ADVANCED CHEMISTRY

QUANTUM MECHANICS H.W. - WINTER - WEEK 8

from the sheet

① $[\hat{L}^2, \hat{L}_x] = [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 , \hat{L}_x]$

$= \underbrace{[\hat{L}_x^2 , \hat{L}_x]}_{0} + [\hat{L}_y^2, \hat{L}_x] + [\hat{L}_z^2 , \hat{L}_x]$

$= [\hat{L}_y \cdot \hat{L}_y , \hat{L}_x] + [\hat{L}_z \cdot \hat{L}_z , \hat{L}_x]$

$= \hat{L}_y [\hat{L}_y, \hat{L}_x] + [\hat{L}_y, \hat{L}_x]\hat{L}_y +$
$\hat{L}_z [\hat{L}_z , \hat{L}_x] + [\hat{L}_z , \hat{L}_x]\hat{L}_z$

$= \hat{L}_y (-i\hbar \hat{L}_z) + (-i\hbar \hat{L}_z)\hat{L}_y + \hat{L}_z (i\hbar)\hat{L}_y$
$+ (i\hbar \hat{L}_y) L_z$

$= -i\hbar \cancel{\hat{L}_y\hat{L}_z} - i\hbar \cancel{\hat{L}_z\hat{L}_y} + i\hbar \cancel{\hat{L}_z\hat{L}_y} + i\hbar \cancel{\hat{L}_y\hat{L}_z}$

$= \underline{\underline{0}}$

(2) $$R = \frac{m_e e^4}{2(4\pi\varepsilon_0)^2 \hbar^2}$$

$$= \frac{(9.109\times10^{-31}\,kg)(1.602\times10^{-19}\,C)^4}{2(4\pi\times 8.854\times10^{-12}\,C^2N^{-1}m^{-2})^2(1.055\times10^{-34}\,Js)^2}$$

$$= 2.177144\times10^{-18}\ \frac{kg\,\cancel{C^4}}{\cancel{C^4}\,N^{-2}\,m^{-4}\,J^2\,s^2}$$

$$\frac{kg\,\cancel{N^2}\,m^4}{\cancel{N^2}\,m^2\,s^2} = kg\,m^2\,s^{-2} - J$$

$$= 2.177144\times10^{-18}\,J \times \frac{1\,eV}{1.602\times10^{-19}\,J} = \underline{\underline{13.59016\ eV}}$$

$= \underline{\underline{\tfrac{1}{2}\ \text{Hartree}}} = \underline{\underline{1\ \text{Rydberg}}}$

~~½ Hartree~~

~~ease~~ $$2.177144\times10^{-18}\,J\times\left[\frac{1}{(6.626\times10^{-34}\,Js)(2.998\times10^{10}\,cm\,s^{-1})}\right]$$

$$= \underline{\underline{1.095983\times10^{5}\ cm^{-1}}}$$

(3) $$a_0 = \frac{\hbar^2(4\pi\varepsilon_0)}{m_e\,e^2} = \frac{(1.055\times10^{-34}\,Js)^2(4\pi\times8.854\times10^{-12}\,C^2N^{-1}m^{-2})}{(9.109\times10^{-31}\,kg)(1.602\times10^{-19}\,C)^2}$$

$$= 5.29735\times10^{-11}\ \frac{J^2\,s^2\,\cancel{C^2}\,N^{-1}\,m^{-2}}{kg\,\cancel{C^2}} \qquad \frac{\cancel{N^2}\,m^2\,s^{2}\,N^{-1}\,m^{-}}{kg}$$

$$= 5.29735\times10^{-11}\,m \times 10^{10}\,\text{Å} \qquad \frac{N\,s^2}{kg} \quad \frac{kg\,m\,\cancel{s^{-2}}\,\cancel{s^2}}{kg}$$

