Physical Systems Midterm – EM & QM - Thus.26.April.2007 – TESC - EJZ

This is a take-home exam. You may use your notes, text, and tables of integrals.
Each section is designed to take 2 hours; please limit yourself to 4 hours per section (perhaps less on EM, more on QM).

SHOW YOUR WORK neatly, and include units where appropriate, to receive full credit.
Please circle or underline your answers for clarity.
Express answers in *simplest exact form* whenever possible.
Order-of-magnitude estimates are usually fine for numerical problems.

(please sign legibly) Zita - Solutions

I affirm that I have worked this exam using only my notes (including homework), text, and table of integrals as resources – no calculators, computers, or other outside resources.

Electromagnetism: *Please note the time you spent on this section: ____ hours*

1. Lorentz
2. Faraday
3. ~~Gauss,~~ Ohm, Ampere, Poynting
4. Discussion

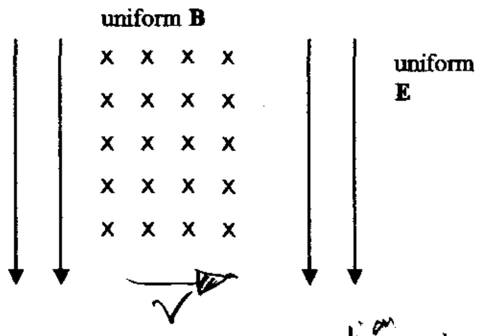
Quantum Mechanics: *Please note the time you spent on this section: ____ hours*

1. Notation
2. Eigenvalues and eigenvectors
3. Polynomial derivative operator
4. Time-dependent particle in a box

ELECTROMAGNETISM section

EM1. A mass spectrometer measures the mass of charged particles by first determining their velocity and then deflecting them with a magnetic field.

(a) Velocity selector: When a charged particle travels through crossed E and B fields, it will be undeflected (and will continue to travel in a straight line) only if its velocity has a certain relationship to E and B. This setup effectively "selects" particles with velocity v.



(a) Derive the relationship between E, v, and B that will select undeflected particles of velocity v.

undeflected: $\frac{dv}{dt} = 0 = \Sigma F = F_E + F_B$

$F_E = -F_B$

$qE = -q v \times B$

$E = -v \times B$

in direction shown, or:

$V = \frac{E}{B}$ $\left(v = \frac{E \times B}{B^2} \right)$

(b) If the electric field E doubles, how will the selected particle's ENERGY change? It will:
(show your work)

Halve - stay the same – double – quadruple – other

Energy = Kinetic energy in this case $= \frac{1}{2} m v^2$

Energy $\propto v^2 \propto E^2_{\text{Electric Field}}$

Now that we know the speed v of the particle from the E and B settings, we'd like to determine the mass m of the particle – which is the point of a mass spectrometer.

Magnetic deflection:

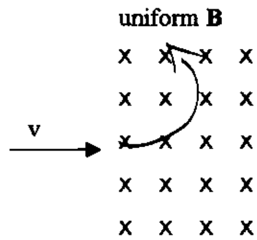
When a charged particle of velocity v enters a region of perpendicular magnetic field (B) (with E=0 now), **how is it deflected**?

(c) Sketch the path of a positively charged particle.

(d) Derive an expression for its radius of curvature in terms of q, v, and B.

(e) Find the mass of the particle in terms of measurable quantities and the charge q. (r, E, B)

The r can be measured.



