CALCULATED FICTION

MATH WORKSHOP 9

November 27, 2007

Work on this workshop in groups of 3 or 4.

PART I: HOMEWORK

As always.

PART II: L'EXAM

As you did once before, come get a fresh copy of the exam for your group and go over each problem. Once everyone in the group understands the group's answer to a problem, write it down on the fresh copy; turn in your group's exam (with all of your names on it) when you're done. Ask for help if you need it on any of the questions. This should take only 20-30 minutes if you stay focused.

PART III: CLOCKS OR WHATEVER

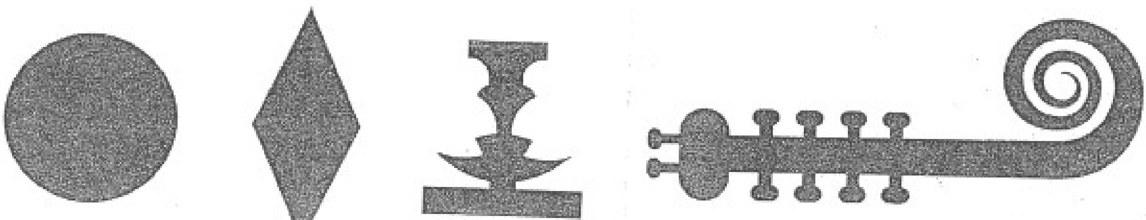
ACTIVITY

1. Do the excursion on pages 415-416 of ME.

PART IV: STREEEEETCH IT OUT

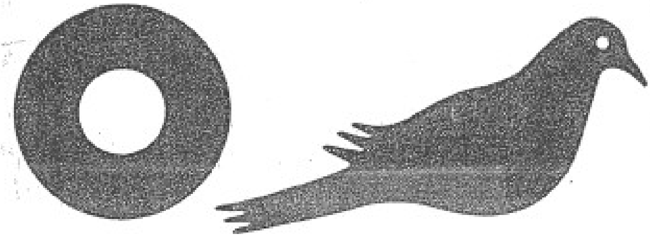
Topology, formally, is the study of properties of figures that endure when the figures are subjected to continuous transformations. Intuitively, this means properties that endure when the figures are stretched or squashed but not cut or sewn together. In topology, we get to pretend that everything is made out of some infinitely stretchy material; very flexible rubber is a reasonable stand-in.

From the figure below it is clear that a continuous transformation can destroy all the usual geometric properties of the disk, such as its shape, area, and perimeter. Therefore these are *not* topological properties of the disk. Any topological properties the disk has must be shared with the other shapes in the figure, since they are all topologically equivalent.



Although topology seems very different from Euclidean geometry, the two are fundamentally alike. In Euclidean geometry there are again a basic group of transformations, called congruences or rigid motions. The subject matter of Euclidean geometry is determined by the congruences, just as the subject matter of topology is determined by the continuous transformations. Congruences preserve things like distance and angles; Euclidean geometry can be defined as the study of properties of figures that endure when the figures are subjected to rigid motions.

Here are two figures that are topologically equivalent to each other but *not* topologically equivalent to the disk:



No matter how or how much we stretch or squash these figures, they will still have exactly one hole in them. The number of holes a figure has is one of its *topological invariants*.

ACTIVITY

2. Draw four plane figures that are not topologically equivalent to each other.

3. Now consider the capital letters of the alphabet, written in some nice blocky font. Which letters are topologically equivalent to which others? Put them into groups by topological equivalence.

Of course there's topological equivalence for 3-dimensional surfaces as well (and 4-dimensional, and 5-dimensional, and...) The principle is exactly the same: we can stretch or squash as much as we like as long as we don't cut or sew. (Topologists do sometimes get to cut and sew, but we'll leave that to the professionals.)

