a

average position for a particle in a 1-d box

$$\langle x \rangle = \int_0^a \psi_x^* \hat{x} \psi_x \, dx$$

$$= \int_0^a \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}\right)x.\right] x \left[\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}\right)x\right] dx$$

$$\langle x \rangle = \left(\frac{2}{a}\right) \int_0^a x \sin^2\left(\frac{n\pi}{a}\right)x \, dx \quad \text{— eq}^n \; ①$$

Let $V = x$ $\quad u' = \frac{du}{dx} = \sin^2\left(\frac{n\pi}{a}\right)x$

$V' = 1$

Then eq$^{\text{s}}$ ① $\Rightarrow$ $\langle x \rangle = \frac{2}{a} \int_0^a V u' \, dx$ — eq^n ②

$$(uV)' = Vu' + V'u$$

$$\Rightarrow Vu' = (uV)' - V'u$$

$$\Rightarrow \int Vu' dx = \int (uV)' dx - \int V' u \, dx$$

$$\int Vu' dx = uV - \int V' u \, dx \quad \text{— eq}^n \; ③$$

using eq^{ns} ② and ③

$$\langle x \rangle = \frac{2}{a} \int_0^a Vu' dx = \frac{2}{a} \left[(uV)_0^a - \int_0^a V' u \, dx \right] \quad \text{— eq}^n \; ④$$

$$u' = \sin^2\left(\frac{n\pi}{a}\right)x = \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{a}\right)x\right]$$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

b

$$u' = \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{a}\right)x\right]$$

$$u = \int \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{a}\right)x\right] dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos\left(\frac{2n\pi}{a}\right)x \, dx$$

$$u = \frac{1}{2}\left[x - \left(\frac{a}{2n\pi}\right)\sin\left(\frac{2n\pi}{a}\right)x\right]$$

$$\therefore uv = \frac{1}{2}\left[x^2 - \left(\frac{a}{2n\pi}\right) x \sin\left(\frac{2n\pi}{a}\right)x\right] \quad \text{— eq}^{n} \text{ (5)}$$

$$v'u = 1\left[\frac{1}{2}\left\{x - \left(\frac{a}{2n\pi}\right)\sin\left(\frac{2n\pi}{a}\right)x\right\}\right]$$

$$\int v'u\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\left(\frac{a}{2n\pi}\right)\int \sin\left(\frac{2n\pi}{a}\right)x\,dx \quad \text{— eq}^{n} \text{ (6)}$$

eq^{ns} (4), (5) and (6) $\Rightarrow$

$$\langle x\rangle = \frac{2}{a}\left(uv\right)_0^a - \frac{2}{a}\int_0^a v'u\,dx$$

$$= \frac{2}{a}\left\{\frac{1}{2}\left[x^2 - \left(\frac{a}{2n\pi}\right)x \sin\left(\frac{2n\pi}{a}\right)x\right]_0^a\right\} -$$

$$\frac{2}{a}\left\{\frac{1}{2}\int_0^a x\,dx - \frac{1}{2}\left(\frac{a}{2n\pi}\right)\int_0^a \sin\left(\frac{2n\pi}{a}\right)x\,dx\right\}$$

c

$$\langle x \rangle = \frac{2}{a}\left\{ \frac{1}{2}\left[a^2 - \left(\frac{a}{2n\pi}\right) a \sin\left(\frac{2n\pi}{a}\right) a - 0 \right\} -\right.$$

$$\frac{2}{a}\left\{ \frac{1}{2}\left(\frac{x^2}{2}\right)_0^a - \frac{1}{2}\left(\frac{a}{2n\pi}\right)\left[-\left(\frac{a}{2n\pi}\right) \cos \frac{2n\pi x}{a} \right]_0^a \right\}$$

$$= \frac{2}{a}\left[\frac{a^2}{2}\right] - \frac{2}{a}\left[\frac{1}{2}\left(\frac{a^2}{2}\right) + \frac{1}{2}\left(\frac{a}{2n\pi}\right)^2 \left\{ \cos \frac{2n\pi}{a}, a - \cos 0 \right\} \right]$$

$$= a - \frac{1}{a}\left[\frac{a^2}{2} + \left(\frac{a}{2n\pi}\right)^2 \left\{ \underset{1}{\underset{\shortparallel}{\cos 2n\pi}} - \underset{1}{\underset{\shortparallel}{\cos 0}} \right\} \right]$$

$$\langle x \rangle = a - \frac{a}{2} = \underline{\underline{\frac{a}{2}}}$$

Average momentum = $\langle P_x \rangle$

$$\langle P_x \rangle = \int_0^a \psi_x^* \hat{P}_x \psi_x \, dx = \underline{\underline{0}}$$

$$\langle P_x^2 \rangle = \int_0^a \psi_x^* \hat{P}_x^2 \psi_x \, dx = \underline{\underline{\frac{n^2 \pi^2 \hbar^2}{a^2}}}$$