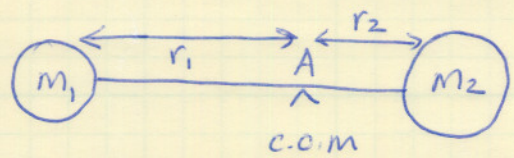
ADVANCED CHEMISTRY

QUANTUM MECHANICS HW - WINTER - WEEK 7

From the worksheet



$$r_1 + r_2 = R \quad \text{—} \ (1)$$

$$r_1 m_1 = r_2 m_2 \quad \text{—} \ (2)$$ Center of mass condition

$$(1) \Rightarrow \quad r_1 = R - r_2 \quad \text{—} \ (3)$$

Substitute (3) in (2)

$$(R - r_2) m_1 = r_2 m_2$$

$$R m_1 - r_2 m_1 = r_2 m_2$$

$$R m_1 = r_2 m_1 + r_2 m_2$$

$$= r_2 (m_1 + m_2)$$

$$\therefore \quad r_2 = \frac{m_1 R}{(m_1 + m_2)}$$

Substitute the value of r_2 in (3)

$$r_1 = R - \frac{m_1 R}{(m_1 + m_2)} = R\left[1 - \frac{m_1}{(m_1 + m_2)}\right]$$

$$r_1 = R\left[\frac{(m_1+m_2)-m_1}{(m_1+m_2)}\right] = \frac{R\,m_2}{m_1+m_2}$$

$$r_1 = \frac{m_2\,R}{(m_1+m_2)}$$

Chapter 9

(30)

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\ \hat{x}\ \psi_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} \psi_1(x).\ x.\ \psi_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} \underbrace{(\text{odd } f^n)(\text{odd } f^n)}_{\text{even fn.}}\ (\text{odd } f^n)\ dx$$

$$= \int_{-\infty}^{\infty} (\text{even } f^n)(\text{odd } f^n)\,dx = \int_{-\infty}(\text{odd } f^n)\,dx$$

$$= \underline{\underline{0}}$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\,.\,\hat{x}^2\,.\,\psi_1(x)\,dx = \int_{-\infty}^{\infty} \psi_1^*(x).\ x^2.\ \psi_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{4\alpha^3}{\pi}\right)^{1/4} x\,e^{-\alpha x^2/2}\,.\ x^2\,.\left(\frac{4\alpha^3}{\pi}\right)^{1/4} x\ e^{-\alpha x^2/2}\,dx$$

$$= \left(\frac{4\alpha^3}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} x^4\ e^{-\alpha x^2}\,dx$$

$$= \left(\frac{4\alpha^3}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} u^2 e^{-\alpha u}$$

Let $u = x^2$
$du = 2x\,dx$

$$\langle \Delta x \rangle = \left[\langle x^2 \rangle - \langle x \rangle^2 \right]^{1/2}$$

(32) $$E_J = \frac{J(J+1)\hbar^2}{2I}$$

when $J=1$ $E_1 = \frac{2\hbar^2}{2I}$ when $J=2$ $E_2 = \frac{6\hbar^2}{2I}$

$$\Delta E = E_2 - E_1 = \frac{6\hbar^2}{2I} - \frac{2\hbar^2}{2I} = \frac{4\hbar^2}{2I} = \frac{2\hbar^2}{I}$$

$$= \frac{2(1.055 \times 10^{-34}\ J\,s)^2}{(2.644 \times 10^{-47}\ kg\,m^2)} = 8.419 \times 10^{-22}\ J$$

$$\frac{J^2 s^2}{kg\,m^2} = \frac{(kg\,m^2 s^{-2})^2 s^2}{kg\,m^2} = kg\,m^2 s^{-2} = J$$

$$\Delta E = h\nu$$

$$\nu = \frac{\Delta E}{h} = \frac{8.419 \times 10^{-22}\ J}{6.626 \times 10^{-34}\ J\,s}$$

$$= \underline{\underline{1.2706 \times 10^{+12}\ s^{-1}}}$$

$$c = \nu\lambda \Rightarrow \frac{1}{\lambda} = \frac{\nu}{c} \Rightarrow \tilde{\nu} = \frac{\nu}{c}$$

$$\tilde{\nu} = \frac{1.2706 \times 10^{12}\ s^{-1}}{2.99 \times 10^{8}\ m\,s^{-1}} = \underline{\underline{4249.623\ m^{-1}}}$$