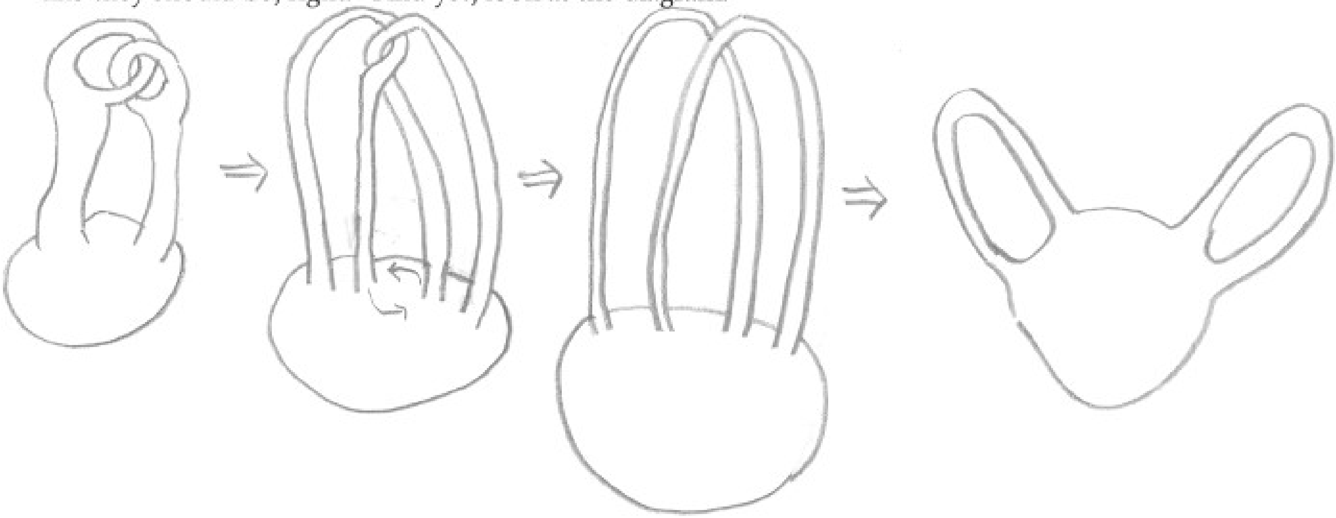
ACTIVITY

4. Draw four 3-dimensional figures that are not topologically equivalent to each other.

5. Consider some familiar objects from a topological perspective. Which of the following objects are topologically equivalent: hand iron, baseball bat, pretzel, telephone handset, rubber band, chair (consider various types), funnel, pair of scissors, Frisbee.

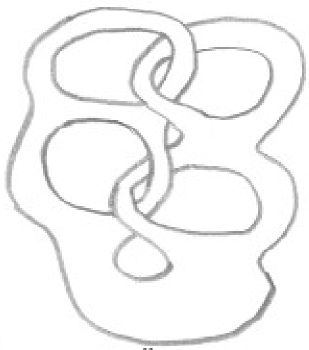
6. Draw a sequence of pictures that shows how a donut and a coffee cup are topologically equivalent. (This is why they say a topologist is someone who can't tell the difference between a donut and a coffee cup.) For an example of such a sequence, look ahead just a bit.

It's probably harder to do continuous transformations in your head in 3 dimensions than it is in 2. Consider the question of whether a sphere with two holes in which the holes are linked (see diagram) is topologically equivalent to a sphere with two *un*linked holes. It doesn't seem like they should be, right? And yet, look at the diagram.



ACTIVITY

7. Consider the 4-holed sphere with holes linked as in the picture. How much can you unlink it? Draw pictures showing how to unlink it as far as you can.



MÖBIYOU + MÖBIME = MÖBIUS

Now for that old topological standby, the Möbius strip (or loop, or band)! This surface was first discovered by August Ferdinand Möbius in 1858. Möbius was a mathematician and professor of astronomy whose work in topology revolutionized the field of non-Euclidean geometry. He was also the first astronaut and he invented break dancing and he *is standing right behind you.*

Let's start by making our own Möbius strips. Start with a longish strip of paper. Give it a half twist and tape the ends together. And voilá! Möbiustastic.

ACTIVITY

8. Make your own Möbius strip. It's "fun".

Now let's see what's so interesting about the Möbius strip. We keep hearing that these things have only one side and only one edge; it's time to see for ourselves what all the fuss is about.

ACTIVITY

9. Verify that your Möbius strip has only one side. To do this, pick a point in the middle of the strip and start drawing around the strip, tracing the line that an ant might follow if it walked around the strip (the long way, naturally). Where does your line end up?

10. Now verify that your strip has only one side. To do this, pick a point on the edge and run a highlighter along the edge. Where do you end up?

These are useful properties in the real world; giant Möbius strips have been used as conveyor belts (to make them last longer, since "each side" gets the same amount of wear) and as continuous-loop recording tapes (to double the playing time). And of course M.C. Escher used Möbius strizzles in his art all the time.

Now let's see what happens when we cut along the Möbius strip.

ACTIVITY

11. Cut your Möbius strip along the line you drew before. Before you do it, what do you expect to end up with? What *do* you end up with? Be specific and careful about identifying what object or objects you get.

12. Make another Möbius strip. This time, instead of cutting down the middle, start your cut 1/3 of the way across and keep cutting until you get back to where you started. Before you make the cut, what do you expect to end up with? What *do* you end up with? Again, be careful and specific in your identification.

13. Now make a different shape as follows: take a strip of paper as before, but give it a *full* twist (instead of a half twist) and then tape the ends together. How many sides does it have? How many edges does it have?

14. Now cut this new shape down the middle. Before you start, predict what you'll end up with. What *do* you end up with?

15. Now make another copy of the new shape and cut around, but this time start your cut 1/3 of the way across. Before you do any cutting, predict what you'll end up with. What *do* you end up with?