(4) $$\psi_{2p_{-1}} + \psi_{2p_{+1}} = \underbrace{\frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta}_{A} \left[e^{-i\phi} + e^{i\phi}\right]$$

$$= A\left[\sin\phi - \cancel{i\cos\phi} + \sin\phi + \cancel{i\cos\phi}\right]$$

$$= 2A\sin\phi$$

$$= \frac{2}{8\sqrt{\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta \sin\phi$$

$$\frac{1}{\sqrt{2}}\left[\psi_{2p_{-1}} + \psi_{2p_{+1}}\right] = \frac{\sqrt{2}}{8\sqrt{\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta \sin\phi$$

$$= \frac{\sqrt{2}}{4\times 2\sqrt{\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta \sin\phi$$

$$= \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta \sin\phi$$

$$= \underline{\underline{\psi_{2p_y}}}$$

$$\psi_{2p_{+1}} - \psi_{2p_{-1}} = \underbrace{\frac{1}{8\sqrt{\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta}_{A} \left[e^{i\phi} - e^{-i\phi}\right]$$

$$= A\left[\sin\phi + i\cos\phi - (\sin\phi - i\cos\phi)\right]$$

$$= A\left[2i\cos\phi\right]$$

$$\frac{1}{i\sqrt{2}}\left[\psi_{2p_{+1}} - \psi_{2p_{-1}}\right] = \frac{2A}{\sqrt{2}}\cos\phi$$

$$= \frac{\sqrt{2}}{\sqrt{\cancel{2}}} \frac{1}{8\sqrt{\pi}} \left(\frac{z}{a}\right)^{5/2} r\, e^{-zr/2a} \sin\theta \cos\phi$$

$$\frac{1}{i\sqrt{2}}\left(\psi_{2p_{+1}} - \psi_{2p_{-1}}\right) = \frac{1}{4\sqrt{2\pi}}\left(\frac{Z}{a}\right)^{5/2} r\, e^{-Zr/2a} \sin\theta \cos\phi$$

$$= \underline{\underline{\psi_{2p_x}}}$$

Chapter 10

(3)

$$E = \frac{-\mu Z^2 e^4}{2(4\pi\varepsilon_0)^2 \hbar^2 n^2}$$ for a H-like atom or ion

$\mu \approx m_e$

$$= \underbrace{\frac{-m_e e^4}{2(4\pi\varepsilon_0)^2 \hbar^2}}_{= R \,=\, 13.5902\text{ eV}} \left(\frac{Z^2}{n^2}\right)$$

$$E = -(13.5902\text{ eV})\frac{Z^2}{n^2}$$ Ground state $n = 1$

$\underline{\underline{He^+}}$ $E = -(13.5902\text{ eV})(4) = \underline{\underline{-54.3608\text{ eV}}}$

$\underline{\underline{Li^{2+}}}$ $E = -(13.5902\text{ eV})(9) = \underline{\underline{-122.3118\text{ eV}}}$

$\underline{\underline{Be^{3+}}}$ $E = -(13.5902\text{ eV})(16) = \underline{\underline{-217.4432\text{ eV}}}$

$\underline{\underline{C^{5+}}}$ $E = -(13.5902\text{ eV})(36) = \underline{\underline{-489.2472\text{ eV}}}$

(2) For the H atom $E = -(13.5902\ eV)\frac{1}{n^2}$ $Z = 1$

(a) 3d orbital $n = 3$

$$E = -1.5100\ eV$$

(b) 4f orbital, $n = 4$ $E = -0.8494\ eV$

(c) 4p orbital $n = 4$ $E = -0.8494\ eV$

(d) 6s orbital $n = 6$ $E = -0.3775\ eV$

(4) $E = -\frac{\mu e^4 Z^2}{2(4\pi\varepsilon_0)^2\hbar^2 n^2} = \frac{-e^4}{2(4\pi\varepsilon_0)^2\hbar^2}\left(\frac{\mu Z^2}{n^2}\right)$

$$\frac{e^4}{2(4\pi\varepsilon_0)^2\hbar^2} = \frac{(1.602\times10^{-19}\,C)^4}{2\left[4\pi\times8.854\times10^{-12}\,C^2N^{-1}m^{-2}\right]^2(1.055\times10^{-34}\,Js)^2}$$

