Music and Mathematics

From Pythagoras to Fractals

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Music and mathematics: an overview

Susan Wollenberg

Mathematics and music have traditionally been closely connected. The seventeenth century has been seen by historians as a crucial turning-point, when music was changing from science to art, and science was moving from theoretical to practical. Many connections between science and music can be traced for this period. In the nineteenth and twentieth centuries, the development of the science of music and of mathematical approaches to composition further extended the connections between the two fields. Essentially, the essays in this book share the concern of commentators throughout the ages with the investigation of the power of music.

Musicke I here call that *Science*, which of the Greeks is called Harmonie... Musicke is a Mathematical Science, which teacheth, by sense and reason, perfectly to judge, and order the diversities of soundes hye and low.

JOHN DEE (1570)

The invitation to write an introduction to this collection offered a welcome opportunity to reflect on some of the historical, scientific, and artistic approaches that have been developed in the linking of mathematics and music. The two have traditionally been so closely connected that it is their separation that elicits surprise. During the late sixteenth and early seventeenth centuries when music began to be recognized more as an art and to be treated pedagogically as language and analysed in expressive terms, it might have been expected to lose thereby some of its scientific connotations; yet in fact the science of music went on to develop with renewed impetus.

This introduction sets out to explore, via a variety of texts, some of the many historical and compositional manifestations of the links between mathematics and music. (This endeavour cannot be other than selective: the field is vast, ranging from ancient theory and early developments in structure such as those of the medieval motet, to the new ideas of post-tonal music and experimental musical techniques explored over the past century.) In what follows, the field is viewed particularly from the perspective of a music historian with a special interest in the history of music in its educational dimension.

In the traditional arrangement of knowledge and teaching in universities, music was one of the seven liberal arts, along with the other quadrivium subjects of arithmetic, geometry and astronomy. This woodcut dates from 1504.

Aspects of notation and content

In contemplating the two disciplines, mathematics and music (and taking music here essentially to mean the Western 'Classical' tradition), it is clear to the observer from the outset that they share some of their most basic properties. Both are primarily (although not exclusively) dependent on a specialized system of notation within which they are first encoded by those who write them, and then decoded by those who read (and, in the case of music, perform) them. Their notations are both ancient and modern, rooted in many centuries of usage while at the same time incorporating fresh developments and newly-contrived systems to accommodate the changing patterns of mathematical and musical thought.

Musical notation can be traced back to the ancient Greek alphabet system. A series of significant stages came in the development of notations within both the Western and Eastern churches during the medieval period. In the eleventh to thirteenth centuries more precise schemes were codified, including Guido d'Arezzo's new method of staff notation and the incorporation of rhythmic indications. By the time of the late sixteenth and the seventeenth centuries, most of the essential features of musical notation as it is commonly understood today were in place within a centrally established tradition. Subsequent additions were mainly in the nature of surface detail, although of considerable importance, as with the expanded range of performance instructions in the nineteenth century. The twentieth century, with its emphasis on experimental music, saw a precipitate rise in new forms of notation. In a comparable way, mathematical notation has developed over a period of at least 2500 years and, in doing so, has inevitably drawn from various traditions and sources.

In music, the relationship between notation and the content it conveys is sometimes more complex than might at first appear. Notation has not invariably fulfilled the role merely of servant to content. While it is generally true that notational schemes evolved in response to the demands posed by new ideas and new ways of thinking, it is also possible that experiments in notation may have been closely fused with the development of such ideas, or may even have preceded—and inspired—their creation. In mathematics, too, the relationship has subtle nuances. Notation developed in one context could prove extremely useful in another (seemingly quite different) context. (A well-known example of this is the use of tensor notation in general relativity.) In one notable case, notation formed part of the focus of a professional dispute, when a prolonged feud developed between Newton and Leibniz as to which of them invented the differential calculus, together with the different notation used by each.

notation

In the course of their history, mathematics and music have been brought together in some curious ways. The Fantasy Machine demonstrated in 1753 by the German mathematician Johann Friedrich Unger to the Berlin Academy of Sciences, under Leonhard Euler's presidency, was designed to preserve musical improvisations; in the words of an English inventor, the Revd John Creed, on whose behalf a similar idea was presented to the Royal Society in London in 1747, with this device the 'most transient Graces' could be 'mathematically delineated'. Unger claimed to have had the idea as early as 1745, although Charles Burney (in his essay 'A machine for recording music') attributed priority of invention to Creed. Although it aroused considerable interest and support among the intelligentsia, and 'was tried out by several well-known musicians' in the mid-eighteenth century, the machine was ultimately not a success.

Music as science: the historical dimension

Throughout the history of mathematical science, mathematicians have felt the lure of music as a subject of scientific investigation; an intricate network of speculative and experimental ideas has resulted. Taking a historical view, Penelope Gouk has voiced her concern that such terms as 'mathematical sciences' are 'routinely used as essentially unproblematical categories which are self-evidently distinct from the arts and humanities . . . Since music is today regarded as an art rather than a science, it is hardly surprising that the topic should be disregarded by historians of science'. Her book remedies this situation with resounding success, inviting a reconsideration of the way joint histories are told.

Within the scope of a work based primarily on seventeenth-century England, Gouk's references range from Pythagoras (in particular, Pythagorean tuning and the doctrine of universal harmony that 'formed the basis of the mathematical sciences') to René Descartes ('the arithmetical foundation of consonance') and beyond. Descartes' *Compendium* (1618) was translated as *Renatus Des-Cartes excellent compendium of musick and animadversions of the author* (1653) by the English mathematician William Brouncker. Brouncker himself was 'the first English mathematician to apply logarithms (invented c.1614) to the musical division'. Thus he entered into a scientific dialogue with the work of Descartes, contesting the latter's findings.

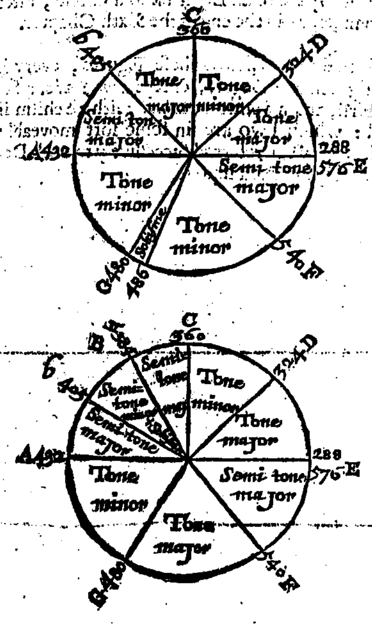
At the period when music was changing from science to art (retaining a foot in both camps), science itself was moving from theoretical to practical. The seventeenth century has been seen by historians as a crucial turning-point, with the emergence of a 'recognizable scientific community' and the institutionalization of science. The founding of the Royal Society of London in 1660 formed a key point in the development

1600s

RENATUS DES-CARTES
EXCELLENT
COMPENDIUM
OF
MUSICK:
WITH
Necessary and Judicious
ANIMADVERSIONS
Thereupon.

By a Person of HONOUR.

London, Printed by *Thomas Harper*, for *Humphrey Moseley*, and are to bee sold at his Shop at the Signe of the *Princes Armes* in S. *Pauls* Church-Yard, and by *Thomas Heath* in *Coven Garden*. 1653.



