m\omega}{2\hbar}}\langle x|X|x\rangle + i\sqrt{\frac{1}{2m\omega\hbar}}\langle x|P|x'\rangle$$
$$= \sqrt{\frac{m\omega}{2\hbar}}\,x'\,\delta(x-x') + i\sqrt{\frac{1}{2m\omega\hbar}}(-i\hbar)\,\delta(x-x')\frac{d}{dx}$$

$$\text{So } \int_{-\infty}^{\infty}\langle x|a|x'\rangle\langle x|n\rangle\,dx' = \sqrt{\frac{m\omega}{2\hbar}}\int x'\delta(x-x')\,\psi_n(x)\,dx' + \hbar\sqrt{\frac{1}{2m\omega\hbar}}\int \delta(x-x')\frac{d}{dx}\psi_n(x)\,dx$$
$$= \sqrt{\frac{m\omega}{2\hbar}}\;x\,\psi_n(x) \qquad + \sqrt{\frac{\hbar}{2m\omega}}\;\frac{d}{dx}\psi_n(x)$$

7.5.2 Continued - So far we turned (1) into

$$\langle x|a|n\rangle = \sqrt{n}\,\langle x|n-1\rangle$$

(2) $$\sqrt{\frac{m\omega}{2\hbar}}\, x\, \psi_n(x) + \sqrt{\frac{\hbar}{2m\omega}}\, \frac{d}{dx}\psi_n(x) = \sqrt{n}\; \psi_{n-1}(x)$$

Now subsitute $y = \sqrt{\frac{m\omega}{\hbar}}\, x$, $dy = \sqrt{\frac{m\omega}{\hbar}}\, dx$

$$\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy} = \sqrt{\frac{m\omega}{\hbar}}\frac{d}{dy}$$

Then (2) becomes

$$\sqrt{n}\,\psi_{n-1}(y) = \frac{y}{\sqrt{2}}\psi_n(y) + \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\omega}{\hbar}}\frac{d}{dy}\psi_n(y) = n\psi_{n-1}(y) = \frac{y}{\sqrt{2}}\psi_n(y) + \frac{1}{\sqrt{2}}\frac{d}{dy}\psi_n(y)$$

where we can write (7.3.22) in terms of y:

$$\psi_n(y) = \left[\frac{C}{2^n n!}\right]^{1/2} e^{-y^2/2} H_n(y) \quad \text{where } C = \sqrt{\frac{m\omega}{\pi\hbar}}$$

Simplify:

$$\sqrt{n}\,\psi_{n-1}(y) = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{C}{2^n n!}}\left[y + \frac{d}{dy}\right] e^{-\frac{y^2}{2}} H_n(y)$$

$$= \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{C}{2^n n!}}\left[y e^{-y^2/2}\cancel{H_n(y)} - y e^{-y^2/2}\cancel{H_n(y)} + e^{-y^2/2}\frac{dH_n(y)}{dy}\right]$$

$$\sqrt{n}\,\psi_{n-1}(y) = \sqrt{n}\left[\frac{C}{2^{n-1}(n-1)!}\right]^{1/2}\cancel{e^{-y^2/2}} H_{n-1}(y) = \left[\frac{C}{2^{n+1}n!}\right]^{1/2}\cancel{e^{-y^2/2}}\frac{dH_n(y)}{dy}$$

$$\frac{dH_n}{dy} = H_{n-1}\sqrt{n}\left[\frac{2^{n+1}n!}{2^{n-1}(n-1)!}\right]^{1/2} = H_{n-1}\sqrt{n}\left[2^2 n\right]^{1/2}$$

$$\frac{dH_n}{dy} = H_{n-1}\sqrt{n}\,(2\sqrt{n}) = 2n\,H_{n-1}$$ ✓

QM HW 3b Thus. 19 April 2007 - Shankar 7.4.5

Exercise 7.4.5.* At $t = 0$ a particle starts out in $|\psi(0)\rangle = 1/2^{1/2}(|0\rangle + |1\rangle)$. (i) Find $|\psi(t)\rangle$; (ii) find $\langle X(0)\rangle = \langle\psi(0)|X|\psi(0)\rangle$, $\langle P(0)\rangle$, $\langle X(t)\rangle$, $\langle P(t)\rangle$; (iii) find $\langle\dot{X}(t)\rangle$ and $\langle\dot{P}(t)\rangle$ using Ehrenfest's theorem and solve for $\langle X(t)\rangle$ and $\langle P(t)\rangle$ and compare with part (ii).