$\vec{F} = q\vec{v} \times \vec{B}$ - It is bent up into a circular path of radius r

$F = ma$

$qvB = m\frac{v^2}{r}$

(b) $r = \frac{mv^2}{qvB} = \frac{mv}{qB}$

(c) $m = \frac{rqB}{v} = \frac{rqB}{E/B} = \frac{rqB^2}{E}$

(f) If the magnetic field B in the deflection region **doubles, how will the RADIUS of curvature of the particle's path change?** It will:
(show your work)

Halve - stay the same – double – quadruple - other

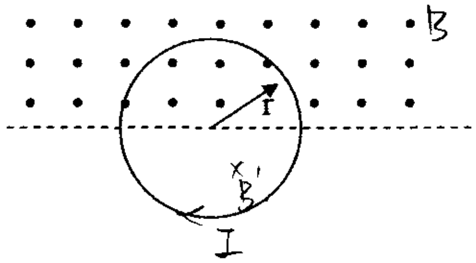
Stronger field → tighter orbit.

EM2. Half of a conducting loop of radius r lies in a uniform magnetic field that is directed out of the page.
The field magnitude is given by $B = 4t^2 + 2t + 3$, with B in teslas and t in seconds.

(a) What is the direction of the emf induced around the loop by the field? Explain.

(b) Find an expression for the emf induced in the loop at any time t.

(c) Find the magnitude of the current induced in a loop with resistance 2 ohms, and radius $= 0.1\,m$ **at time t=10s.**



$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \bar{B}\cdot d\bar{A}$$

$$A = \frac{1}{2}(\pi r^2)$$

$$\frac{dB}{dt} = \frac{d}{dt}(4t^2+2t+3) = 8t+2 \left(\frac{T}{s}\right)$$

(a) B is increasing out of the page
Lenz's law: Induced $\mathcal{E}$ OPPOSES THE INCREASE IN FLUX.
Induced B' is opposite B: INTO the page
This requires CLOCKWISE induced I & $\mathcal{E}$

(b) $\mathcal{E} = -A\frac{dB}{dt} = -\frac{\pi r^2}{2}(8t+2) = -\pi r^2(4t+1)$ volts

(c) $\mathcal{E} = IR$ (Ohm's Law) so

$$I = \frac{\mathcal{E}}{R} = \frac{\pi r^2(4t+1)}{R} = \frac{\pi(0.1\,m)^2(4\times 10+1)\left(\frac{T}{s}\right)}{2\text{ ohms}} = \frac{41\pi\times 10^{-2}}{2}$$

units: $\frac{Tm^2}{s}$ = volts, $\frac{\text{volts}}{\text{ohms}}$ = amps

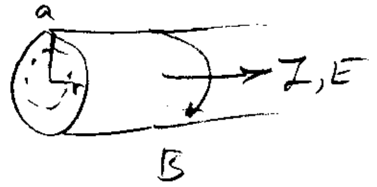
$I \sim 20\pi\times 10^{-2} \sim 60\times 10^{-2} \sim 0.6$ amps

HRW 30-2 p. 797

EM3. A cylindrical conductor of radius a and conductivity σ carries a steady current I distributed uniformly across its cross-section.

(a) Draw the configuration, clearly labeled.
(b) Determine **E** inside the conductor.
(c) Determine **B** ~~just outside~~ inside the conductor.
(d) Determine the Poynting vector at the surface of the conductor. What is its direction and magnitude?
(e) Integrate over the Poynting vector to show that the rate at which electromagnetic energy enters the conductor is equal to the Ohmic dissipation rate. (Hint: resistance = resistivity × length/area)

(a)



$\frac{\text{current}}{\text{area}} = \text{constant} = J = \frac{I}{\pi a^2}$

$\frac{I(r)}{\pi r^2} = \frac{I}{\pi a^2} \rightarrow \underline{I(r) = I\frac{r^2}{a^2}}$

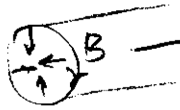
(b) Ohm's Law: $V = IR$

$E = J\eta = \frac{J}{\sigma} = \frac{I}{\pi a^2 \sigma} = E = \underline{\frac{I(r)}{\pi r^2 \sigma}}$

(c) $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{(r)} = B \cdot 2\pi r \rightarrow \underline{B(r) = \frac{\mu_0 I(r)}{2\pi r}} = \frac{\mu_0 I r^2}{2\pi r a^2}$