Their Proportions, Denominations,

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506.25	9.375	16	15	Semitone major; or Degree minor
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364.5 360	6.750 6.666	40 3	27 2	Fifth — Schisme Fifth; or Diapente
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324 320	6000 5.925	5 27	3 16	Sixth major; or diapente + Tone minor Diapente + tone major
303.75 300	5.625 5.555	16 9	9 5	Seventh minim; or Diapason — Tone major Seventh minor; or Diapason — Tone minor
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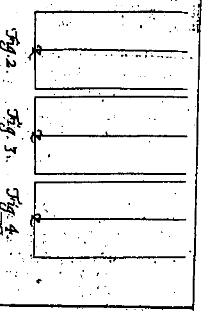
270 .. or	5000 .. is to	2	1	Eight, or Diapason

Their proper



Differences.

1	2	Eighth, or Diapason	
25	48	Semi-Eighth	Semitone minor, Diesis major, or Chromatica.
8	15	Seventh major	Diesis minor, or Enharmonica, as 125 to 128 Δ
5 9	9 16	Seventh minor Seventh minime	Semitone minor. Schisma, i.e. as 81 to 80 Δ. as 80 to 81 ∇
16 3	27 5	Fifth + Second major Sixth major	Semitone, or Limma Pythag. as 243 to 256. ∇ Schisme, or Comma major
5	8	Sixth minor	Semitone minor.
			Semitone major.
2 27	3 40	Fifth Fifth — Schisme	Schisme.
45 32	64 45	Semififth Tritone	Semitone minor. Comma minus i.e. as 2048 to 2025 Δ
20 3	27 4	Fourth + Schisme Fourth	Semitone minor. Schisme.
4	5	Third major	Semitone major.
5 27	6 32	Third minor Third minor — Schisme	Semitone minor Schisme } Semitone med. as 135 : 128. Δ
8 9	9 10	Tone major Second minor	Semitone pythag. as 256 to 243. Δ Schisme.
15 24	16 25	Semitone major Semitone minor	Semitone minor Diesis minor } Semit. max. as 27 to 25. Δ Semitone minor.
25	10	As an Unisone, to its graver.	

tions, Denominations, Differences.

540	A	10.000	540	A	10.000	540	A	10.000
510.3	B	9.450	509.7	B	9.439	509.2	B	9.429
482.2	C	8.929	481.1	C	8.909	480.1	C	8.891
455.7	D	8.438	454.1	D	8.409	452.7	D	8.384
430.6	E	7.974	428.6	E	7.937	426.9	E	7.905
406.9 405.9	F	7.535 7.517	404.5	F	7.492	402.5	F	7.454
384.5	G	7.120	381.8	G	7.071	379.6	G	7.029
363.3	H	6.728	360.4	H	6.674	357.9	H	6.628
343.3 339.3	I	6.358 6.283	340.2	I	6.300	337.5	I	6.250
324.4	K	6.008	321.1	K	5.946	318.2	K	5.893
306.6	L	5.677	303.1	L	5.612	300.1	L	5.557
289.7	M	5.365	286.1	M	5.297	282.9	M	5.240
273.7 269.1	N	5.069 4.983	270 .	N	5.000	266.8	N	4.941
258.7	O	4.790				251.6	O	4.659
244.5	P	4.527				237.2	P	4.393
231.0	Q	4.278				223.7	Q	4.142
218.3	R	4.042						
206.3	S	3.820						

Four pages from Lord Brouncker's translation, *Renatus Des-Cartes excellent compendium of musick.*

of modern scientific enquiry. The leading scientific thinkers who gathered under its auspices focused some of their attention on music. Gouk notes that 'the [Royal] Society's most overt interest in musical subjects occurred within the presidencies of Brouncker and Moray, both of whom . . . were competent musicians and keen patrons of music'. Practitioners of both mathematics and music could learn much from each other's work.

Ideas such as those of musical tunings were constantly subject to review in the light of new theories. Musical issues occupied a central, not peripheral, position in science: 'the conceptual problems involved in the division of musical space were among the most important challenges faced by seventeenth-century mathematicians and natural philosophers'. As Gouk observes, Newton in the mid-1660s 'learned all that had been developed by modern mathematicians such as Descartes, Oughtred and Wallis' regarding the musical scale, and especially the division of the scale, and 'rapidly went beyond them in his own studies'. Important discoveries of this period generally included the observation that 'pitch can be identified with frequency'. The seventeenth century saw the beginnings of modern acoustical science: the new science of sound.

The work of Mersenne has also been seen as representing 'a significant milestone in the emergence of modern science, just like the musical laws that he established'. Mersenne's writings—notably, his *Harmonie universelle* (1636) and *Harmonicorum libri* (see Chapter 2)—became available in England. (Gouk notes 'how rapidly Mersenne's work on musical acoustics was assimilated in England'.) Mersenne's belief that 'the universe was constructed according to harmonic principles expressible through mathematical laws' provided an impetus

The legend of Pythagoras's discovery of a relation between musical notes and hammer weights, as portrayed in a twelfth-century manuscript.

for mathematicians such as Newton. Personal contacts and correspon ence among scientists further created and consolidated intellectu connections at this period.

Educationally, the influential tradition of Boethius (*c.*480–524), cas ing a long shadow over the following centuries, and based in its turn o Pythagoras and Plato, aligned music with arithmetic, astronomy an geometry in the *quadrivium*, while grammar, rhetoric and logic forme the language-based *trivium*. When the seven Gresham Professorship were founded in the City of London in 1596 to provide free adult edu cation, their subjects included music, 'physic', geometry and astron omy. At the opposite end of the educational spectrum, Henry Savile' 1619 foundation of the University Chair in mathematics at Oxfor University included, in its stipulations of the new professor's duties that he was to expound on 'canonics, or music' as one of the quadrivial disciplines. Music was taught at the universities as a science, while it was examined (in the form of the B.Mus. and D.Mus. degrees) as an art, by means of the submission of a composition.

Among the spate of Professorships endowed during the early decades of the seventeenth century, William Heather's founding of the Chair in Music at Oxford (1627) recognized this duality with its provision for the regular practice of music as well as lectures on the science of music. In doing so, Heather reflected Thomas Morley's two-fold division in his *Plaine and easie introduction to music* (1597):

> *Speculative is that kinde of musicke* which by Mathematical helpes, seeketh out the causes, properties, and natures of soundes . . . content with the onlie contemplation of the Art, *Practical* is that which teacheth al that may be known in songs, eyther for the understanding of other mens, or making of one's owne . . .

Scientific musical enquiry, analytical listening (or listening with understanding), and the art of composition, are all equally acknowledged as valid activities here.

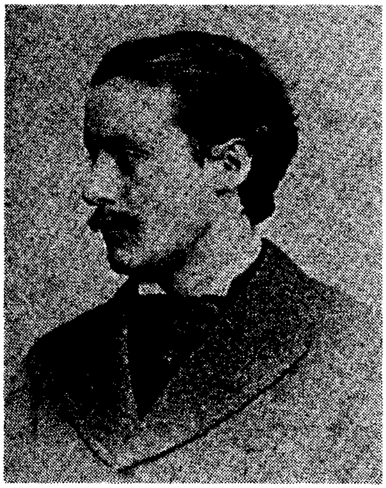
William Heather, founder of the Heather Chair in Music at Oxford University, and William Crotch, a later holder of that Chair.