$\Psi(t=0)$ has equal parts $|n=0\rangle$ and $|n=1\rangle$; $\Psi(x,t=0) = \frac{|n=0\rangle + |n=1\rangle}{\sqrt{2}} = \Psi(0)$

(i) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left[e^{-iE_0t/\hbar}|n=0\rangle + e^{-iE_1t/\hbar}|n=1\rangle\right]$

where $E_n = (n+\frac{1}{2})\hbar\omega \rightarrow E_0 = \frac{1}{2}\hbar\omega$, $E_1 = \frac{3}{2}\hbar\omega$.

(ii) $\langle X(0)\rangle = \langle\Psi(0)|X|\Psi(0)\rangle$ where $X = \sqrt{\frac{\hbar}{2m\omega}}(a+a^+)$

$$= \left(\frac{1}{\sqrt{2}}\right)^2\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0| + \langle 1|\right)\left|(a+a^+)\right|\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|(a+a^+)|0\rangle + \langle 1|(a+a^+)|0\rangle\right) + \langle 0|a+a^+|1\rangle + \langle 1|a+a^+|1\rangle$$

$$\langle 0|(a+a^+)|0\rangle = \langle 0|a|0\rangle + \langle 0|a^+|0\rangle \rightarrow 0 + \sim\langle 0|1\rangle = 0$$

$a|0\rangle = 0$

$$\langle 1|(a+a^+)|0\rangle = \langle 1|a|0\rangle + \underline{\langle 1|a^+|0\rangle} = 0 + \sqrt{1}\,\langle 1|1\rangle = 1$$

$$\langle 0|a+a^+|1\rangle = \underline{\langle 0|a|1\rangle} + \langle 0|a^+|1\rangle = \sqrt{1}\langle 0|0\rangle + \sim\langle 0|2\rangle$$

$= 1 + 0$

$$\langle 1|a+a^+|1\rangle = \langle 1|a|1\rangle + \langle 1|a^+|1\rangle$$

$\sim\langle 1|0\rangle + \sim\langle 1|2\rangle = 0$

$$\langle X(0)\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}(1+1) = \sqrt{\frac{\hbar}{2m\omega}} \quad (1)$$

$$p = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^+ - a)\ , \quad \psi(0) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\langle P(0)\rangle = i\sqrt{\frac{m\omega\hbar}{2}}\Big[(\langle 0| + \langle 1|)(a^+ - a)(|0\rangle + |1\rangle)\Big]\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\Big[\langle 0|a^+ - a|0\rangle + \langle 1|a^+ - a|0\rangle + \langle 0|a^+ - a|1\rangle + \langle 1|a^+ - a|1\rangle\Big]$$

$$\langle 0|a^+ - a|0\rangle = \langle 0|a^+|0\rangle - \langle 0|a|0\rangle \sim \langle 0|1\rangle + 0 = 0$$

$$\langle 1|a^+ - a|0\rangle = \underline{\langle 1|a^+|0\rangle} - \langle 1|a|0\rangle = \sqrt{1}\langle 1|1\rangle - 0 = 1$$

$$\langle 0|a^+ - a|1\rangle = \langle 0|a^+|1\rangle - \underline{\langle 0|a|1\rangle} = \sqrt{2}\langle 0|2\rangle - \sqrt{1}\langle 0|0\rangle$$

$$= 0 - 1$$

$$\langle 1|a^+ - a|1\rangle = \langle 1|a^+|1\rangle - \langle 1|a|1\rangle = 0$$

$$\langle P(0)\rangle = \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}(1-1) = 0$$ ✓ 2