(34) $^{23}Na^{35}Cl$ $$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{(23)(35)}{(23+35)}\ amu$$

$$= \frac{23 \times 35}{(23+35)}\ amu \times \frac{1.661 \times 10^{-27}\ kg}{amu}$$

$$= \underline{\underline{2.3054 \times 10^{-26}\ kg}}$$

$$I = \mu R^2 = \left(2.3054 \times 10^{-26}\ \text{kg}\right)\left(236\ \text{pm} \times \frac{\text{m}}{10^{12}\ \text{pm}}\right)^2$$

$$= \underline{\underline{1.284 \times 10^{-45}\ \text{kg m}^2}}$$

$$E_J = \frac{J(J+1)\hbar^2}{2I}$$

$$\frac{\hbar^2}{2I} = \frac{\left(1.055 \times 10^{-34}\ \text{J s}\right)^2}{2\left(1.284 \times 10^{-45}\ \text{kg m}^2\right)} = 4.334 \times 10^{-24}\ \text{J}$$

$$E_1 = \frac{2\hbar^2}{2I} = \underline{\underline{8.668 \times 10^{-24}\ \text{J}}}$$

$$E_2 = \frac{6\hbar^2}{2I} = \underline{\underline{2.6005 \times 10^{-23}\ \text{J}}}$$

(54) For a one dimensional harmonic oscillator

$$V_x = \frac{1}{2} k_x x^2$$

$$E_x = \left(v_x + \frac{1}{2}\right) h\nu_x \qquad v_x = 0, 1, 2, --$$

for a 3-dimensional harmonic oscillator

$$V = \frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2 + \frac{1}{2} k_z z^2$$

$$E_{tot} = E_x + E_y + E_z = \left(v_x + \frac{1}{2}\right) h\nu_x + \left(v_y + \frac{1}{2}\right) h\nu_y + \left(v_z + \frac{1}{2}\right) h\nu_z$$

Zero point energy is when

$$v_x = v_y = v_z = 0$$

$$E = \frac{1}{2}h\nu_x + \frac{1}{2}h\nu_y + \frac{1}{2}h\nu_z$$

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$$E = \left(v_x + \frac{1}{2}\right)h\nu_x + \left(v_y + \frac{1}{2}\right)h\nu_y + \left(v_z + \frac{1}{2}\right)h\nu_z$$

$$\nu_x = \frac{1}{2\pi}\sqrt{\frac{k_x}{\mu}} \qquad \nu_y = \frac{1}{2\pi}\sqrt{\frac{k_y}{\mu}} \qquad \nu_z = \frac{1}{2\pi}\sqrt{\frac{k_z}{\mu}}$$

For an isotropic harmonic oscillator

$$k_x = k_y = k_z$$

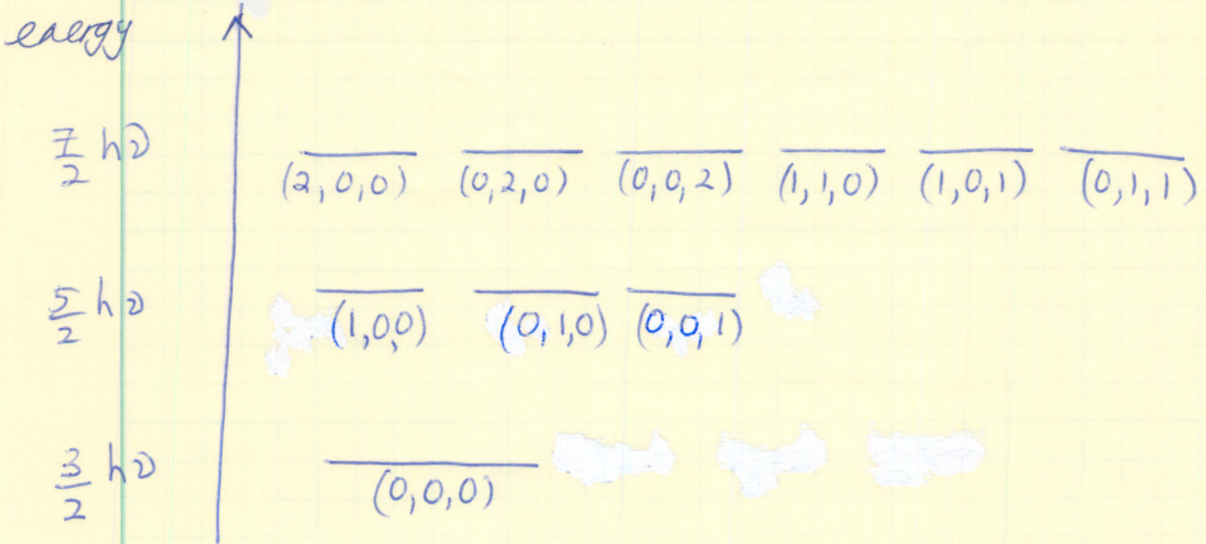
$$\Rightarrow \nu_x = \nu_y = \nu_z = \nu \text{ (say)}$$