In his pioneering lectures published in 1831, the Heather Professor of Music, William Crotch was in no doubt as to music's position in the scheme of things: from the outset of his 'Chap. 1: Introductory', he asserted that 'Music is both an art and a science'. Crotch followed this opening gambit with a long and particularly apposite quotation from the work of Sir William Jones:

> Music . . . belongs, as a science, to an interesting part of natural philosophy, which, by mathematical deductions . . . explains the causes and properties of sound . . . but, considered as an art, it combines the sounds which philosophy distinguishes, in such a manner as to gratify our ears, or affect our imaginations; or, by uniting both objects, to captivate the fancy, while it pleases the sense; and speaking, as it were, the language of nature, to raise corresponding ideas and connections in the mind of the hearer. It then, and then only, becomes fine art, allied very nearly to poetry, painting, and rhetoric . . .

While Crotch went on to state that 'The science of music will not constitute the subject of the present work', he nevertheless used this as a device to launch into a discussion of the merits of such an enquiry, strongly recommending 'the study ... of the science of music ... to every lover of the art', and pursuing some of its ramifications at considerable length before concluding that 'enough ... has now been said, to induce the lover of music to study the science, which, it will be remembered, is not the proper subject of this work'. After some ten pages of discussion the reader might well have forgotten this assertion, or be inclined to question it; and it is clear that Crotch felt it inappropriate to offer to the public a didactic treatise on music without paying any consideration to its scientific dimension, even though his primary purpose in presenting these lectures was an aesthetic one ('being the improvement of taste').

Some social and educational connections



Hubert Parry, as an Oxford undergraduate, and William Donkin, Savilian Professor of Astronomy.

In the more informal sphere, the history of cultural life is liberally scattered with examples of musical mathematicians and scientists. The group of intellectuals and artists to which C. P. E. Bach belonged in eighteenth-century Hamburg, and which included J. J. C. Bode (translator of, among other works, Sterne's *A sentimental journey*), met regularly at the house of the mathematician J. G. Büsch; 'many were keen amateur musicians, including Bode who played the cello in the regular music-making at Büsch's house'. C. P. E. Bach's biographer, Hans-Günter Ottenberg, has written of 'the friendly atmosphere and liberal exchange of ideas which took place at the home of the mathematician Johann Georg Büsch ...', quoting Reichardt's description of these gatherings, which evidently possessed a certain cachet: 'not everyone was admitted to the inner circle which would not infrequently assemble for a pleasant evening's entertainment apart from the wider academic community'. Ottenberg stresses that C. P. E. Bach was 'one of Büsch's closer acquaintances'.

In nineteenth-century Oxford, Hubert Parry, as an undergraduate, frequented the home of Professor Donkin (Savilian Professor of Astronomy) where the Donkins—a highly musical family altogether—held chamber-music gatherings. For Parry these occasions and the opportunities they provided, both for getting to know the chamber music repertoire and for composing his own efforts in the genre, were enormously stimulating. The Donkins were influential figures in Oxford's musical life during the second half of the nineteenth century.

It was in this period, too, that the academic status of music, in the shape of the Oxford musical degrees, acquired greater weight. The succession of Heather Professors of Music at Oxford and their assistants voiced their hopes for the development of the subject within the

University, including serious consideration given to the science of music; for example, the set texts for the D.Mus. at Oxford included Helmholtz (see Chapter 5), and others, on acoustics. The evidence presented by Sir Frederick Ouseley (then Heather Professor) to the University of Oxford Commission in 1877 included a 'Proposal for establishing a Laboratory of Acoustics' (apparently this plan was never realised); Ouseley envisaged that such a laboratory 'might work in with the scientific side of a school of technical music' and would have 'more direct relations with the school of physics in the University'.

Holders of music degrees from Oxford during this period (qualifications that were considerably coveted in the musical profession) did not all follow primarily musical careers; William Pole FRS (b.1814, B.Mus. 1860, D.Mus. 1867) was Professor of Civil Engineering at University College, London, as well as organist of St Mark's, North Audley Street. Among those who took the B.Mus. at Oxford, in addition to the ordinary BA, was J. Barclay Thompson of Christ Church (B.Mus. 1868), who became University Reader in Anatomy. More recent scientist-musicians have included the mathematically trained musicologist Roy Howat, whose work on the golden section in Ravel's music, among other topics, has attracted wide interest.

Mathematics and music: the compositional dimension

While music has fascinated mathematical scientists as a subject of enquiry, musicians have been attracted by the possibilities of incorporating mathematical science into their efforts, most notably in the fields of composition and analysis. The fundamental parameters of music—pitch, rhythm, part-writing, and so on—and the external ordering of musical units into a set, have lent themselves to systematic arrangement reflecting mathematical planning. Much has been written about the mathematical aspects of particular compositional techniques—for example Schoenberg's method of serialism (see Chapter 8)—and individual works have frequently been analysed in terms of their mathematical properties, among other aspects.

The possibilities of mathematical relationships not only within a single piece, but also between a number of pieces put together to form a set, are well documented. These sorts of schemes may be expressed in the findings of musical analysts, possibly by reconstructing notional systems of composition, and, further, by examining both the known and the speculative symbolic associations, as well as the mathematical ramifications, of such structural procedures. This is found most obviously in the case of number symbolism, which may be perceived as governing the musical relationships of an individual piece or a whole set of pieces.

Contrapuntal techniques in music have traditionally been treated mathematically and identified with qualities of rigour. Among the prime examples in these latter two categories—the compositional set, and rigorous counterpoint—must be counted the works of J. S. Bach, with their mirror canons and fugues, their ordering by number (as with the *Goldberg variations*), and their emphasis on combinatorial structures. At a distance of over 200 years, Paul Hindemith's cycle of fugues and interludes for piano, the *Ludus tonalis*, with its 'almost geometric design', its pairs of pieces mirroring each other (see Chapter 6), provides a modern echo of these contrapuntal ideas very much in the Bach tradition, as well as building on techniques developed in Hindemith's theoretical writings. It has been suggested, moreover, that Hindemith 'identified... closely with Kepler', whose life and work formed the subject of Hindemith's last full-length opera, *The harmony of the world* (1956–7).

'Scientific' music has not, however, always been appreciated by musical scientists. Christiaan Huygens, for instance, expressed a wish that composers 'would not seek what is the most artificial or the most difficult to invent, but what affects the ear most', professing not to care for 'accurately observed imitations called "fugues" ', or for canons, and claiming that the artists who 'delight in them' misjudge the aim of music, 'which is to delight with sound that we perceive through the ears, not with the contemplation of art'. Huygens here articulated the tension between 'scientific' construction in musical composition, on the one hand, and music's expressive effect, on the other. The balance between these two aspects, and more widely between the scientific basis of the art of music and its aesthetic applications, has been a source of fascination for scholars, and indeed continues to be so, as the essays in this book serve collectively to demonstrate. Their shared concern is essentially the investigation of the power of music, which has preoccupied commentators throughout the ages, from antiquity to our own time.