$= \frac{\mu_0 I r}{2\pi a^2}$

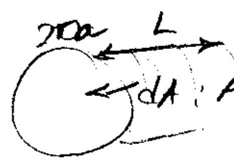
(d)



$S = \frac{1}{\mu_0} E \times B$ is <u>INWARD</u>

$S(a) = \frac{1}{\mu_0}\frac{I}{\pi a^2 \sigma}\frac{\mu_0 I}{2\pi a} = \underline{\frac{I^2}{2\pi^2 \sigma a^3}} = \frac{\text{power}}{\text{area}}$

(e) Power $= \int \vec{S}\cdot d\vec{A}_{\text{cylinder}}$



dA: flow into cylinder

$= S(a)\cdot 2\pi a \cdot L = \frac{I^2\, 2\pi a L}{2\pi^2 \sigma a^3} = \frac{I^2 L}{\pi \sigma a^2} = \frac{I^2 L}{\sigma A_{disk}}$

(e) Ohmic power dissipated $= I^2 R$ where $R = \frac{L}{\sigma \cdot A_{disk}}$ ✓ since conductivity $= \frac{1}{\text{resistivity}}$ 4

Giancoli 32-52/p.809

QUANTUM MECHANICS section

QM1. Consider two arbitrary numbers α and β, two arbitrary kets $|\psi\rangle$ and $|\phi\rangle$, and an arbitrary operator Ω. **Classify the following expressions** as real numbers, pure imaginary numbers, general complex numbers, bras, kets, operators, or nonsense. If nonsense, explain why, briefly. *Choose the description that most accurately describes the expression.*

a. $\langle\phi|\psi\rangle$ = complex number

b. $\alpha|\psi\rangle+\Omega|\phi\rangle$ = ket

c. $\langle\psi|i\psi\rangle$ = imaginary

d. $\alpha|\psi\rangle+\beta^*\langle\phi|$ = nonsense (bra + ket)

e. $\langle\psi|\Omega^\dagger\Omega|\psi\rangle$ = real

f. $|\psi\rangle\langle\phi\rangle$ = nonsense

g. $\Omega\langle\phi|\psi\rangle$ = $\Omega\cdot$ number = operator

h. $\langle\psi|\Omega$ = bra

QM2. Consider the operator below.

a. Show that it is Hermition. $\Omega = \Omega^\dagger$

b. Find its eigenvalues and a set of orthonormal eigenvectors.

(a) $\Omega \leftrightarrow \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix} \xrightarrow{\text{transpose}} \begin{bmatrix} 1 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 1 \end{bmatrix} \xrightarrow{\text{conjugate}} \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix} = \Omega^\dagger = \Omega$ ✓

(b) Eigenvalues solve $\det(\Omega - I\omega) = 0 = \begin{vmatrix} 1-\omega & 0 & i \\ 0 & 2-\omega & 0 \\ -i & 0 & 1-\omega \end{vmatrix}$

$0 = (1-\omega)[(2-\omega)(1-\omega)-0] - 0 + i(0+i(2-\omega))$

$= (1-\omega)(2-\omega)(1-\omega) - (2-\omega)$

$= (2-\omega)[(1-\omega)^2 - 1] = (2-\omega)[1-2\omega+\omega^2-1] = (2-\omega)(\omega^2-2\omega)$

$= (2-\omega)(\omega-2)\,\omega$

Eigenvalues are $\underline{\omega=0}$, $\underline{\omega=2}$, $\underline{\omega=2}$

Eigenvectors $|\omega\rangle = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfy $(\Omega-\omega I)|\omega\rangle = 0$

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-\omega & 0 & i \\ 0 & 2-\omega & 0 \\ -i & 0 & 1-\omega \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

for $\omega = 0$,

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\rightarrow a = -ic$, $\rightarrow b = 0$, $\rightarrow ia = c$