$$= 2.3901\times10^{12}\ \frac{\cancel{C^4}}{\cancel{C^4}N^{-2}m^{-4}J^2s^2}$$

$$= 2.3901\times10^{12}\ m^2s^{-2}$$

$$= \frac{N^2m^4}{N^2m^2s^2} = m^2s^{-2}$$

$$E = -(2.3901\times10^{12}\ m^2s^{-2})\left(\frac{\mu Z^2}{n^2}\right)$$

$$\mu_H = \frac{m_N \cdot m_e}{m_N + m_e} = \frac{(1.673\times10^{-27})(9.109\times10^{-31})\ kg}{(1.673\times10^{-27}+9.109\times10^{-31})}$$

$$\mu_H = 9.1040\times10^{-31}\ kg$$

In the deuterium atom the nucleus contains a proton and a neutron.

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$$\mu_D = \frac{(m_p + m_n)\, m_e}{(m_p + m_n) + m_e}$$

$$= \frac{(1.673\times10^{-27} + 1.675\times10^{-27})\; 9.109\times10^{-31}}{(1.673\times10^{-27} + 1.675\times10^{-27} + 9.109\times10^{-31})}\ \text{kg}$$

$$\mu_D = 9.1065\times10^{-31}\ \text{kg}$$

$$E_H = -(2.3901\times10^{12}\ m^2 s^{-2})(9.1040\times10^{-31}\ \text{kg})\ \frac{1^2}{1^2}$$

$$E_H = \underline{-2.1759\times10^{-18}\ J}$$

$$E_D = -(2.3901\times10^{12}\ m^2 s^{-2})(9.1065\times10^{-31}\ \text{kg})\ \frac{1^2}{1^2}$$

$$= \underline{-2.1765\times10^{-18}\ J}$$

Ionization potential of $H = \underline{\underline{2.1759\times10^{-18}\ J}}$

of $D = \underline{\underline{2.1765\times10^{-18}\ J}}$

<u>for H</u>

$$E_{n_1} = -2.1759\times10^{-18}\ \frac{J}{n_1^2} \qquad E_{n_2} = -2.1759\times10^{-18}\ \frac{J}{n_2^2}$$

$$\frac{\Delta E}{hc} = \frac{1}{\lambda} = \frac{E_2 - E_1}{hc} = \frac{2.1759\times10^{-18}\ J}{(6.626\times10^{-34}\ Js)(2.99\times10^{8}\ ms^{-1})}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

$$= 1.0983\times10^{7}\ m^{-1}\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

For the Balmer series first line $n_1 = 2$ $n_2 = 3$

$$\frac{1}{\lambda} = 1.0983 \times 10^{7} \, m^{-1} \left[\frac{1}{4} - \frac{1}{9} \right] = 1.5254 \times 10^{6} \, m^{-1}$$

$$\lambda = 6.5557 \times 10^{-7} \, m = \underline{\underline{655.57 \, nm}}$$

<u>For D</u>

$$E_{n_1} = -2.1765 \times 10^{-18} \frac{J}{n_1^2} \qquad E_{n_2} = -2.1765 \times 10^{-18} \frac{J}{n_2^2}$$

$$\frac{1}{\lambda} = \frac{2.1765 \times 10^{-18}}{(6.626 \times 10^{-34} Js)(2.99 \times 10^{8} ms^{-1})} \left[\frac{1}{n_1^2} - \frac{1}{n_2^2} \right]$$

For the Balmer series, first line $n_1 = 2$ $n_2 = 3$

$$\frac{1}{\lambda} = 1.5258 \times 10^{6} \, m^{-1}$$

$$\lambda = 6.5538 \times 10^{-7} \, m = \underline{\underline{655.38 \, nm}}$$

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$$\mu = \frac{m_p \, m_{muon}}{m_p + m_{muon}} = \frac{(1.673 \times 10^{-27} kg)(200 \times 9.109 \times 10^{-31} kg)}{[1.673 \times 10^{-27} + (200 \times 9.109 \times 10^{-31})] kg}$$