PART II

The mathematics of musical sound

The science of musical sound

Charles Taylor

This chapter complements the others by describing practical demonstrations and experiments. In recent years a good deal has been said about the differences in experiments in an elementary physics laboratory, mathematical theory, and real musical instruments. In fact there are no real differences except those arising from too simplistic an approach.

Sound of any kind involves changes of pressure in the air around us; for example, in ordinary speech the pressure just outside the mouth increases and decreases by not more than a few parts in a million. But to be detected by our ears and brains as sound, these changes have to be made fairly rapidly. This can be demonstrated easily by inflating a balloon and then gently squeezing it between thumb and finger. This creates quite large pressure changes without any attendant sound; but inserting a pin creates a change that can very readily be heard.

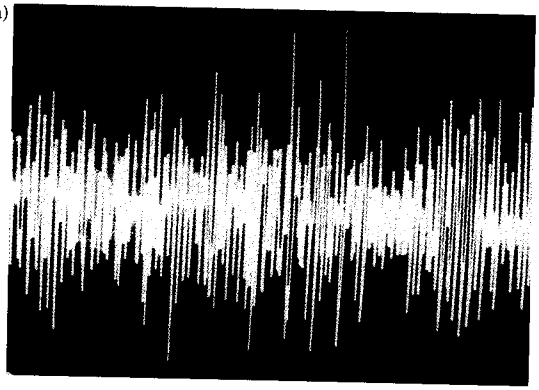
Scientists study the nature of the pressure changes using a cathode-ray oscillograph that draws a graph of the pressure as a function of time. It is interesting to look at the traces corresponding to a wide variety of sounds and try to relate what is perceived by the ear–brain system with what is simultaneously perceived by the eye–brain system. It proves to be impossible to make any but the broadest generalizations about a sound by observing only its oscillograph trace. As an example, it is not easy to differentiate between the oscillograph traces of the end of the first movement of Mendelssohn's *Violin concerto*, a symphony orchestra 'tuning up', and the chatter of an audience waiting for a lecture to begin (see overleaf), although aurally they are completely different.

One of the most astonishing properties of the human brain is that of recognizing sounds in a split second. For example, if a dozen subjects are all asked to repeat the same word, an audience has no difficulty in understanding what is being said. But for each one of the twelve, the corresponding oscillograph traces is completely different and it is virtually impossible to find common features.

So here we have two different ways of presenting the same information: the brain has little problem in interpreting the aural form, but the visual form presents far greater difficulties.

Charles Taylor demonstrating an oboe to some children.

(a)



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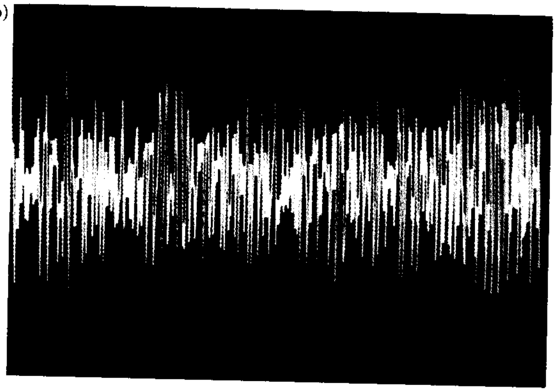

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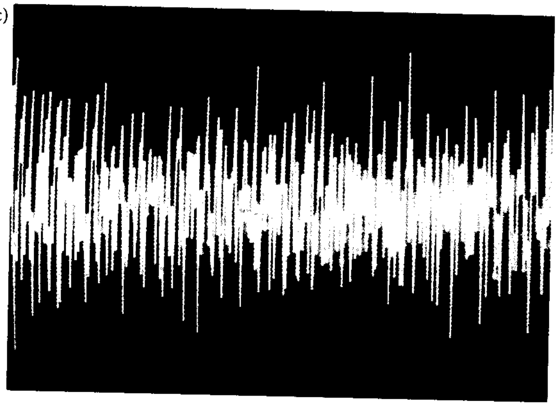


Oscillograph traces of three different sounds:
(*a*) the end of the first movement of the Mendelssohn violin concerto;
(*b*) a symphony orchestra tuning up;
(*c*) an audience waiting for a lecture to begin.

The part played by the brain

Having introduced the topic of aural perception, we next elaborate on its remarkable features, since it affects practically every experiment done in the field of musical acoustics.

We first notice that the pressure of the air can only have a single value at a particular point at a particular time. So, if you listen to a large orchestra of seventy players, each instrument creates its own characteristic changes in pressure, but they all add together to produce a single sequence of changes at the ear and there is only one graph of pressure against time that represents the sum of the changes produced by all the instruments. Yet, with surprisingly little effort, a member of the audience can listen at will to the different instruments. The problem of disentangling these instrumental components from the single graph would be extraordinarily difficult for a computer, unless it were given all kinds of clues about the nature of each different instrument, but the human ear–brain system performs the miracle in a fraction of a second.

One of the factors that makes this possible is the learning ability of the brain. Stored in our brains we all have the characteristic features of all the various instruments that we have heard before and these can be drawn on subconsciously to aid the disentangling process.

An interesting example of this learning process is as follows. A recording of synthetic speech can be created by first imitating the raw sound of the vocal chords by means of an interrupted buzz on one note, adding chopped white noise to represent ss, sh and ch sounds, and then introducing just one formant for each vowel. An audience is unable to recognize the sentence that has been synthesized. However, having been told what the sentence was, they have no difficulty in recognizing it on a second hearing.

This ability of the brain, both to memorize sounds, and to identify similar sounds in the memory banks at great speed, is vital to our existence, but is also a great nuisance in psycho-acoustic research. Its importance lies in the way that we can rapidly identify sounds that indicate danger, in the way that we learn to speak as babies, and in the way that we can adapt to very distorted sounds and in many other activities. Adaptation to distorted sounds is illustrated if one listens to messages being relayed over 'walkie-talkie' systems to the police, to pilots in flying displays, and in other circumstances where those used to the system have no difficulty in understanding the messages, but outsiders find the speech hard to follow.

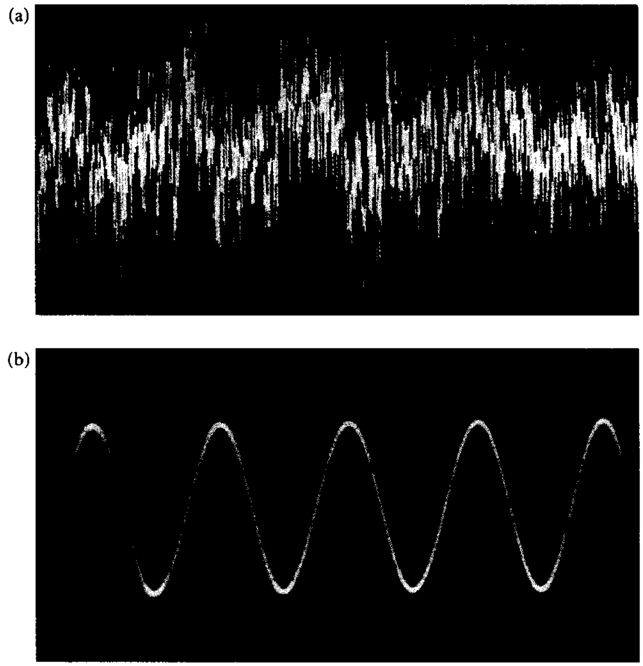
The problem in psycho-acoustic research arises because the very act of performing the first experiment produces changes in the memory banks of the subject. For example, consider an experiment on pitch perception where a participant is asked to compare groups of sounds and to say which is the highest in pitch. Once the first group of sounds has been heard it is impossible for the subject to ignore those sounds, and the response, at a latter stage, even to the same group of sounds, is very rarely the same.