$$E = \left[\left(v_x + \frac{1}{2}\right) + \left(v_y + \frac{1}{2}\right) + \left(v_z + \frac{1}{2}\right)\right]h\nu$$

v_x	v_y	v_z	energy	degeneracy
0	0	0	$\frac{3}{2}h\nu$	1
1	0	0	$\frac{5}{2}h\nu$	3
0	1	0		
0	0	1		
1	1	0	$\frac{7}{2}h\nu$	3
1	0	1		

v_x	v_y	v_z	energy	degeneracy
~~1~~	~~1~~	1	~~$\frac{9}{2} h\nu$~~	~~1~~
2	0	0	$\frac{7}{2} h\nu$	3
0	2	0		
0	0	2		

The degeneracies of the first 10 states are 1, 3 and 6 as shown below



(59) $^{14}N^{16}O$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{(14)(16)}{(14+16)} \text{ amu} \times \frac{1.661 \times 10^{-27} \text{ kg}}{1 \text{ amu}}$$

$$= \underline{\underline{1.2402 \times 10^{-26} \text{ kg}}}$$

$$I = \mu R^2 = (1.2402 \times 10^{-26} \text{ kg})(115.1 \times 10^{-12} \text{ m})^2$$

$$\underline{\underline{I = 1.643 \times 10^{-46} \text{ kg m}^2}}$$

$$E_J = \frac{J(J+1)\hbar^2}{2I}$$

$$\frac{\hbar^2}{2I} = \frac{(1.055 \times 10^{-34} \text{ Js})^2}{2(1.643 \times 10^{-46} \text{ kg m}^2)} = 3.3871 \times 10^{-23} \text{ J}$$

$$E_0 = \underline{\underline{0}} \text{ J} \qquad E_1 = \frac{2\hbar^2}{2I} = \underline{\underline{6.7742 \times 10^{-23} \text{ J}}}$$

$$E_2 = \frac{6\hbar^2}{2I} = \underline{\underline{2.0323 \times 10^{-22} \text{ J}}}$$

(60) $$\int Y_0^{0*} Y_0^0 \, d\tau = \int \left(\frac{1}{4\pi}\right)^{1/2} \left(\frac{1}{4\pi}\right)^{1/2} d\tau = \int \left(\frac{1}{4\pi}\right) d\tau$$

Y_ℓ^m are functions of θ and ϕ only

$$\therefore \int d\tau = \int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$\int Y_0^{0*} Y_0^0 \, d\tau = \int \frac{1}{4\pi} \, d\tau = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi} \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = \frac{1}{4\pi} \Big[-\cos\theta\Big]_0^{\pi} \Big[\phi\Big]_0^{2\pi}$$

$$= \frac{1}{4\pi} \Big[-\cos\pi + \cos 0\Big] (2\pi - 0)$$

$$\begin{array}{c|c} S & A \\ \hline T & C \end{array}$$

$$= \frac{1}{4\pi} \left(1 + 1\right) (2\pi) = 1$$

$\therefore$ Y_0^0 is normalized

$$\int Y_1^{-1*} Y_1^{-1} \, d\tau = \int \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\phi} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\phi} \, d\tau$$

$$= \left(\frac{3}{8\pi}\right) \int \sin^2\theta \, d\tau = \frac{3}{8\pi} \int_0^{\pi} \sin^2\theta \, \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{3}{8\pi} \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$= \frac{3}{8\pi} \left[\int_0^{\pi} \frac{1}{4} \left(3\sin\theta - \sin 3\theta\right) d\theta \right] \Big[\phi\Big]_0^{2\pi}$$

$$= \frac{3}{} \; \frac{1}{} \; (2\pi) \left[3 \int_0^{\pi} \sin\theta \, d\theta - \int_0^{\pi} \sin 3\theta \, d\theta \right]$$

$$= \frac{3}{16}\left[3\left(-\cos\theta\right)_0^{\pi} + \frac{1}{3}\left[\cos 3\theta\right]_0^{\pi}\right]$$

$$= \frac{3}{16}\left[3\left(+1-(-1)\right) + \frac{1}{3}\left(\cos 3\pi - \cos 0\right)\right]$$

$$= \frac{3}{16}\left[6 + \frac{1}{3}\left(-1-1\right)\right] = \frac{3}{16}\left(6 - \frac{2}{3}\right) = \frac{3}{16}\left(\frac{18-2}{3}\right)$$