$|\omega=0\rangle = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}\Big/\sqrt{2}$ or $\begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix}\Big/\sqrt{2}$

QN 2 - continued

for $\omega = 2$, $\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1-2 & 0 & i\\ 0 & 2-2 & 0\\ -i & 0 & 1-2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}-1 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & -1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$ $\begin{array}{l} a = ic \\ \to b = \text{anything} \\ \to ia = -c \end{array}$

$$|\omega = 2\rangle = \begin{bmatrix}1\\1\\-i\end{bmatrix}\Big/\sqrt{3} \quad \text{or} \quad \begin{bmatrix}i\\1\\1\end{bmatrix}\Big/\sqrt{3} \quad \text{or} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}i\\0\\1\end{bmatrix}\Big/\sqrt{2} \quad \text{or} \quad \begin{bmatrix}1\\0\\-i\end{bmatrix}\Big/\sqrt{2}$$

Lots of options. Good thing, because we need two orthogonal $|\omega = 2\rangle$ eigenvectors since the $\omega = 2$ eigenvalue is degenerate. We could pick one, and construct the second, or we could check two of these for orthogonality. Let's try $|\omega_1 = 2\rangle = \begin{bmatrix}1\\1\\-i\end{bmatrix}\Big/\sqrt{3}$ and $|\omega_2 = 2\rangle = \begin{bmatrix}1\\0\\-i\end{bmatrix}\Big/\sqrt{2}$. They're orthogonal if $\langle\omega_1 = 2|\omega_2 = 2\rangle = 0$

$$\langle\omega_1 = 2|\omega_2 = 2\rangle = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & +i\end{bmatrix}\begin{bmatrix}1\\0\\-i\end{bmatrix} = \frac{1}{\sqrt{6}}(1 + 0 + 1) \neq 0 \quad \text{oh oh -}$$

In fact, none of those five are mutually orthogonal! We'll have to construct another $|\omega_2 = 2\rangle$, either using Gram Schmidt, or by inspection. I'll use $|\omega_1 = 2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}i\\0\\1\end{bmatrix}$

Then $\langle\omega_1 = 2|\omega_2 = 2\rangle \sim \begin{bmatrix}-i & 0 & 1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = -ia + c = 0$

when $c = ia$ and $b =$ anything, e.g. $|\omega_2 = 2\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\i\end{bmatrix}$

Here we go, one of several possible bases for Ω:

$$|\omega = 0\rangle = \begin{bmatrix}1\\0\\i\end{bmatrix}\Big/\sqrt{2}, \quad |\omega_1 = 2\rangle = \begin{bmatrix}i\\0\\1\end{bmatrix}\Big/\sqrt{2}, \quad |\omega_2 = 2\rangle = \begin{bmatrix}1\\1\\i\end{bmatrix}\Big/\sqrt{3}$$

To check that this is an orthonormal set, I could test each dot product OR construct $U = \begin{bmatrix} w_0 & w_1 & w_2 \end{bmatrix}$

and see if $U^{+}\Omega U \longrightarrow$ diagonalizes Ω with eigenvalues on the diagonal.

I think it's easier to check the inner products.

$\langle w=0 | w_1=2 \rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \langle 1\ 0\ -i | i\ 0\ 1 \rangle = \frac{1}{2}(i+0-i) = 0$

$\langle w=0 | w_2=2 \rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \langle 1\ 0\ -i | 1\ 1\ i \rangle = \frac{1}{\sqrt{6}}(1+0-1) = 0$

$\langle w_1=2 | w_2=2 \rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} \langle -i\ 0\ 1 | 1\ 1\ i \rangle = \frac{1}{\sqrt{6}}(-i+0+i) = 0$ ✓

all orthogonal.

This orthonormal eigenbasis is not <u>unique</u> of course.

I could have chosen two other eigenvectors for $|w=0\rangle$ and $|w=2\rangle$...