Another of the many remarkable properties of the brain that plays a part in our aural perception is that of ignoring sounds which are of no importance to us. If a series of sounds—such as a baby crying, a dog barking or a fire engine's siren—were played while someone continued to speak, then the listeners will continue to hear what is being said, because they rapidly identify the extra sounds as of no personal relevance.

Differences between music and noise

The above examples are of relatively complex sounds and it is difficult to draw clear scientific distinctions between music and noise with sounds of this complexity. The two simplest kinds of sounds that occur in studies of sound are white noise and a pure tone. Musically useful sounds consist of mixtures and modifications of these two basic kinds of sound—pure tones and noise.

The oscillograph trace of white noise shows no element of regularity at all. The only variable parameter is that of the amplitude, which



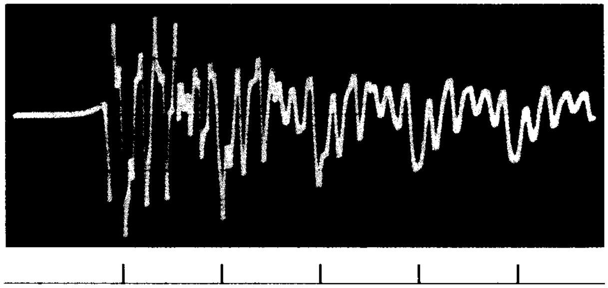
(*a*) oscillograph trace for white noise
(*b*) oscillograph trace for a pure tone

corresponds to the loudness of the sound. (It is possible, of course, to 'colour' noise by filtering out various frequency components, but then it can no longer be described as 'white'.)

The oscillograph trace for a pure tone is that of a sine wave and is completely regular. There are now two parameters that matter: the amplitude which, as before, relates to the loudness, and the frequency which relates to the pitch of the note. Many textbooks tend to keep these two parameters separate, but in fact they are linked, again because of the mechanism of perception. For example, listen to a pure tone of frequency 440 Hz (the note with which the tuning of orchestral instruments is checked) at a relatively low amplitude. Then, without changing the frequency, increase the amplitude very suddenly. The loudness will increase, and many people will also detect a change in pitch. With a fairly large audience one usually finds that about a half hear the pitch go down, rather fewer hear it go up, and a few hear no change. This is a dynamic effect that only occurs with sudden changes and only with fairly pure tones.

Sources of musical sounds

Many common objects have a natural frequency of vibration that can be excited by striking or blowing. All kinds of tubes, or vessels with a narrow opening, for example, will emit a sound if the opening is struck with the flat of the hand. In this case it is the air that is vibrating and, if



Oscillograph trace from a tube when a cork in one end is suddenly withdrawn; the vertical lines correspond to the natural resonant frequency of the tube.

the natural frequency lies within the sensitivity range of the human ear (about 30–18000 Hz in young people, although the upper sensitivity declines rapidly with age), a musical sound is heard. When a cork is suddenly withdrawn from the end of a tube a compression wave travels back and forth in the air in the tube. Although its amplitude rapidly decays, as shown above, the time taken for each transit determines a discernible musical pitch in the short-lived sound.

In order to convert this into a usable musical instrument, we must feed in energy to keep the wave travelling up and down for as long as the note is required. This can be done by blowing across the end with such a speed that the edge tone generated as the air jet strikes the edge has an oscillatory frequency that matches the natural frequency of the tube. Alternatively a reed can be used. All reeds are, in effect, taps that allow pulses of air to pass through at a well-defined frequency, which again can be made to match that of the pipe. The lips form the reeds in the brass family of instruments, single or double strips of cane form the reeds of the woodwinds.

Similar arguments can be applied to the vibration of strings. Transverse vibration of strings can be excited by striking (as in the piano or clavichord), or by plucking (as in the harp, guitar or harpsichord). But to convert such short-lived notes into those of much longer duration, energy must be fed in to maintain the vibration. In modern electric guitars various forms of electronic feedback can be used, but the traditional method, used in the orchestral string family, is by bowing. This depends on the difference between the static and dynamic frictional properties of resin. Powdered resin adheres to the microscopic scales of the horse-hair used in bows: when the bow is placed on the string the static friction is high but when moved to one side the string sticks to it and is also moved to the side. Eventually the restoring forces created in the string overcome the static friction, and the string starts to slip back to its neutral position. Dynamic friction, which is very much lower than the static friction, allows the string to move easily under the bow, overshoot the neutral position, come to rest, and then be picked up once more by the static frictional force to repeat the cycle.

Harmonics, overtones, and privileged frequencies

Although most objects have a natural vibration frequency, the real situation is much more complicated. An easy way to approach these complications is to think of a child's swing. The oscillation can be kept going by giving a slight push once in every cycle of the swing—but the timing is all important and it is just as easy to bring the swing to a standstill if the push is applied at the wrong moment. The right moment is just after the swing has started to accelerate from one of the extreme positions and the push must obviously be in the same direction as the movement of the swing. But the swing can also be kept going if a push is given every second time the swing reaches the optimum position, or every third time, and so on. Equally, if the person pushing gives a push (some of which, of course, will not connect with the swing) at twice the natural frequency of the swing, or at three times the natural frequency, the pushes that connect with the swing will still be at the right frequency to maintain the oscillation.

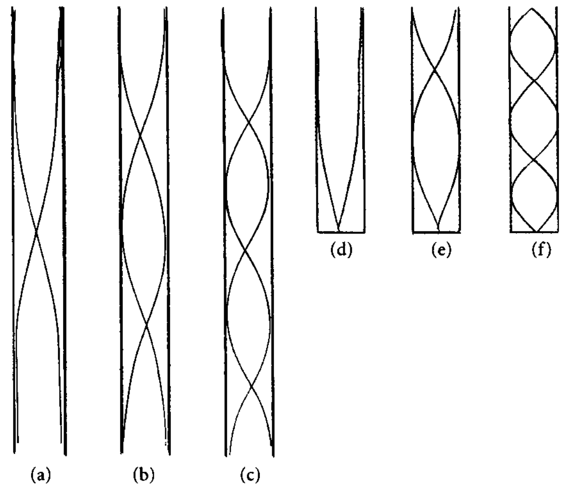
Consider the tube discussed earlier. The oscillation can be maintained if the hand is repeatedly slapped on the end of the tube at its natural frequency f. But, as with the swing, it could equally well be excited at frequencies $2f$, $3f$, $4f$, $5f$, etc. and also at frequencies $\frac{1}{2}f$, $\frac{1}{3}f$, $\frac{1}{4}f$, etc. Indeed, it can also be excited at $\frac{2}{3}f$, $\frac{3}{4}f$, $\frac{5}{3}f$, and at many other possible frequencies.