$$\mu = \underline{\underline{1.6429 \times 10^{-28} \, kg}}$$

$$R = \frac{\mu e^4}{2(4\pi\varepsilon_0)^2 \hbar^2}$$

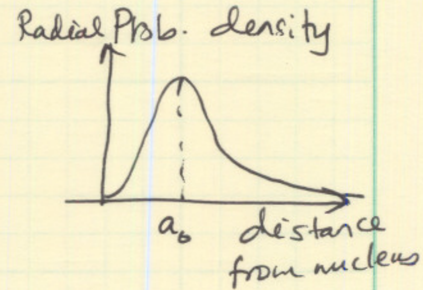
where e = charge of muon

μ = reduced mass of muonium atom

$$R = \frac{(1.6429 \times 10^{-28}\,\text{kg})(1.602 \times 10^{-19}\,\text{C})^4}{2(4\pi \times 8.854 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2})^2 (1.055 \times 10^{-34}\,\text{Js})^2}$$

$$= \underline{\underline{3.9267 \times 10^{-16}\ \text{J}}}$$

$$E = \frac{-\mu e^4}{2(4\pi\varepsilon_0)^2\hbar^2}\left(\frac{Z^2}{n^2}\right) = \underline{\underline{-R\frac{Z^2}{n^2}}}$$



most probable radius for a 1s orbital $\Big\} = a_0$

$$a_0 = \frac{\hbar^2 (4\pi\varepsilon_0)}{m_{muon}\, e^2} \qquad e = \text{charge of muon}$$

$$a_0 = \frac{(1.055 \times 10^{-34}\,\text{Js})^2 (4\pi \times 8.854 \times 10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2})}{(200 \times 9.109 \times 10^{-31}\,\text{kg})(1.602 \times 10^{-19}\,\text{C})^2}$$

$$= 2.6488 \times 10^{-13}\,\text{m} \times \frac{10^{12}\,\text{pm}}{\text{m}} = \underline{\underline{0.26488\ \text{pm}}}$$

⑧ $$\frac{1}{\lambda} = R\left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

Paschen series

$n_1 = 3 \qquad \underbrace{n_2 = 4, 5, 6}_{\text{for the first three lines}}$

$$\frac{1}{\lambda} = 1.097 \times 10^{7}\,\text{m}^{-1}\left[\frac{1}{9} - \frac{1}{16}\right]$$

$$= 5.3326 \times 10^{5}\,\text{m}^{-1}$$

$$\lambda = 1.87524 \times 10^{-6}\,\text{m} = \underline{\underline{1.87524\ \mu\text{m}}}$$

$10^{-6}\,\text{m} = 1\,\mu\text{m}$

for the second line

$$\frac{1}{\lambda} = 1.097\times10^{7}\,m^{-1}\left(\frac{1}{9}-\frac{1}{25}\right) = 7.8009\times10^{5}\,m^{-1}$$

$$\lambda = 1.28191\times10^{-6}\,m = \underline{\underline{1.28191\,\mu m}}$$

Third line $$\frac{1}{\lambda} = 1.097\times10^{+7}\,m^{-1}\left(\frac{1}{9}-\frac{1}{36}\right) = 9.14167\times10^{5}\,m^{-1}$$

$$\lambda = 1.09389\times10^{-6}\,m = \underline{\underline{1.09389\,\mu m}}$$

(9) $$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$$ $n_2 > n_1$

$n_2 = 100 \quad n_1 = 99$

$$= 1.097\times10^{7}\,m^{-1}\left[\frac{1}{99^2}-\frac{1}{100^2}\right]$$

$$= 22.27354\,m^{-1}$$

$$\lambda = 4.48963\times10^{-2}\,m = \underline{\underline{4.48963\,cm}}$$

(10) $$R = \mu\frac{e^4}{2(4\pi\varepsilon_0)^2\hbar^2} = \mu A$$ $\frac{e^4}{2(4\pi\varepsilon_0)^2\hbar^2} = A$ (say)