QM3. Consider the 3-dimensional space of all functions of the form f(x) = ax^{2} + bx + c (that is, all polynomials of second order or less). This is a valid vector space: if you add two such functions, you get a function of the same type; if you multiply such a function by a scalar, you get a function of the same type; and all the vector space axioms are satisfied under these operations. This is a 3D space in that specification of the three parameters a, b, c completely specifies the polynomial. Now define the inner product of two kets in this space to be:

$\int_{-1}^{1} f^{*}(x)g(x)dx = \langle f \mid g \rangle$ (Note that we are only considering the range from -1 to 1.)

a. **Show that the polynomials below are an orthonormal basis** for this space.

$$P_0(x) = \sqrt{\tfrac{1}{2}},\ P_1(x) = \sqrt{\tfrac{3}{2}}x,\ P_2(x) = \sqrt{\tfrac{5}{8}}\left(3x^2 - 1\right)$$

b. The derivative operator sends a ket in this space to a ket (of lower order) in this space. **Find the matrix representing the derivative operator** in the basis given above.
Hint: matrix elements are $\langle i \mid D \mid j \rangle = \int_{-1}^{1} P_i \frac{\partial P_j}{\partial x}\, dx$

c. **Show that this matrix does what it is supposed to** this way:
(i.) Choose an arbitrary polynomial.
(ii) Express it as a column vector in the basis given above.
(iii.) Operate on it with the derivative operator (matrix).
(iv.) Show that the resulting column vector indeed represents the function which is the derivative of your original function.

(a)

$$\langle 0|1\rangle = \int_{-1}^{1} P_0^* P_1\, dx = \int_{-1}^{1} \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{3}{2}}\, x\, dx = \frac{\sqrt{3}}{2}\frac{x^2}{2}\Big|_{-1}^{1} = \frac{\sqrt{3}}{4}\left(1^2 - (-1)^2\right) = 0 \checkmark$$

$$\langle 0|2\rangle = \int_{-1}^{1} P_0^* P_2\, dx = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{5}{8}}\int_{-1}^{1}(3x^2-1)\,dx = \sqrt{\tfrac{5}{2\cdot 8}}\left(\frac{3x^3}{3} - x\right)\Big|_{-1}$$
$$= \sqrt{\tfrac{5}{2\cdot 8}}\left[1^3 - (-1)^3 - \{1-(-1)\}\right] = \sqrt{\tfrac{5}{2\cdot 8}}\left[1+1-\{1+1\}\right] = 0 \checkmark$$

$$\langle 1|2\rangle = \int P_1^* P_2\, dx = \sqrt{\tfrac{3}{2}}\sqrt{\tfrac{5}{8}}\int_{-1}^{1} x(3x^2-1)\,dx \rightarrow \int (3x^3 - x)\,dx = \frac{3x^4}{4} - \frac{x^2}{2}\Big|_{-1}^{1}$$
$$\rightarrow \frac{3}{4}\left(1^4 - (-1)^4\right) - \frac{1}{2}\left(1^2 - (-1)^2\right) = \frac{3}{4}(1-1) - \frac{1}{2}(1-1) = 0 \checkmark$$

So they are orthogonal. Now check norms:

$$\langle 0|0\rangle = \int_{-1}^{1} P_0^* P_0\, dx = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2}}\int dx = \frac{1}{2}x\Big|_{-1}^{1} = \frac{1}{2}(1-(-1)) = \frac{1}{2}\cdot 2 = 1 \checkmark$$

$$\langle 1|1\rangle = \int_{-1}^{1} P_1^* P_1\, dx = \sqrt{\tfrac{3}{2}}\sqrt{\tfrac{3}{2}}\int x^2\, dx = \frac{3}{2}\frac{x^3}{3}\Big|_{-1}^{1} = \frac{1}{2}\left(1^3 - (-1)^3\right) = \frac{1}{2}\cdot 2 = 1 \checkmark$$

$$\langle 2|2\rangle = \int_{-1}^{1} P_2^* P_2\, dx = \sqrt{\tfrac{5}{8}}\sqrt{\tfrac{5}{8}}\int_{-1}^{1}(3x^2-1)^2\,dx = \frac{5}{8}\int(9x^4 - 6x^2 + 1)\,dx = \frac{5}{8}\left(\frac{9x^5}{5} - \frac{6x^3}{3} + x\right)\Big|_{-1}^{1}$$