$= 1$ $\therefore$ Y_1^{-1} is normalized

$$\int Y_1^{0*} Y_1^{0}\, d\tau = \int \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \cdot \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta\, d\tau$$

$$= \left(\frac{3}{4\pi}\right) \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \left(\frac{3}{4\pi}\right) \int_0^{\pi} -\cos^2\theta\, d(\cos\theta) \int_0^{2\pi} d\phi$$

$$= \left(\frac{3}{4\pi}\right)\left(-\left[\frac{\cos^3\theta}{3}\right]_0^{\pi}\right)(2\pi) = -\left(\frac{1}{4\pi}\right)\left(\cos^3\pi - \cos^3 0\right)(2\pi)$$

$$= -\left(\frac{1}{4\pi}\right)\left(-1-1\right)(2\pi) = -\left(\frac{1}{4\pi}\right)(-2)(2\pi) = 1$$

$\therefore$ Y_1^{0} is normalized

$$\int Y_1^{1*} Y_1^1 \, d\tau = \int \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\phi} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\phi} \, d\tau$$

$$= \left(\frac{3}{8\pi}\right) \int \sin^2\theta \, d\tau = \left(\frac{3}{8\pi}\right) \int_0^{\pi} \sin^2\theta \, \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \left(\frac{3}{8\pi}\right) \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{2\pi} d\phi$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$= \left(\frac{3}{8\pi}\right) \left[\frac{1}{4} \int_0^{\pi} (3\sin\theta - \sin 3\theta) \, d\theta\right] (2\pi)$$

$$= \left(\frac{3}{8\pi}\right)\left(\frac{1}{4}\right)(2\pi) \left[\int_0^{\pi} 3\sin\theta \, d\theta - \int_0^{\pi} \sin 3\theta \, d\theta\right]$$

$$= \frac{3}{16} \left[3 \left(-\cos\theta\right)_0^{\pi} + \frac{1}{3} \left(\cos 3\theta\right)_0^{\pi}\right]$$

$$= \frac{3}{16} \left[-3 \left(\cos\pi - \cos 0\right) + \frac{1}{3} \left(\cos 3\pi - \cos 0\right)\right]$$

$$= \frac{3}{16} \left[-3(-1-1) + \frac{1}{3}(-1-1)\right] = \frac{3}{16} \left[6 - \frac{2}{3}\right]$$

$$= \frac{3}{16} \left(\frac{18-2}{3}\right) = 1$$

Y_1^1 is normalized

$$\int Y_0^{0*}\, Y_1^{-1}\, d\tau = \int \left(\frac{1}{4\pi}\right)^{1/2} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{-i\phi}\, d\tau$$

$$= \left(\frac{3}{32\pi^2}\right)^{1/2} \int_0^{\pi} \sin\theta \, \sin\theta \, d\theta \int_0^{2\pi} e^{-i\phi}\, d\phi$$

$$= \left(\frac{3}{32\pi^2}\right)^{1/2} \int_0^{\pi} \sin^2\theta \, d\theta \int_0^{2\pi} \left(\sin\phi - i\cos\phi\right) d\phi$$

$$\boxed{\begin{aligned} \cos 2\theta &= 1 - 2\sin^2\theta \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \end{aligned}}$$

$$= \left(\frac{3}{32\pi^2}\right)^{1/2} \left[\int_0^{\pi} \frac{1}{2}\left(1 - \cos 2\theta\right) d\theta\right] \left[\int_0^{2\pi} \sin\phi \, d\phi - i \int_0^{2\pi} \cos\phi \, d\phi\right]$$

$$= \left(\frac{3}{32\pi^2}\right)^{1/2} \frac{1}{2} \left[\int_0^{\pi} d\theta - \int_0^{\pi} \cos 2\theta \, d\theta\right] \left[\left(-\cos\phi\right)_0^{2\pi} - i\left(\sin\phi\right)_0^{2\pi}\right]$$

$$= \left(\frac{3}{32}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\right) \left[(\pi) - \frac{1}{2}\left(\sin 2\theta\right)_0^{\pi}\right] \underbrace{\left[-(1-1) - i(0-0)\right]}_{=0}$$

$= \underline{\underline{0}}$ $\quad$ Y_0^0 and Y_1^{-1} are orthogonal.

