2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

The xy terms represent encounters between the two species x and y. An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

4. Let $P(t)$, $Q(t)$, and $R(t)$ represent the populations of flies, frogs, and crocodiles, respectively. All the constants used are positive so that a plus sign means an increase and a minus sign means a decrease in the corresponding growth rate.

"In the absence of frogs, the fly population will grow exponentially and the crocodile population will decay exponentially" gives us $dP/dt = +k_1P$ and $dR/dt = -k_2R$.

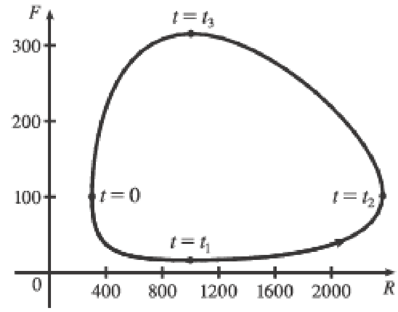
"In the absence of crocodiles and flies, the frog population will decay exponentially" gives us $dQ/dt = -k_3Q$.

"To survive, frogs need to eat flies and crocodiles need to eat frogs" gives us encounters that flies lose, frogs win and lose, and crocodiles win. In terms of the growth rates, this means $dP/dt = -c_1PQ$, $dQ/dt = +c_2PQ - c_3QR$, and $dR/dt = +c_4QR$.

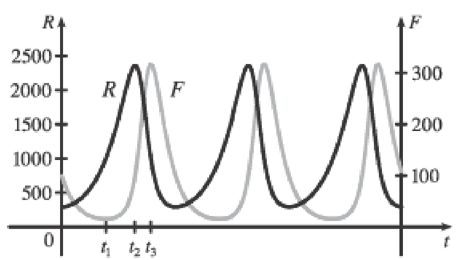
Putting this information together gives us the following system of differential equations.

$$dP/dt = +k_1P - c_1PQ$$
$$dQ/dt = -k_3Q + c_2PQ - c_3QR$$
$$dR/dt = -k_2R + c_4QR$$

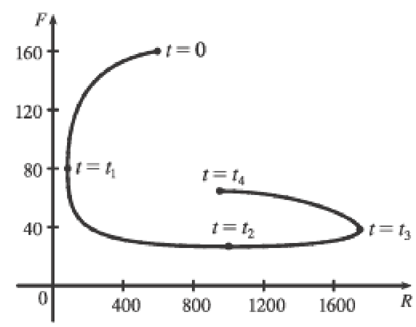
5. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



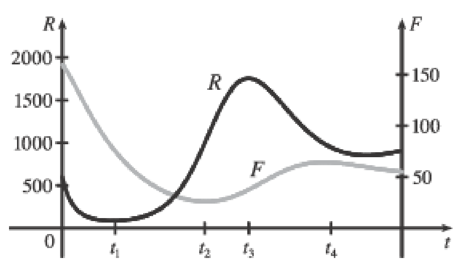
(b)



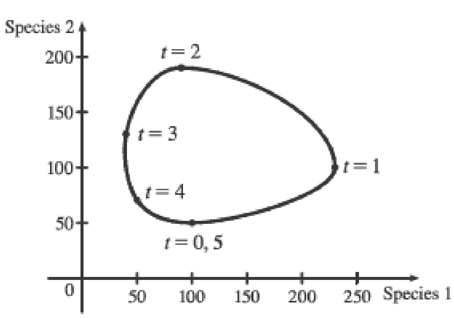
6. (a) At $t = 0$, there are about 600 rabbits and 160 foxes. At $t = t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80. At $t = t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000. At $t = t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750. The curve ends at $t = t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950.



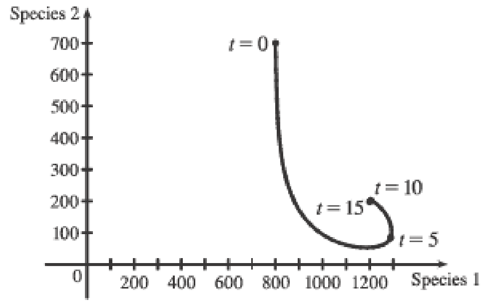
(b)



7.



8.

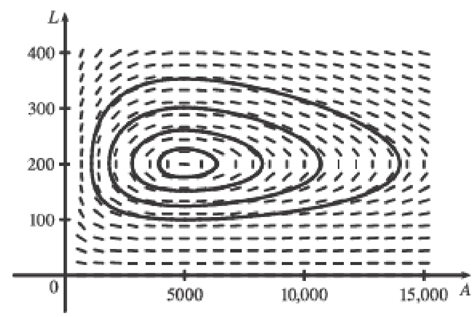

Species 2
700
600
500
400
300
200
100
0
200
400
600
800
1000
1200
Species 1
$t = 0$
$t = 10$
$t = 15$
$t = 5$

10. (a) A and L are constant $\Rightarrow$ $A' = 0$ and $L' = 0$ $\Rightarrow$ $\left.\begin{cases} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{cases}\right\}$ $\Rightarrow$ $\begin{cases} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{cases}$

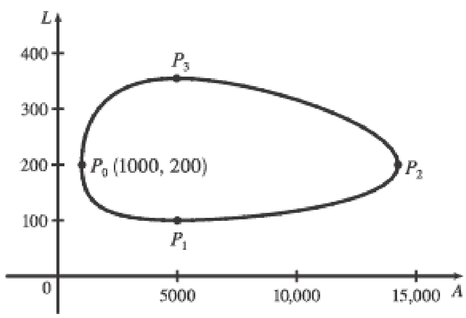
So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point $(5000, 200)$ inside them.