(yes, orthonormal too)

$$= \frac{5}{8}\left(\frac{9}{5}\cdot 2 - 2\cdot 2 + 2\right) = \frac{9}{4} + \frac{5}{4}(-2+1) = \frac{9}{4} - \frac{5}{4} = \frac{4}{4} = 1 \checkmark$$

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(b) $P_0 = \sqrt{\frac{1}{2}}$ $\quad P_1 = \sqrt{\frac{3}{2}}\, x$ $\quad P_2 = \sqrt{\frac{5}{8}}(3x^2-1)$

$\langle i | D | j \rangle = \int P_i \frac{\partial P_j}{\partial x}\, dx$

$\frac{\partial P_0}{\partial x} = 0$ $\quad \frac{\partial P_1}{\partial x} = \sqrt{\frac{3}{2}}$ $\quad \frac{\partial P_2}{\partial x} = \sqrt{\frac{5}{8}}(6x) = \frac{1}{2}\sqrt{\frac{5}{2}}$ [illegible] $= 3\sqrt{\frac{5}{2}}\, x = \frac{\partial P_2}{\partial x}$

$\langle 0 | D | 0 \rangle = \int P_0 \frac{\partial P_0}{\partial x} dx = \int \sqrt{\frac{1}{2}} \cdot 0\, dx = 0$, in fact, [illegible]

$\langle 1 | D | 0 \rangle = \langle 2 | D | 0 \rangle = 0$ [illegible]

$\langle 0 | D | 1 \rangle = \int \sqrt{\frac{1}{2}} \frac{\partial P_1}{\partial x} dx = \int \sqrt{\frac{1}{2}}\sqrt{\frac{3}{2}}\, dx = \frac{\sqrt{3}}{2} x \Big| = \frac{\sqrt{3}}{2}(1-(-1)) = \sqrt{3}$

$\langle 1 | D | 1 \rangle = \int \sqrt{\frac{3}{2}}\, x \frac{\partial P_1}{\partial x} dx = \int \sqrt{\frac{3}{2}}\, x \sqrt{\frac{3}{2}}\, dx$ [illegible] $= \frac{3}{4}(1^2-(-1)^2) = 0$

$\langle 2 | D | 1 \rangle = \int \sqrt{\frac{5}{8}}(3x^2-1)\sqrt{\frac{3}{2}}\, dx = \frac{\sqrt{5 \cdot 3}}{\sqrt{8 \cdot 2}}(x^3 - x)\Big|_{-1}^{1} = \frac{\sqrt{15}}{4}\left[(1^3-(-1)^3)-(1-(-1))\right]$

$= \frac{\sqrt{15}}{2}\left[\ldots - \ldots\right] = 0$

$\langle 0 | D | 2 \rangle = \int \sqrt{\frac{1}{2}} \frac{\partial P_2}{\partial x} dx = \int \sqrt{\frac{1}{2}}\, 3\sqrt{\frac{5}{2}}\, x\, dx = \frac{3\sqrt{5}}{2} \frac{x^2}{2}\Big| = \frac{3\sqrt{5}}{4}(1^2-(-1)^2) = 0$

$\langle 1 | D | 2 \rangle = \int \sqrt{\frac{3}{2}}\, x\, 3\sqrt{\frac{5}{2}}\, x\, dx = \frac{3\sqrt{15}}{2}\int x^2 dx = \frac{3\sqrt{15}}{2}\frac{x^3}{3}\Big| = \frac{\sqrt{15}}{2}(1^3-(-1)^3)$