If your predictions were right every time, you should probably become a topologist. The vans are waiting outside.

Now cut two identical longish strips of paper that are about one inch wide. Place one strip on top of the other and then give both strips together a half-twist and tape the ends together, top to top and bottom to bottom (but not top to bottom or bottom to top). You should now have something sort of like two Möbius loops nested right next to each other.

ACTIVITY

16. Take something topologically equivalent to a pencil and place it between the two loops. Move the pencil around the loop one time until it returns to the place where it started. What has happened to the pencil?

17. What do you think would happen if you pulled the two strips apart (without damaging them, of course)? Pull them apart and find out. What do you end up with?

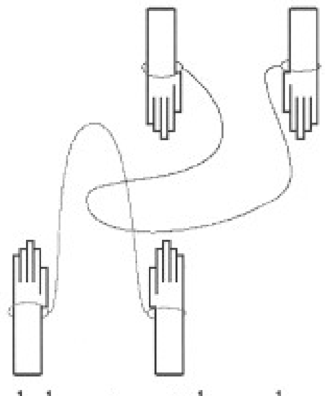
Incidentally, it's possible that our universe is a giant *n*-dimensional equivalent of a Möbius strip. What would this mean? Well, if you live in the surface of a Möbius strip and you walk around the strip once, when you get back your orientation is reversed. (Hmmm...) Our equivalent would be that if you cruised in a straight line far enough in some direction you'd end up back where you started, only you'd be left-right reversed! Sounds like a wacky Jim Carrey movie to me. And since mirror-image sugar molecules taste sweet but aren't usable by our bodies, if you were carrying a can of soda with you when you left, when you got back it'd be *diet* soda.

TOPOLOGICAL FUNS

See how you do at these stretchriffic puzzles. In all cases, there's an answer that doesn't involve cheating; that's the one you should look for. (Nobody will think you're clever for cheating, so don't bother.) These are tricky for most people, so don't fret if you don't solve them.

ACTIVITY

18. *The Handcuffs Puzzle.* For this puzzle, you need two people and two longish pieces of string, each long enough that the person wearing it can easily step over it if they want to. Each person puts on a complete set of handcuffs. Before putting them on, they loop their handcuffs around each other so they are tied together, as in the diagram below.



Now the two people have to get themselves apart, subject to these rules:

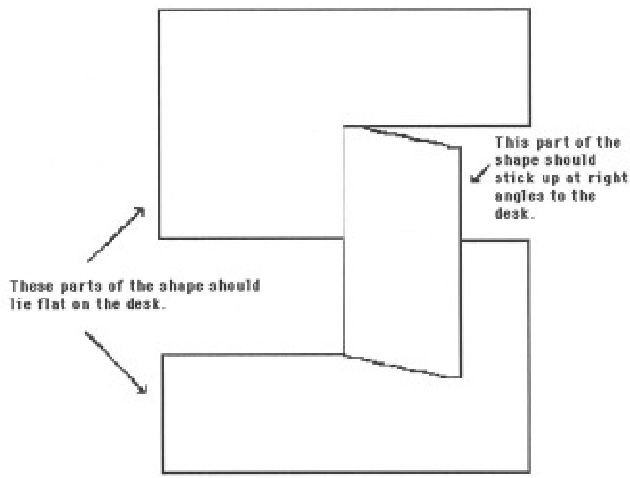
(1) The handcuffs cannot be removed.

(2) You may not break, cut, saw through, bite through, or in any other way damage the string. Damaging each other is also forbidden.

So how do you do it?

19. *The Perplexing Pencil Puzzle.* Take a garment with a buttonhole to Brian/Steven/Taiyo for this one. Your task is to remove the pencil and string from the garment without damaging the pencil, the string, or the garment, and without untying the string or shortening the pencil (for example, by sharpening it). How do you do it?

20. Can you make the 3-dimensional shape pictured below from a single sheet of paper, without using any tape or glue?



THE WRITE-UP

Your final math write-up for the quarter is due at 5pm tomorrow (Wednesday). Each student must submit a write-up individually. On each item of your write-up, indicate whom you worked with on that item.

As before, the point here is to show us how well you can explain your mathematical work. **If you just write down the answer, you're not actually doing the assignment.** It's fine to write up problems that you don't have complete solutions for as long as you have something interesting to say about them. More often, you'll probably want to choose problems for which you're confident of the solutions; show us your best efforts at writing up those solutions clearly and neatly.

For this final write-up, choose the three items from this workshop that you can write up the best. Shoot for variety and quality. Remember that your choice of items to write up is an important element of your work.

REFERENCES

Stuff for this workshop was boosted from the following sources:

A Combinatorial Introduction to Topology by Michael Henle
http://scidiv.bcc.ctc.edu/Math/Mobius.html
http://www.aimsedu.org/Puzzle/
http://ccins.camosun.bc.ca/~jbritton/jbpuzzle.htm