The frequencies commonly discussed in connection with musical instruments are f, $2f$, $3f$, $4f$, etc., which are usually termed *harmonics* (see Chapter 1). In practice, because of end effects, the effect of the diameter of a pipe, and many other complications, a simple tube will not resonate precisely at the harmonic frequencies—but in spite of this musicians still tend to call them harmonics. Scientists know them as *overtones*.

The remaining frequencies of the type $\frac{1}{2}f$, $\frac{1}{3}f$, or $\frac{3}{2}f$, $\frac{5}{3}f$, etc., are known as *privileged frequencies* (and strictly speaking, the harmonics are privileged frequencies as well). The following table shows a list of the harmonics and privileged frequencies for a tube open at both ends with a basic natural frequency of 240 Hz: the numbers in bold type are the true harmonics.

120	**240**	360	**480**	600	**720**	840	**960**	1080	**1200**
60	120	180	**240**	300	360	420	**480**	540	600
40	80	120	160	200	**240**	280	320	360	400
30	60	90	120	150	180	210	**240**	270	300
24	48	72	96	120	144	168	192	216	**240**

Notice that some of the privileged frequencies (such as 120 and 60) occur more than once, and if the table were still further extended others would occur. These frequencies are easier to excite than the ones that occur only once.



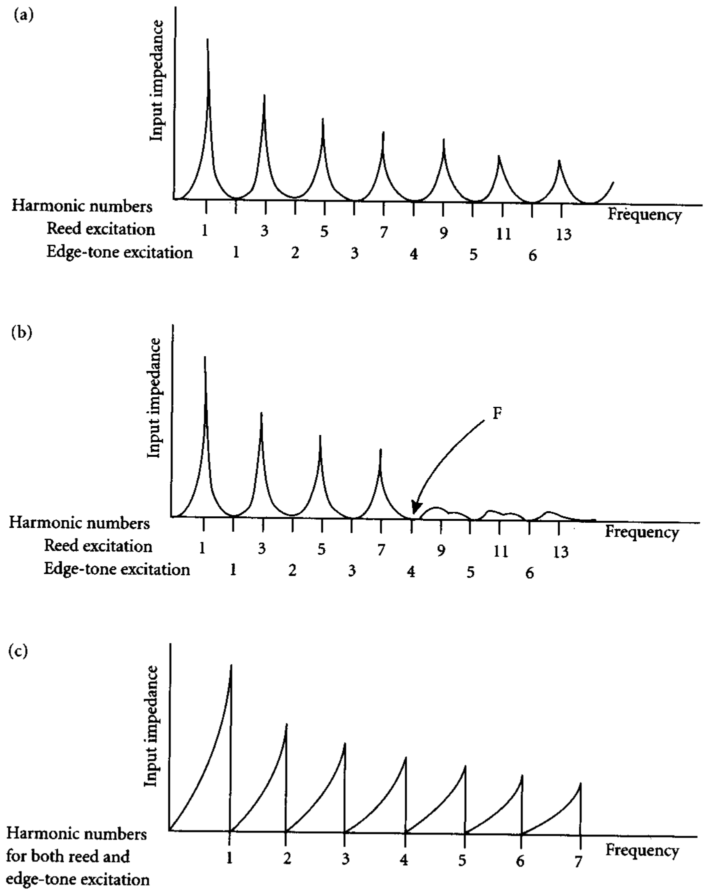
Traditional diagrams showing the graphs of the displacements in open and closed pipes:
(*a*) open, frequency *f*
(*b*) open, frequency 2*f*
(*c*) open, frequency 3*f*
(*d*) closed, frequency *f*
(*e*) closed, frequency 3*f*
(*f*) closed, frequency 5*f*.

Impedance view of the behaviour of tubes

In the elementary approach to the behaviour of vibrations in tubes, use is made of the fact that a compression becomes an expansion on reflection at an open end, but stays a compression when reflected from the end of a closed pipe (see above). This must obviously be so, as the reflection at the end of an open tube must add to the outgoing wave to produce no excess pressure, and must therefore be an expansion. For the closed pipe there is obviously maximum pressure at the end.

Problems arise if the pipe is not precisely cylindrical for the whole of its length and it is then no longer possible to draw convincing diagrams based on the simple theory. Measurement of the input impedance of a tube as a function of frequency leads to a more satisfactory argument.

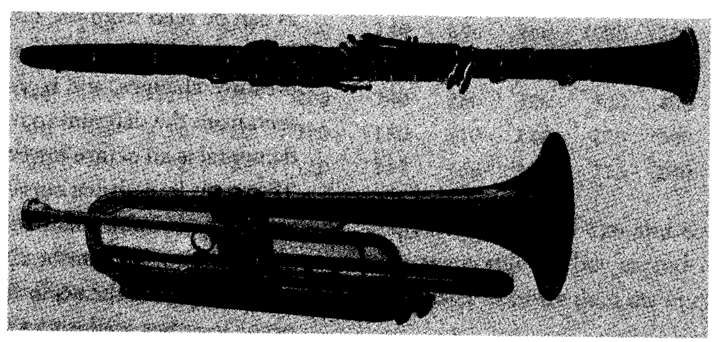
The figure (*a*) overleaf shows such a diagram based on the work of Arthur Benade. The difference in behaviour between edge-tone instruments and reeds can be explained without assumptions about open or closed ends. Edge tone excitation involves only small changes in pressure, although the displacements are high. Thus it is a low impedance device (analogous to a low-voltage high-current electrical device) and, as can be seen from the diagram, this leads to a full series of harmonics. A reed, on the other hand, involves relatively low air flow but high pressure changes, and is thus a high impedance device, which can be seen from the diagram to involve only the odd harmonics, but the fundamental is an octave lower than that for a low impedance instrument. The input impedance curve for a pipe with a series of side holes (as in most woodwind instruments) is shown in figure (*b*); the existence of a cut-off frequency can be clearly seen. The input impedance curve for a conical pipe is shown in figure (*c*). Notice that the peaks and troughs occur together at almost exactly the same frequency; thus it no longer



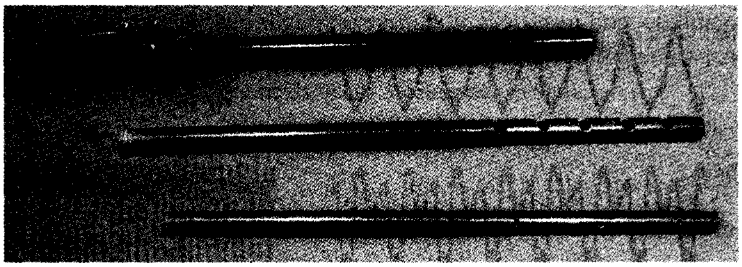
(*a*) Curve showing the relationship between input impedance and frequency for a plain cylindrical pipe.
(*b*) As (*a*), but with a regular series of side holes.
(*c*) Input impedance plotted against frequency for a conical pipe.

matters whether the pipe is excited by edge tones or reeds, and the full series of harmonics is always produced.

Such diagrams have also been used by Benade to explain the behaviour of trumpets, which seem to produce a full series of harmonics in spite of being largely cylindrical and closed by the player's mouth. Changes are produced in the input impedance curve, first by the addition of the bell and then by the addition of the mouthpiece, and the result is a full sequence of harmonics.



