1 Spacetime Level 1: Dealing with speed

The first step in solving spacetime problems is to be sure you are comfortable dealing with speed. Imagine that you are standing at the beginning of a long hallway. The hallway has special detectors and computers which allow you to determine the position of any object in the hallway at any time.

1.1 How fast?

The first type of problem you should be able to solve is to determine the speed of an object if you are given the distance it travels during a certain duration. Calculate how fast the object would have to move to get to each of the following coordinates from the origin (0, 0).

(1.5 s, 150,000 km) (3.6 s, 3.2 s) (100,000 m, 1 km) (5.5 m, 4.4 m)

________ ________ ________ ________

Commentary: *Make sure you see how easy it is to determine the speed relative to the speed of light. Expressing your answers as a fraction of the speed of light (even the first answer) allows you to check to make sure the speed you are calculating is reasonable. What would be an unreasonable answer?*

1.2 How far?

The second type of problem you should be able to solve is to determine how far an object will go if you are given the speed of the object and the amount of time that passes. Fill out the following coordinates for an object moving at 100,000 km/s along the hallway (which we will call the x-direction).

(0, 0) (6.3 s, ________) (0.039 s, ________) (4.5 m, ________) (330 m, ________)

Commentary: *To answer these questions, you don't need to use relativity. You won't need the equation for the spacetime interval because you are not comparing measurements in two different frames. You are dealing with only one frame of reference. Everything is relative to the hallway.*

So what do you do? You can use the speed formula, but you should also be able to do it just by looking at the situation. The object is going one-third the speed of light. How far would light go in each of the times listed? So how far will this object go? Check that the coordinates make sense for an object moving at the given speed.

1.3 At what time?

Finally, you should be able to determine the amount of time which passes if you are given the speed of an object and the distance it traveled. Fill out the following coordinates for an object moving at 250,000 km/s along the hallway (again, the x-direction).

(0, 0) (________, 55 m) (________, 150 km) (________, 25 yr) (________, 60 s)

2 Spacetime Level 2: Visualizing the rocket frame

After getting comfortable with speed, coordinated, and using the same units to measure both distance and duration, you need to practice "seeing" what happens in another frame.

Again, imagine that there is a long hallway with special detectors and computers which determine the position of any object in the hallway at any time. But this time, you are inside the object which is moving down the hallway.

2.1 How fast?

According to the measurements in the hallway, the object you are in is traveling at two-thirds of light speed. To you, however, it is the hallway which appears to move. How fast do the detectors in the hallway seem to zoom past you? This is a very important thing to figure out and then remember. The speed of the object is measured relative to the hallway, so it is the same as the speed of the hallway measured relative to the object. Different things are perceived to be moving, but there is only one relative speed.

2.2 Where in space?

The position of events is different in the two frames because they are moving relative to each other. To make this more "visible," consider three lights in the walls of the hallway – one at the beginning, one at the end, and one in the middle. Each light flashes whenever an object in the hallway is directly in front of it.

So, from the perspective of the hallway, as the object you are in enters the hallway, the first light will flash. Later, as the object moves past the middle of the hallway, that light will flash. Finally, as the object passes by the end of the hallway, the light there will flash.

How does this look from inside the object? Sitting in the object and looking out, you would see the beginning of the hallway come rushing by, and the light at the beginning of the hallway would flash right in front of you as it passes by. If you decide to make this event the origin of your spacetime coordinates, you can say that it happened at $(0, 0)$.

A little later, you see the middle of the hallway pass by, and the middle light flashes just as it goes by you. What coordinates would you assign to that event? Well, you didn't move. You're still at the same place as the previous event. It's just a little later in time. Right? So the coordinates would be $(t, 0)$ where t is the amount of time that has passed.

Finally, the end of the hallway passes by. Again, you see a light flash right next to you. You haven't moved, so the spacetime coordinates of this last event would be $(2t, 0)$ where t is still the amount of time that passed between entering the hallway and reaching the middle.

Make sure you understand why the x-coordinate is zero for each of these events, as measured in the "rocket frame."

3 Spacetime Level 3: Comparing frames

If you stay in one frame or another, you don't need to worry about special relativity or the spacetime interval. You can just make your own observations and go about your own business. But if you want to compare the observations in one frame to observations made in another frame which is moving relative to the first frame, you must consider special relativity. And the spacetime interval is one easy way to do this.

So, again, let's imagine a high-tech hallway. According to people at rest relative to the hallway, it is 60 m long and has a light at the beginning and the end. Each light will flash when any object is directly in front of it. An object moves at 250,000 km/s down the hallway (in the x-direction).

3.1 Duration

As the object enters the hallway, the light at the beginning of the hallway flashes. We can choose this event as the origin and give it the spacetime coordinates (0, 0). A little later, a second event occurs as the object passes the end of the hallway and the light there flashes. From the perspective of the hallway, what are the spacetime coordinates of this second event?