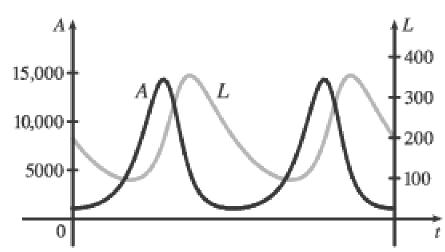
(d)



At $P_0(1000, 200)$, $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0, there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000, 100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14,250, 200)$.

Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000, 355)$, and as we pass through P_2, the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0, where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.



Hughes-Hallett 11.11

1. The characteristic equation is $r^2 + 4r + 3 = 0$, so $r = -1$ or -3.
Therefore $y(t) = C_1e^{-t} + C_2e^{-3t}$.

2. The characteristic equation is $r^2 + 4r + 4 = 0$, so $r = -2$.
Therefore $y(t) = (C_1t + C_2)e^{-2t}$.

11. If we try a solution $z(t) = Ae^{rt}$ then

$$r^2 + 2r = 0$$

which has solutions $r = 0$ and $r = -2$ so that the general solution is of the form

$$y(t) = A + Be^{-2t}$$

14. The characteristic equation is

$$r^2 + 5r + 6 = 0$$

which has the solutions $r = -2$ and $r = -3$ so that

$$y(t) = Ae^{-3t} + Be^{-2t}$$

The initial condition $y(0) = 5$ gives

$$A + B = 5$$

and $y'(0) = 1$ gives

$$-3A - 2B = 1$$

so that $A = -11$ and $B = 16$ and

$$y(t) = -11e^{-3t} + 16e^{-2t}$$

23. The characteristic equation is $r^2 + 2r + 2 = 0$, so $r = -1 \pm i$.
Therefore $p(t) = C_1e^{-t}\cos t + C_2e^{-t}\sin t$.
$p(0) = 0 = C_1$ so $p(t) = C_2e^{-t}\sin t$
$p(\pi/2) = 20 = C_2e^{-\pi/2}\sin\frac{\pi}{2}$ so $C_2 = 20e^{\pi/2}$
Therefore $p(t) = 20e^{\frac{\pi}{2}}e^{-t}\sin t = 20e^{\frac{\pi}{2}-t}\sin t$.

Problems

25. (a) $x'' + 4x = 0$ represents an undamped oscillator, and so goes with (IV).
(b) $x'' - 4x = 0$ has characteristic equation $r^2 - 4 = 0$ and so $r = \pm 2$. The solution is $C_1e^{-2t} + C_2e^{2t}$. This represents non-oscillating motion, so it goes with (II).
(c) $x'' - 0.2x' + 1.01x = 0$ has characteristic equation $r^2 - 0.2 + 1.01 = 0$ so $b^2 - 4ac = 0.04 - 4.04 = -4$, and $r = 0.1 \pm i$. So the solution is

$$C_1e^{(0.1+i)t} + C_2e^{(0.1-i)t} = e^{0.1t}(A\sin t + B\cos t).$$

The negative coefficient in the x' term represents an amplifying force. This is reflected in the solution by $e^{0.1t}$, which increases as t increases, so this goes with (I).
(d) $x'' + 0.2x' + 1.01x$ has characteristic equation $r^2 + 0.2r + 1.01 = 0$ so $b^2 - 4ac = -4$. This represents a damped oscillator. We have $r = -0.1 \pm i$ and so the solution is $x = e^{-0.1t}(A\sin t + B\cos t)$, which goes with (III).

44 $\frac{d^2y}{dt^2} = -\frac{dx}{dt} = y$ so $\frac{d^2y}{dt^2} - y = 0$.
Characteristic equation $r^2 - 1 = 0$, so $r = \pm 1$.
The general solution for y is $y = C_1e^t + C_2e^{-t}$, so $x = C_2e^{-t} - C_1e^t$.

Hughes-Hallett 11.10

2. If $y(t) = 3\sin(2t) + 2\cos(2t)$ then

$$y' = 6\cos(2t) - 4\sin(2t)$$
$$y'' = -12\sin(2t) - 8\cos(2t) = -4(3\sin(2t) + 2\cos(2t)) = -4y$$

as required.