Clarinet and Trumpet.



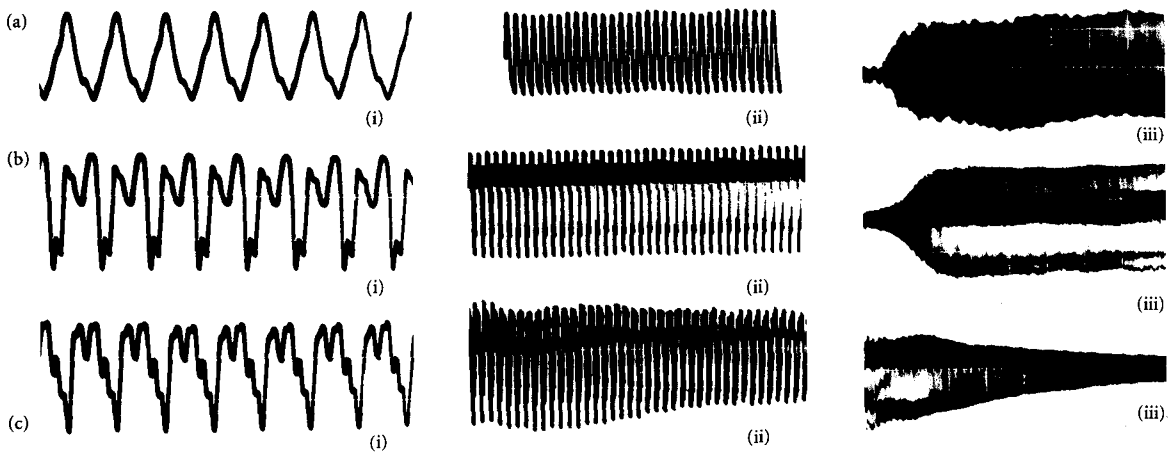
Three plastic tubes used to illustrate the influence of the finger holes on the tone quality of a clarinet: in (*a*), the tube has no side holes and the tone is muffled; in (*b*), there are five large holes and the tone is much more clarinet-like; in (*c*), there are three small holes and the tube is unplayable; however, if one or more of the holes are covered, a note can be sounded.

Turning a cylindrical tube into a clarinet

Just as the trumpet involves departures from a plain cylindrical tube, so does a clarinet. The important departures are the side holes which, even when closed, produce regular 'bumps' in the bore, and these have a profound effect. There are at least three functions that have to be performed by the side holes if a clarinet is to behave like a real musical instrument. The first, and most obvious, is that they determine the vibrating length, and hence the pitch of a given note. Secondly, they radiate the sound (very little of which emerges from the bell, as can be demonstrated easily with a microphone and oscilloscope) and, being arranged in a regular sequence, are frequency-sensitive. Thirdly, there must be a balance between the energy reflected back towards the reed to keep the oscillation going, and the energy radiated away. The position and spacing of the holes has a considerable influence on this. A clarinet maker must be able to adjust at least these three functions independently and, in order to do this, makes use of the positions of the holes, their diameter, the wall thickness at the hole, and the bore diameter throughout the length. The diameter varies along the whole length and is adjusted either by using a reamer to enlarge it slightly at a particular place, or by using a special brush to paint lacquer on the wall to reduce the bore.

The quality of a musical sound

At quite an early stage in the study of musical physics, it was thought that a vibrating body that could vibrate at a number of discrete harmonic frequencies could probably vibrate in several at once, and that the resulting combination of a number of harmonics could be the source of variations in quality. The wave-forms of various instruments were studied, and it seemed clear that their regular waveforms could be subject to Fourier analysis; thus, if the harmonics could be generated in the right proportions, the sound of any instrument could be imitated. The Hammond and Compton electronic organs, both developed in 1932, used the principle of harmonic mixture to determine tone quality. But, as is now well known, the sound of these organs was noticeably 'electronic', and we need to ask why.



Oscillograph traces for notes on (*a*) a flute; (*b*) a clarinet; (*c*) a guitar: in (i), the trace lasts for one hundredth of a second; in (ii), the trace lasts for one tenth of a second; in (iii), the trace lasts for one second.

The oscillograph traces for three instruments (flute, clarinet and guitar) are obviously different, and over a period of $\frac{1}{100}$ second all three appear to be fairly regular. However if we look at traces lasting $\frac{1}{10}$ second, or 1 second, it immediately becomes obvious that they are far from regular. It turns out that it is these departures from regularity that tell the brain that a 'real' instrument is involved, rather than an electronically synthesized sound. Nowadays, of course, synthesizers have become so sophisticated that departures from regularity can be imitated.

There are many causes of these variations in real instruments, but probably the most significant from the point of view of recognition by the brain is the way in which the note is initiated. Most instruments involve at least two coupled systems: the strings of a violin and the body, the reed of a clarinet and the pipe, the lips of a player and the trumpet itself, and so on. When any coupled system begins to oscillate, one of the systems begins to drive the other in forced vibration. Because of the inertia of the second system, there is a time delay in the commencement of the forced vibration and it may take as much as $\frac{1}{10}$ second before the whole settles down. But this first tenth of a second is crucial: it is called the *starting transient* and every instrument has its own characteristic transient. It is the transient that the brain recognizes, and so permits a listener to identify the various instruments in a combination. Mathematically, the solution of the differential equation for the coupled system is the sum of two parts, the steady state part and the transient part.

Combinations of notes

The phenomenon of beats (rises and falls in amplitude) is well known, and can be demonstrated most easily by sounding the same note on two recorders and then slightly covering the first open hole on one of them to flatten its note slightly. If the two notes have frequencies of 480 Hz and 477 Hz, the beats occur 3 times per second.

Trace for notes of frequencies 480 and 477 Hz, sounded simultaneously.

In fact, because of non-linearities in the ear–brain system, a note corresponding in frequency to the difference of the two sounding notes can be heard; this is known as the *difference tone*. The result can be very complicated as there are also sum tones, and there are secondary sum and difference tones between the primary sum and difference tones.