$\cancel{Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}}$ ~~even function~~

$\cancel{Y_1^0(\theta) = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta}$

~~To test if~~ $\cancel{Y_1^0}$ ~~is even or odd replace~~ $\cancel{\theta}$ ~~with~~ $\cancel{(\pi-\theta)}$

$\cancel{Y_1^0(\pi-\theta) = \left(\frac{3}{4\pi}\right)^{1/2} \cos(\pi-\theta) = \left(\frac{3}{4\pi}\right)\left(-\cos\theta\right) = -Y_1^0(\theta)}$

$$\int Y_0^{0*} Y_1^0 \, d\tau = \int \left(\frac{1}{4\pi}\right)^{1/2} \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \, d\tau$$

$$= \frac{\sqrt{3}}{4\pi} \int_0^{\pi} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\sqrt{3}}{4\pi} \left[\int_0^{\pi} \cancel{\cos} \sin\theta \, d(\sin\theta) \right] \left[\phi\right]_0^{2\pi}$$

$$= \frac{\sqrt{3}}{4\pi} \left[\frac{\sin^2\theta}{2} \right]_0^{\pi} (2\pi) = \frac{\sqrt{3}}{4\pi} \left[\underbrace{\frac{\sin^2\pi - \sin^2 0}{2}}_{=0} \right] (2\pi)$$

$$= 0$$

$\therefore$ Y_0^0 and Y_1^0 are orthogonal.

$$\int Y_0^{0*} Y_1^1 \, d\tau = \int \left(\frac{1}{4\pi}\right)^{1/2} \left(\frac{8}{3\pi}\right)^{1/2} \sin\theta \, e^{i\phi} \, d\tau$$

$$= \left(\frac{8}{12\pi^2}\right)^{1/2} \int_0^{\pi} \sin\theta \cdot \sin\theta \, d\theta \int_0^{2\pi} e^{i\phi} \, d\phi$$

$$= \underbrace{\left(\frac{8}{12\pi^2}\right)^{1/2} \left(\int_0^{\pi} \sin^2\theta \, d\theta\right)}_{=A} \left[\frac{e^{i\phi}}{i}\right]_0^{2\pi} = \frac{A}{i} \left[\sin\phi + i\cos\phi\right]_0^{2\pi}$$

$$= \frac{A}{i} \left[\sin 2\pi + i\cos 2\pi - \sin 0 - i\cos 0\right] =$$

$$= \frac{A}{i} \left[0 + i - 0 - i\right] = 0$$

$\therefore$ Y_0^0 and Y_1^1 are orthogonal

$$\int Y_1^0\, Y_1^1\, d\tau = \int \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \;\; e^{i\phi}\, d\tau$$

$$= \underbrace{\left(\frac{9}{32\pi^2}\right)^{1/2} \int_0^{\pi} \cos\theta \sin\theta \, d\theta}_{= A} \int_0^{2\pi} e^{i\phi}\, d\phi$$

$$= A \int_0^{2\pi} e^{i\phi}\, d\phi = \frac{A}{i}\left[e^{i\phi}\right]_0^{2\pi} = \frac{A}{i}\left[\sin\phi + i\cos\phi\right]_0^{2\pi}$$

$$= \frac{A}{i}\left[\sin 2\pi + i\cos 2\pi - \sin 0 - i\cos 0\right]$$

$$= \frac{A}{i}\left[0 + i - 0 - i\right] = 0$$

$\therefore$ Y_1^0 and Y_1^1 are orthogonal

$$\int Y_1^{0*} Y_1^{-1} d\tau = \int \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta \cdot \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\phi} \, d\tau$$

$$= \left(\frac{9}{32\pi^2}\right)^{1/2} \left[\int_0^{\pi} \cos\theta \sin\theta \cdot \sin\theta \, d\theta\right] \left[\int_0^{2\pi} e^{-i\phi} \, d\phi\right]$$