$= \frac{\sqrt{15}}{2}(1+1) = \sqrt{15}$

$\langle 2 | D | 2 \rangle = \int \sqrt{\frac{5}{8}}(3x^2-1)\, 3\sqrt{\frac{5}{2}}\, x\, dx = \frac{15}{4}\int (3x^3 - x)\, dx$

$= \frac{15}{4}\left(\frac{3x^4}{4} - \frac{x^2}{2}\right)\Big| = $ [illegible] $- \frac{1}{2}(1^2-(-1)^2))$

$= \frac{15}{4}$ [illegible] $= 0$

$$D = \begin{pmatrix} \langle 0|D|0\rangle & \langle 0|D|1\rangle & \langle 0|D|2\rangle \\ \langle 1|D|0\rangle & \langle 1|D|1\rangle & \langle 1|D|2\rangle \\ \langle 2|D|0\rangle & \langle 2|D|1\rangle & \langle 2|D|2\rangle \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$$

PM 3 ⓒ Let's choose $\left[\sqrt{\tfrac{8}{5}}P_2 = (3x^2-1)\right] + \left[\sqrt{\tfrac{2}{3}}P_1 = x\right]$

$$\underline{f = \sqrt{\tfrac{8}{5}}P_2 + \sqrt{\tfrac{2}{3}}P_1} \quad = (3x^2-1)+x \quad \underline{= (3x^2+x-1)}$$

$f = \begin{bmatrix} 0 \\ \sqrt{2/3} \\ \sqrt{8/5} \end{bmatrix}$ in our polynomial basis. $\underline{\frac{\partial f}{\partial x} = 6x+1}$

$$Df = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3\cdot5} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \sqrt{2/3} \\ \sqrt{8/5} \end{bmatrix} = \begin{bmatrix} 0+\sqrt{3}\cdot\sqrt{2/3}+0 \\ 0+0+\sqrt{3\cdot5}\sqrt{8/5} \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{24} \\ 0 \end{bmatrix}$$

$$Df = \sqrt{2}\,P_0 + \sqrt{24}\,P_1 + 0\,P_2$$

$$= \sqrt{2}\sqrt{\tfrac{1}{2}} + \sqrt{2\cdot3\cdot4}\sqrt{\tfrac{3}{2}}\,x$$

$$= 1 + 3\cdot2\,x = 1+6x \checkmark$$

QM4. Consider the **particle-in-a-box** problem. Imagine that at t=0, the state of the particle is

$$|\psi(0)\rangle = \sqrt{\tfrac{1}{2}}|1\rangle + \sqrt{\tfrac{1}{2}}|3\rangle,$$

where $|1\rangle$ and $|3\rangle$ represent the energy eigenstates in the n=1 and n=3 levels, respectively.

(0) You need E_n and $\psi(x,t)$ first.

(a) **Find the position probability density** $|\psi(x,t)|^2$ as a function of time, the easiest way you know how. Show your work and explain your reasoning.

(b) **Describe qualitatively and with diagrams** how $|\psi(x,t)|^2$ changes with time.

(c) If the particle is an electron, and the width of the box is 3Å, **what is the oscillation frequency** of the probability density pattern in Hz?

$\frac{3}{2}\lambda = L$ $\quad \frac{n\lambda}{2} = L$ $\quad p = \frac{h}{\lambda} = \frac{nh}{2L}$

$\lambda = L$ $\quad \hbar = \frac{h}{2\pi} \to h = 2\pi\hbar$

$\frac{\lambda}{2} = L$ $\quad k_n = \frac{2\pi}{\lambda_n} = \frac{2\pi n}{2L} = \frac{n\pi}{L}$