When the nucleus is extremely heavy

$$\mu = \frac{m_e m_N}{m_e+m_N} = \frac{m_N \cdot m_e}{m_N} = m_e$$

$\therefore$ when the nucleus is extremely heavy

$$R_\infty = \mu A = m_e A$$

for H atom $R_H = \mu_H A$

$$\frac{R_H}{R_\infty} = \frac{\mu_H A}{m_e A} = \frac{\mu_H}{m_e} = \frac{1}{m_e}\left[\frac{m_e \cdot m_N}{m_e + m_N}\right]$$

$$= \frac{m_N}{m_e + m_N}$$

$$R_H = R_\infty \left(\frac{m_N}{m_e + m_N}\right)$$

$$R_H = 10973731.534\ m^{-1}\left[\frac{1.673 \times 10^{-27}\ kg}{9.109 \times 10^{-31}\ kg + 1.673 \times 10^{-27}\ kg}\right]$$

$$= \underline{\underline{1.0967759 \times 10^{7}\ m^{-1}}}$$

$$R_D = R_\infty \left(\frac{m_N}{m_e + m_N}\right)$$

$$= 10973731.534\ m^{-1}\left[\frac{(1.673 \times 10^{-27} + 1.675 \times 10^{-27})\ kg}{(9.109 \times 10^{-31} + 1.673 \times 10^{-27} + 1.675 \times 10^{-27})\ kg}\right]$$

$$R_D = \underline{\underline{1.0970747 \times 10^{7}\ m^{-1}}}$$

(11) $$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

$$\int \psi_{100}^{*}\, \psi_{100}\, d\tau = \int \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\ \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\ d\tau$$

$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^{3} \int_0^\infty \int_0^\pi \int_0^{2\pi} e^{-2Zr/a_0}\ r^2 \sin\theta\ dr\, d\theta\, d\phi$$

$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 \int_0^\infty e^{-2Zr/a_0}\, r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 \int_0^\infty r^2 e^{-2Zr/a_0}\, dr \left[\left(-\cos\theta\right)_0^\pi \left(\phi\right)_0^{2\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 \left[\frac{2!}{\left(\frac{2Z}{a_0}\right)^3}\right]\left(-\cos\pi+\cos 0\right)\left(2\pi\right) \qquad \boxed{\int_0^\infty x^n e^{-kx}\, dx = \frac{n!}{k^{n+1}}}$$

$$= \left(\frac{1}{\pi}\right)\left(\frac{Z}{a_0}\right)^3\left(\frac{a_0}{2Z}\right)^3 2\left(1+1\right)\left(2\pi\right) = \cancel{8}\left(\frac{Z^3}{a_0^3}\right)\left(\frac{a_0^3}{\cancel{8}\,Z^3}\right) = 1$$

∴ ψ_{100} is normalized.

(12) $$\int \psi_{100}^* \psi_{200}\, d\tau = \int \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} \frac{1}{4\sqrt{2}\,\pi}\left(\frac{Z}{a_0}\right)^{3/2}\left(2-\frac{Zr}{a_0}\right)e^{-Zr/2a_0}\, d\tau$$

$$= \frac{1}{4\sqrt{2}\,(\pi)^{3/2}}\left(\frac{Z}{a_0}\right)^3\int\left(2-\frac{Zr}{a_0}\right)e^{-3Zr/2a_0}\, d\tau$$

$$= \frac{1}{4\sqrt{2}\,(\pi)^{3/2}}\left(\frac{Z}{a_0}\right)^3\int\left(2-\frac{Zr}{a_0}\right)e^{-3Zr/2a_0}\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\sqrt{2}\,(\pi)^{3/2}}\left(\frac{Z}{a_0}\right)^3\int_0^r\left(2r^2-\frac{Zr^3}{a_0}\right)e^{-3Zr/2a_0}\, dr\, \underbrace{\int_0^\pi \sin\theta\, d\theta\int_0^{2\pi} d\phi}_{=(4\pi)}$$

$$= \underbrace{\frac{1}{4\sqrt{2}(\pi)^{3/2}}\left(\frac{Z}{a_0}\right)^3}_{A''}\left[2\int_0^\infty r^2 e^{-3Zr/2a_0}\,dr - \left(\frac{Z}{a_0}\right)\int_0^\infty r^3 e^{-3Zr/2a_0}\,dr\right](4\pi)$$

$$= A''\left[2\,\frac{2!}{\left(\frac{3Z}{2a_0}\right)^3} - \left(\frac{Z}{a_0}\right)\frac{3!}{\left(\frac{3Z}{2a_0}\right)^4}\right](4\pi)$$

$$= 4\pi A\left[4\,\frac{8a_0^3}{27Z^3} - \frac{Z}{a_0}\cdot\frac{6\cdot 16a_0^4}{81\,Z^4}\right]$$

$$= 4\pi A\left[\frac{32a_0^3}{27Z^3} - \frac{32a_0^3}{27Z^3}\right] = 0$$

$\therefore$ ψ_{100} and ψ_{200} are orthogonal.