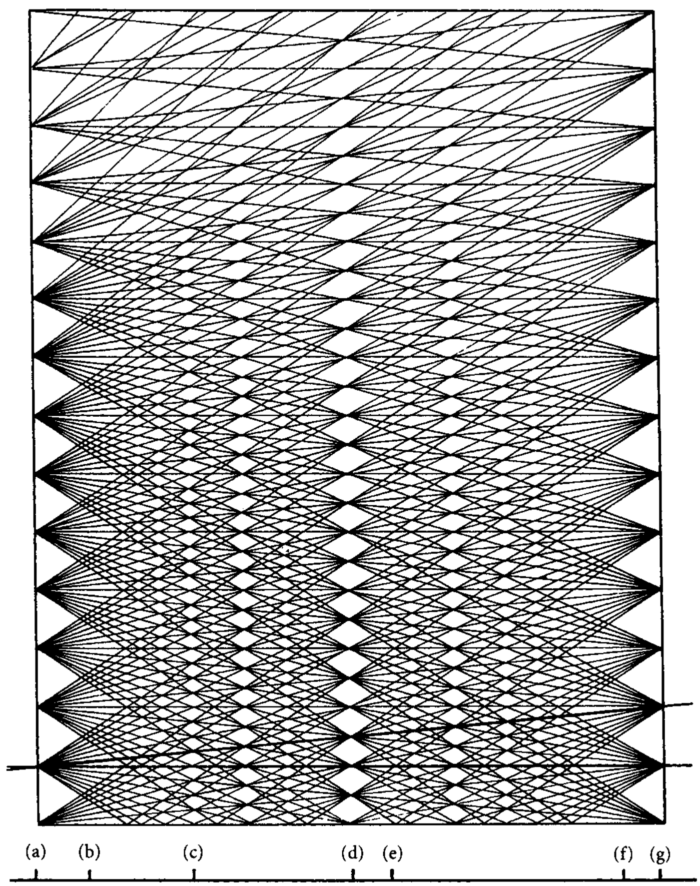
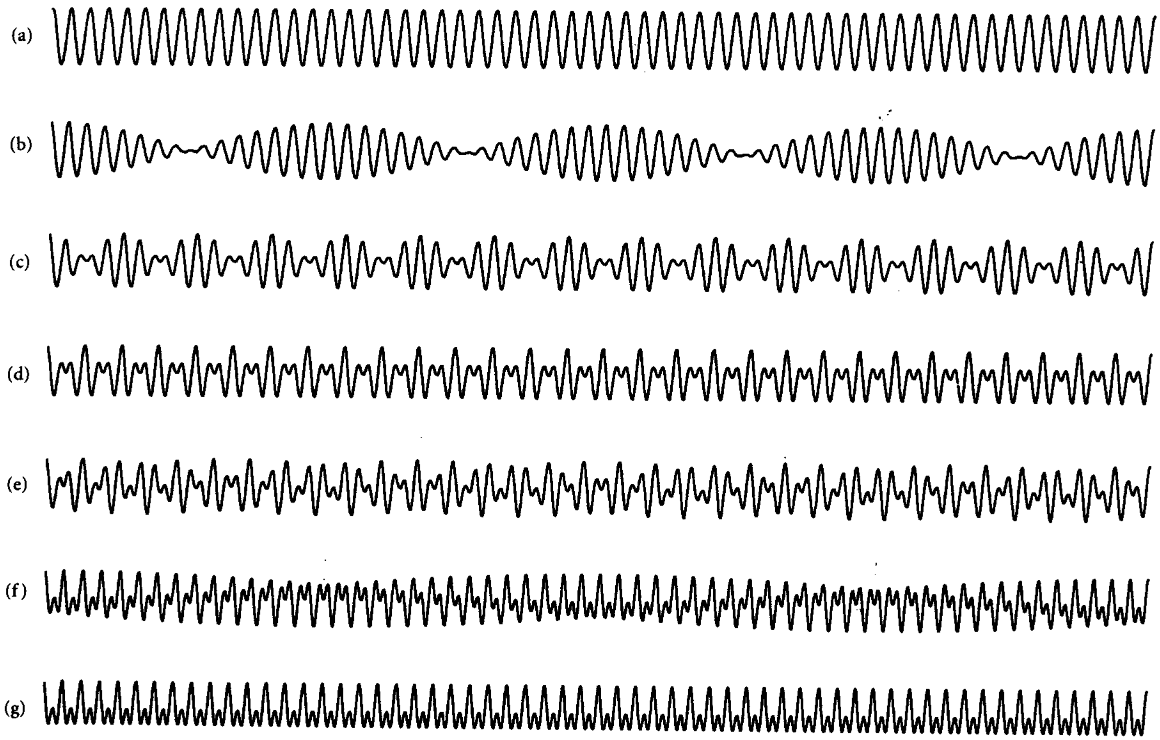


Diagram representing some of the results of adding two pure tones. The horizontal thick line represents a note of fixed frequency; the sloping thick line represents a note whose frequency commences from that of the fixed note and then glides smoothly upwards through one octave. The frequency ratios represented by the lower case letters are: (*a*) 1 : 1; (*b*) 15 : 16; (*c*) 4 : 5; (*d*) 2 : 3; (*e*) 20 : 31; (*f*) 30 : 59; (*g*) 1 : 2.

The above diagram shows the result of performing Helmholtz's hypothetical experiment of sounding one note continuously and bringing a second note from being in tune with the steady note to a pitch an octave higher. The thick lines represent the frequencies of the two



Wave traces of two simultaneously sounded pure tones; the frequency ratios correspond to those listed in the previous caption.

notes actually being sounded, and the thin lines represent all the various possible sum and difference tones. It can be seen that when the ratio of their frequencies is relatively simple, the number of tones present becomes less. It has been suggested that these tones sound pleasant because fewer notes are involved; also if the wave traces are studied, the simpler ratios give less complicated wave forms. The above diagram shows wave traces for pairs of notes with various frequency ratios.

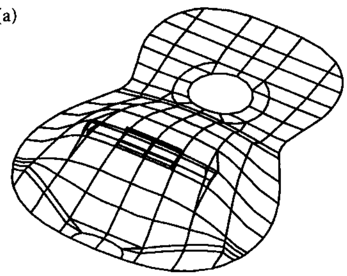
This looks as though it might begin to account for the phenomena of consonance and dissonance. But there are further complications. The ear–brain system is non-linear only for rather loud sounds, but the sum and difference, and dissonance phenomena, occur even for very low amplitudes.

Also, if three tones are sounded together—say, 400, 480 and 560 Hz—a difference tone at 80 Hz can be heard quite clearly, even at low amplitudes: 80 Hz is the fundamental of the series of which the sounding notes are the 5th, 6th and 7th harmonics. Now, if the frequencies are all raised by the same amount to 420, 500, 580 Hz, although the difference is still 80 Hz, the perceived tone is found to go up by about 10 Hz. The three notes are now the 21st, 25th and 29th harmonics of a fundamental of 20 Hz, although this note cannot be heard. This odd phenomenon, sometimes called the *residue effect*, provides yet one more example of the inadequacy of simple theories to explain musical phenomena. Nor is it an inconsequential complication: the tone of the bassoon, for example, can only be explained using these ideas. If a frequency analysis of bassoon tone is made, there is found to be relatively little energy at the fundamental of any given note. Most of the energy lies in the 5th, 6th and 7th harmonics and the ear–brain system 'manufactures' the fundamental using this residue phenomenon.

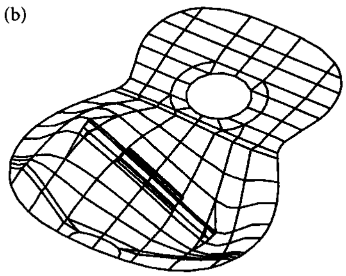
The bodies of stringed instruments

Earlier, we mentioned the coupled system incorporating the strings and bodies in the string family. It turns out that the body of an instrument like a violin or a guitar performs an extraordinarily complicated function in transforming the vibrations of the strings into radiated sound. Stradivari, Guarneri, Amati and others obviously solved the problem of making the right kinds of bodies in a purely empirical way and, although physicists can lend assistance to instrument makers in arriving more rapidly at an acceptable solution, the secret of the success of the Cremona school and others is by no means understood. It is clear that much work remains to be done on the science of musical sound.

(a)



(b)



Computer simulations for two different low-frequency modes of the front plate of a guitar.