In the hallway: Event 2: (____________, ____________)

Commentary: *All of the information so far is given relative to the hallway, so you don't have to use the spacetime interval. Just use the known length of the hallway and the speed of the object.*

From the perspective of the object, as the beginning of the hallway passes by, a light flashes right in front of the object. This event can be chosen as the origin. A little later, a second event occurs as the end of the hallway passes by. Again, a light flashes right in front of the object. From the perspective of the object, what are the spacetime coordinates of this second event?

For the object: Event 2: (____________, ____________)

Commentary: *Now we are jumping to another frame. When can we use the spacetime interval? As soon as we have one coordinate for Event 2 in the new frame, we can get the other coordinate by using the spacetime interval between the first and second events. But how do we get the first coordinate? Can we figure out either where or when the second event happened in this frame? Yes! This situation is just like the "Where in space?" section of Level 2. So we know the x-coordinate of Event 2! And then we can use that and the spacetime interval to get the t-coordinate of Event 2.*

3.2 Length

After we have found the spacetime coordinates for the events as viewed from the object frame, there is one other interesting thing to notice. According to the object, it is the hallway which is moving. What is the speed of the hallway relative to the object?

And how much time passed between the beginning of the hallway passing by and the end of the hallway passing by?

So how long is the hallway, according to the object?

4 Spacetime Level 4: The Twin Paradox

There is really nothing special about the twin paradox. It simply has a good story to go along with the situation you already investigated in Level 3. Instead of an object in a hallway, we send one twin off into space on a rocket while the other twin remains home. To top it all off, we then have the first twin turn around and come back at the same speed he or she went away. This is a nice touch because it puts both twins back in the same place again. But it is also the key to the paradox. But before we get to the paradox, let's just do the math.

4.1 The first trip

We can set the twin paradox up with any distances and speeds we choose, so let's use the same numbers as Wolfson uses in chapter 9 of [**?**]. We want to send one twin to a star that is 20 light-years from Earth. We send the twin off at a speed of $0.8c$ or 80% the speed of light.

To find out how much time passes for the Earthbound twin and the space-trekking twin, we can set up spacetime coordinates. The key here is to recognize that the phrase "20 light-years from Earth" implies a frame of reference – the Earth frame. The next step is to be very clear about what events interest us. In this case, we can define the origin as the event of the space-trekking twin leaving Earth. The second event of interest is his or her arrival at the end of the trip.

Earth frame: Event 1 (leaving Earth): (0, 0)

Earth frame: Event 2 (arriving at star): (t, 20 yr)

Do you see that Event 2 is located 20 years from Earth in the Earth frame? So how much time does it take the space-trekking twin to get there, according to Earth clocks? *Hint:* Use the speed!

$t =$ __________

Now we must investigate the same two events from the rocket frame. Where do these two events occur? After you figure out the x-coordinates for both events, you can easily find the elapsed time by using the spacetime interval. *Hint:* It's just like what you did in Level 3.

Rocket frame: Event 1 (Earth passes by): (__________, __________)

Rocket frame: Event 2 (Star passes by): (__________, __________)

4.2 The second trip

To complete the paradox, the space-trekking twin somehow has to leap off the rocket and get on the star (ouch!) and then leap onto another rocket heading in the opposite direction, back to Earth, at the same speed as the first rocket. Aside from the physical difficulties of doing this, the return trip is identical mathematically to the first trip. In the Earth frame, the distance and speed are the same. So according to Earth's clocks, the full round-trip trek from Earth to star and back will take $2t$, where t is the value you calculated for the first trip.

Similarly, the time that passes for the space-trekking twin is mathematically twice the time that passed during the first trip from Earth to the star. (You could add in some extra time for the space-trekking twin to explore the star once he or she gets there, but that added time would increase the duration of the trip equally on both Earth's clocks and the space-trekking twin's clocks. Why?)

So when the space-trekking twin returns to Earth and meets the Earth-bound twin, they will have experienced different time lapses. One will be older than the other. By how much?

Difference in age of twins after the trip = __________

4.3 The paradox

It may seem that the paradox here is obvious. How can twins be different ages? But that is actually not what is paradoxical about this situation. It is strange, but it is simply what is predicted by special relativity. No, the paradox is this: How do you know which twin is older? From the perspective of the Earthbound twin, it is the other twin who is moving, so it is the other twin who should be younger (according to special relativity). But according to the space-trekking twin, it is the Earthbound twin who is moving at 80% lightspeed in one direction and then coming back at 80% lightspeed. It is therefore the Earthbound twin who should be younger (according to special relativity). But it has to be one or the other. When they get back together, it cannot be that one is older that the other at the same time and place as the other is older than the one! That's the paradox.

What is the way out of the paradox? Recall that it is the space-trekking twin who must do the physically demanding work of jumping out of one spaceship and getting into another going the other direction. Why can't the rocket just turn around? Because special relativity only applies to uniformly moving reference frames. There have to be three reference frames to allow one twin to go away and come back while the other just sits around. One twin uses two frames while the other just stays in one. This difference determines which twin will be older. The one who doesn't jump frames ages more.

There are many other mathematical and visual ways to investigate this paradox. As usual, Wikipedia does a fair job of outlining both the history and the content of the idea. Just do a search on "twin paradox" in Wikipedia.