$E = T + V$, $V = 0$

$E = T = \frac{p^2}{2m} = \left(\frac{nh}{2L}\right)^2 \frac{1}{2m}$

$E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2\hbar^2\pi^2}{2mL^2}$

$E_1 = \frac{\hbar^2\pi^2}{2mL^2}$ $\quad E_3 = 9E_1$

In the ENERGY basis: $\psi(t) = \frac{1}{\sqrt{2}}|1\rangle e^{-iE_1t/\hbar} + \frac{1}{\sqrt{2}}|3\rangle e^{-iE_3t/\hbar}$

In the POSITION basis: $\psi_n(x) = \frac{1}{\sqrt{L}}\cos k_n x$, so

$$\psi(x,t) = \frac{1}{\sqrt{L}}\cos\frac{\pi x}{L}e^{-iE_1t/\hbar} + \frac{1}{\sqrt{L}}\cos\frac{3\pi x}{L}e^{-i9E_1t/\hbar}$$

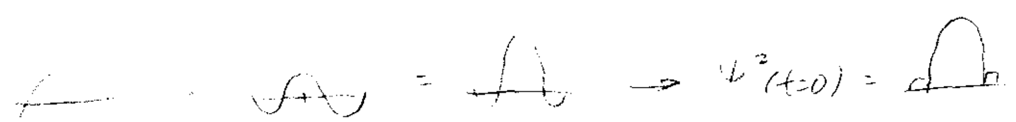
$$|\psi(x,t)|^2 = \psi^*\psi = \left(\frac{1}{\sqrt{L}}\cos\frac{\pi x}{L}e^{+iE_1t/\hbar} + \frac{1}{\sqrt{L}}\cos\frac{3\pi x}{L}e^{+9iE_1t/\hbar}\right)\left(\frac{1}{\sqrt{L}}\cos\frac{\pi x}{L}e^{-iE_1t/\hbar} + \frac{1}{\sqrt{L}}\cos\frac{3\pi x}{L}e^{-9iE_1t/\hbar}\right)$$

$$= \frac{1}{L}\left(\cos^2\frac{\pi x}{L}e^0 + \cos^2\frac{3\pi x}{L}e^0 + \cos\frac{\pi x}{L}\cos\frac{3\pi x}{L}\left[e^{i(E_1-9E_1)t/\hbar} + e^{i(9E_1-E_1)t/\hbar}\right]\right)$$

$$= \frac{1}{L}\left(\cos^2\frac{\pi x}{L} + \cos^2\frac{3\pi x}{L} + \cos\frac{\pi x}{L}\cos\frac{3\pi x}{L}\left[2\cos 8E_1t/\hbar\right]\right)$$

because $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + [\cos(-\theta) + i\sin(-\theta)] = \cos\theta + i\sin\theta + [\cos\theta - i\sin\theta] = 2\cos\theta$ ✓

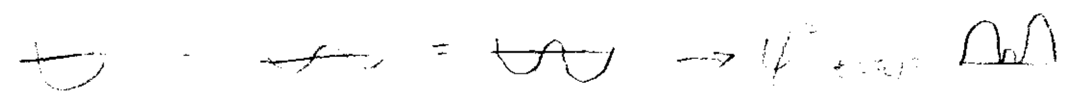
(b) $t=0$:



$|3\rangle$ ... $|1\rangle$ ... is at $\frac{2\pi}{2}$ ...

$|3\rangle$ is at ... (just ... smaller)

$t = \tau$:



(c) $\omega = \left(\frac{E_3 - E_1}{\hbar}\right) = \frac{(3^2-1)E}{\hbar} = \frac{8}{\hbar}\frac{\hbar^2\pi^2}{2mL^2} = \frac{4\hbar\pi^2}{mL^2} \simeq \frac{4(10^{-34}\,J.s)10}{9\cdot 10^{-31}\,kg(3\times 10^{-10}\,m)^2} \simeq \frac{3\cdot 10^{-33+30}}{3\cdot 3\cdot 10^{-20}} \sim 10^{50-33}$! $\sim 10^{17}\,\frac{rad}{s}$