$$= \left(\frac{9}{32\pi^2}\right)^{1/2} \left[\int_0^{\pi} \sin^2\theta \cos\theta \, d\theta\right] \left[\left(-\frac{1}{i}\right) \left(e^{i\phi}\right)_0^{2\pi}\right]$$

$$= \underbrace{\left(\frac{9}{32\pi^2}\right)^{1/2} \left(-\frac{1}{i}\right)}_{=A} \left[\int_0^{\pi} \sin^2\theta \, d(\sin\theta)\right] \left(e^{i\phi}\right)_0$$

$$= A \left[\frac{\sin^3\theta}{3}\right]_0^{\pi} \left(e^{i\phi}\right)_0^{2\pi} = \frac{A}{3} \left[\sin^3\pi - \sin 0\right] \left(e^{i\phi}\right)_0^{2\pi}$$

$$= \frac{A}{3} \left[0 - 0\right] \left(e^{i\phi}\right)_0^{2\pi} = 0$$

$\therefore$ Y_1^0 and Y_1^{-1} are orthogonal to each other.

$$\int Y_1^{1*}\, Y_1^{-1} d\tau = \int \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{-i\phi} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{-i\phi}\, d\tau$$

$$= \left(\frac{3}{8\pi}\right)\left[\int_0^{\pi} \sin^2\theta \cdot \sin\theta\, d\theta\right]\left[\int_0^{2\pi} e^{-2i\phi}\, d\phi\right]$$

$$= \left(\frac{3}{8\pi}\right)\left[\int_0^{\pi} \sin^3\theta\, d\theta\right]\left(\frac{1}{-2i}\right)\left[e^{-2i\phi}\right]_0^{2\pi}$$

$$= \underbrace{\left(\frac{-3}{8\pi i}\right)\left[\int_0^{\pi} \sin^3\theta\, d\theta\right]}_{=A}\left[\sin 2\phi - i\cos 2\phi\right]_0^{2\pi}$$

$$= A\left[(\sin 4\pi - i\cos 4\pi) - (\sin 0 - i\cos 0)\right]$$

$$= A\left\{\left[0 - i(+1)\right] - \left[0 - i\right]\right\} = A\left[\underbrace{(-i) - (-i)}_{=0}\right]$$

$= 0$ $\therefore$ Y_1^{1} and Y_1^{-1} are orthogonal

(61) $\psi \propto \cos\left(\frac{\pi x}{2a}\right)$

$\therefore \psi = N \cos\left(\frac{\pi x}{2a}\right)$ where N = normalization const.

$\psi^* = N \cos\left(\frac{\pi x}{2a}\right) \qquad \int \psi^* \psi \, d\tau = 1$

$$\int \psi^* \psi \, d\tau = \int_{-a}^{a} N^2 \cos^2\left(\frac{\pi x}{2a}\right) dx = 1$$

$$\boxed{\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1 \\ \cos^2\alpha &= \frac{1}{2}\left[\cos 2\alpha + 1\right]\end{aligned}}$$

$$1 = N^2 \int_{-a}^{a} \cos^2\left(\frac{\pi x}{2a}\right) dx$$

$$1 = \frac{N^2}{2} \int_{-a}^{a} \left(\cos 2\left(\frac{\pi x}{2a}\right) + 1\right) dx$$

$$1 = \frac{N^2}{2}\left[\int_{-a}^{a} \cos\left(\frac{\pi x}{a}\right) dx + \int_{-a}^{a} dx\right]$$

$$= \frac{N^2}{2}\left\{\left[-\frac{\sin\left(\frac{\pi x}{a}\right)}{(\pi/a)}\right]_{-a}^{a} + \left(x\right)_{-a}^{a}\right\}$$

$$= \frac{N^2}{2}\left\{\left(-\frac{a}{\pi}\right)\left(\sin\frac{\pi x}{a}\right)_{-a}^{a} + \left(a-(-a)\right)\right\}$$

$$1 = \frac{N^2}{\ } \left\{\left(-\frac{a}{\pi}\right)\left(\sin\pi - \sin(-\pi)\right) + 2a\right\}$$

$$1 = \frac{N^2}{2}\left\{\underbrace{\left(-\frac{a}{\pi}\right)(0-0)}_{=0} + 2a\right\}$$

$$1 = \frac{N^2}{2}\left[2a\right] = N^2 a$$

$$N^2 = \frac{1}{a} \qquad N = \pm\sqrt{\frac{1}{a}}$$

$$\psi = \pm\sqrt{\frac{1}{a}} \cos\frac{\pi x}{2a}$$