$$\langle X(t)\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big[e^{+iE_0t/\hbar}\langle 0| + e^{+iE_1t/\hbar}\langle 1|\Big(a + a^+\Big)\Big|e^{-iE_0t/\hbar}|0\rangle + e^{-iE_1t/\hbar}|1\rangle\Big]$$

we've already seen that the only terms that don't vanish are $\langle 1|a^+|0\rangle$ and $\langle 0|a|1\rangle$

$$\langle x(t)\rangle = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big[e^{\frac{iE_1t}{\hbar}}\langle 1|a^+|0\rangle e^{\frac{-iE_0t}{\hbar}} + e^{\frac{iE_0t}{\hbar}}\langle 0|a|1\rangle e^{-iE_1t/\hbar}\Big]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big[e^{\frac{i(E_1-E_0)t}{\hbar}}\sqrt{1}\langle 1|1\rangle + e^{\frac{i(E_0-E_1)t}{\hbar}}\sqrt{1}\langle 0|0\rangle\Big]$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\Big[e^{\frac{i\hbar\omega t}{\hbar}} + e^{\frac{-i\hbar\omega t}{\hbar}} = e^{i\omega t} + e^{-i\omega t}\Big]$$

$$\langle x(t)\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$ ✓ 3

$$\langle P(t)\rangle = \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left[e^{+\frac{iE_0t}{\hbar}}\langle 0| + e^{\frac{iE_1t}{\hbar}}\langle 1|\;\middle|a^{+}-a\middle|\;e^{-\frac{iE_0t}{\hbar}}|0\rangle + e^{-\frac{iE_1t}{\hbar}}|1\rangle\right.$$

We already saw that the only terms that don't vanish are

$\langle 1|a^{+}|0\rangle = 1$ and $\langle 0|a|1\rangle = 1$

$$\langle P(t)\rangle = \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\; e^{\frac{iE_1t}{\hbar}}\langle 1|a^{+}|0\rangle e^{\frac{-iE_0t}{\hbar}} - e^{\frac{iE_0t}{\hbar}}\langle 0|a|1\rangle e^{-\frac{iE_1t}{\hbar}}$$

$$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left[e^{\frac{i(E_1-E_0)t}{\hbar}} - e^{\frac{i(E_0-E_1)t}{\hbar}}\right]$$

$$= \frac{i}{2}\sqrt{\frac{m\omega\hbar}{2}}\left[e^{i\omega t} - e^{-i\omega t}\right]$$

$$\langle P(t)\rangle = -\sqrt{\frac{m\omega\hbar}{2}}\sin\omega t \quad (4)$$

(iii) Find $\langle\frac{\partial x}{\partial t}\rangle$ and $\langle\frac{\partial p}{\partial t}\rangle$ using Ehrenfest's theorem:

(using (4) found above)

(6.5) $\langle\frac{\partial x}{\partial t}\rangle = \frac{\langle P\rangle}{m} = -\frac{1}{m}\sqrt{\frac{m\omega\hbar}{2}}\sin\omega t = -\sqrt{\frac{\omega\hbar}{2m}}\sin\omega t$

(6.8) $\langle\frac{dP}{dt}\rangle = -\frac{i}{\hbar}\langle[P,H]\rangle = \langle-\frac{\partial H}{\partial X}\rangle$ where $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2X^2$

$\frac{\partial H}{\partial X} = m\omega^2 X$ so, using (3) found above,

$$\langle\frac{\partial P}{\partial t}\rangle = -m\omega^2\langle X(t)\rangle = -m\omega^2\sqrt{\frac{\hbar}{2m\omega}}\cos\omega t = -\omega^2\sqrt{\frac{\hbar m}{2\omega}}\cos\omega t$$

Check: $\langle\frac{\partial P}{\partial t}\rangle = \frac{\partial}{\partial t}\left(-\sqrt{\frac{m\omega\hbar}{2}}\right)\sin\omega t = -\omega\sqrt{\frac{m\omega\hbar}{2}}\cos\omega t$ ✓

same

7.4.5.iii