6. $y = A\cos\alpha t$
$y' = -\alpha A\sin\alpha t$
$y'' = -\alpha^2 A\cos\alpha t$
If $y'' + 5y = 0$, then $-\alpha^2 A\cos\alpha t + 5A\cos\alpha t = 0$, so $A(5-\alpha^2)\cos\alpha t = 0$. This is true for all t if $A = 0$, or if $\alpha = \pm\sqrt{5}$.
We also have the initial condition: $y'(1) = -\alpha A\sin\alpha = 3$. Notice that this equation will not work if $A = 0$. If $\alpha = \sqrt{5}$, then

$A = -\frac{3}{\sqrt{5}\sin\sqrt{5}} \approx -1.705$.
Similarly, if $\alpha = -\sqrt{5}$, we find that $A \approx -1.705$. Thus, the possible values are $A = -\frac{3}{\sqrt{5}\sin\sqrt{5}} \approx -1.705$ and $\alpha = \pm\sqrt{5}$.

Problems

17. ~~13~~. First, we note that the solutions of:
(a) $x'' + x = 0$ are $x = A\cos t + B\sin t$;
(b) $x'' + 4x = 0$ are $x = A\cos 2t + B\sin 2t$;
(c) $x'' + 16x = 0$ are $x = A\cos 4t + B\sin 4t$.
This follows from what we know about the general solution to $x'' + \omega^2 x = 0$.
The period of the solutions to (a) is 2π, the period of the solutions to (b) is π, and the period of the solutions of (c) is $\frac{\pi}{2}$.
Since the t-scales are the same on all of the graphs, we see that graphs (I) and (IV) have the same period, which is twice the period of graph (III). Graph (II) has twice the period of graphs (I) and (IV). Since each graph represents a solution, we have the following:

- equation (a) goes with graph (II)
  equation (b) goes with graphs (I) and (IV)
  equation (c) goes with graph (III)
- The graph of (I) passes through $(0,0)$, so $0 = A\cos 0 + B\sin 0 = A$. Thus, the equation is $x = B\sin 2t$. Since the amplitude is 2, we see that $x = 2\sin 2t$ is the equation of the graph. Similarly, the equation for (IV) is $x = -3\sin 2t$. The graph of (II) also passes through $(0,0)$, so, similarly, the equation must be $x = B\sin t$. In this case, we see that $B = -1$, so $x = -\sin t$.
  Finally, the graph of (III) passes through $(0,1)$, and 1 is the maximum value. Thus, $1 = A\cos 0 + B\sin 0$, so $A = 1$. Since it reaches a local maximum at $(0,1)$, $x'(0) = 0 = -4A\sin 0 + 4B\cos 0$, so $B = 0$. Thus, the solution is $x = \cos 4t$.

18. All the differential equations have solutions of the form $s(t) = C_1\sin\omega t + C_2\cos\omega t$. Since for all of them, $s'(0) = 0$, we have $s'(0) = 0 = C_1\omega\cos 0 - C_2\omega\sin 0 = 0$, giving $C_1\omega = 0$. Thus, either $C_1 = 0$ or $\omega = 0$. If $\omega = 0$, then $s(t)$ is a constant function, and since the equations represent oscillating springs, we don't want $s(t)$ to be a constant function.
Thus, $C_1 = 0$, so all four equations have solutions of the form $s(t) = C\cos\omega t$.
i) $s'' + 4s = 0$, so $\omega = \sqrt{4} = 2$. $s(0) = C\cos 0 = C = 5$. Thus, $s(t) = 5\cos 2t$.
ii) $s'' + \frac{1}{4}s = 0$, so $\omega = \sqrt{\frac{1}{4}} = \frac{1}{2}$. $s(0) = C\cos 0 = C = 10$. Thus, $s(t) = 10\cos\frac{1}{2}t$.
iii) $s'' + 6s = 0$, so $\omega = \sqrt{6}$. $s(0) = C = 4$, Thus, $s(t) = 4\cos\sqrt{6}t$.
iv) $s'' + \frac{1}{6}s = 0$, so $\omega = \sqrt{\frac{1}{6}}$. $s(0) = C = 20$. Thus, $s(t) = 20\cos\sqrt{\frac{1}{6}}t$.

(a) Spring (iii) has the shortest period, $\frac{2\pi}{\sqrt{6}}$. (Other periods are π, 4π, $2\pi\sqrt{6}$)

(b) Spring (iv) has the largest amplitude, 20.

(c) Spring (iv) has the longest period, $2\pi\sqrt{6}$.

(d) Spring (i) has the largest maximum velocity. We can see this by looking at $v(t) = s'(t) = -C\omega\sin\omega t$. The velocity is just a sine function, so we look for the derivative with the biggest amplitude, which will have the greatest value. The velocity function for Spring i) has amplitude 10, the largest of the four springs. (The other velocity amplitudes are $10\cdot\frac{1}{2} = 5$, $4\sqrt{6} \approx 9.8$, $\frac{20}{\sqrt{6}} \approx 8.2$)