(13) (a) $\langle r\rangle = \int \psi_{100}^* \, r \, \psi_{100}\, d\tau$

$$= \int \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\, r\, \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\, d\tau$$

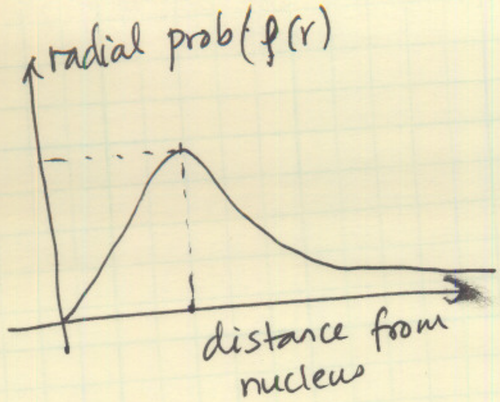
$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 \iiint e^{-2Zr/a_0}\, r.\, r^2 dr \sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^3 \int_0^\infty r^3 e^{-2Zr/a_0}\, dr \underbrace{\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi}_{=4\pi}$$

$$= \frac{1}{\cancel{\pi}}\left(\frac{Z}{a_0}\right)^3\left[\frac{3!}{\left(\frac{2Z}{a_0}\right)^4}\right](4\cancel{\pi}) = \frac{Z^3}{a_0^3}\cdot\frac{\cancel{2}6\cdot a_0^4\cdot 4}{16\,Z^4}$$

$$= \frac{24\,a_0}{16\,Z} = \underline{\underline{\frac{3}{2Z}a_0}}$$

(b)



$$\text{Radial probability} = 4\pi r^2\left[R_{n,l}(r)\right]^2 = \rho(r)$$

at the most probable value of r, there is a maximum in the radial probability $\rho(r)$.

$\therefore$ at the most probable value of r,

$$\frac{d\rho(r)}{dr} = 0$$

$$\rho(r) \text{ for 1s orbital} = 4\pi r^2 R_{1,0}(r)^2$$

$$= 4\pi r^2\left[\frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}\right]^2$$

$$\cancel{\rho(r) = \left(4\sqrt{\pi}\left(\frac{Z}{a_0}\right)^{3/2} r^2 e^{-Zr/a_0}\right)}$$

$$\cancel{\frac{d\rho(r)}{dr} = \frac{d}{dr}\left[\underbrace{4\sqrt{\pi}\left(\frac{Z}{a_0}\right)^{3/2}}_{=A} r^2 e^{-Zr/a_0}\right] = 0}$$

$$\rho(r) = 4\pi r^2 \left(\frac{1}{\pi}\right)\left(\frac{z}{a_0}\right)^3 e^{-2zr/a_0}$$

$$= 4\cancel{\pi}\left(\frac{z}{a_0}\right)^3 r^2 e^{-2zr/a_0}$$

$$\frac{d\rho(r)}{dr} = 4\left(\frac{z}{a_0}\right)^3 \frac{d}{dr}\left[r^2 e^{-2zr/a_0}\right] = 0$$

$$= 4\left(\frac{z}{a_0}\right)^3 \left[2r e^{-2zr/a_0} + r^2\left(\frac{-2z}{a_0}\right)e^{-2zr/a_0}\right] = 0$$

$$r e^{-2zr/a_0}\left[2 - \frac{2zr}{a_0}\right] = 0 \qquad r \neq e^{-2zr/a_0} \neq 0$$

$$\Rightarrow \left[2 - \frac{2zr}{a_0}\right] = 0 \Rightarrow \cancel{r} = \frac{a_0}{z}$$

$$r = a_0/z$$

$\therefore$ most probable distance $(r) = a_0/z$