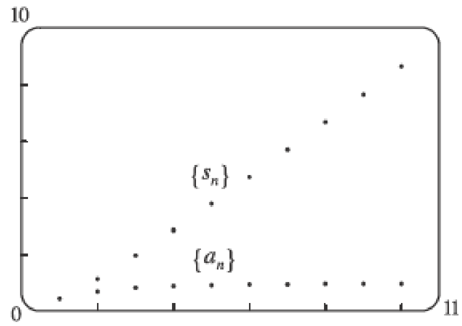
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

5.

n	s_n
1	0.44721
2	1.15432
3	1.98637
4	2.88080
5	3.80927
6	4.75796
7	5.71948
8	6.68962
9	7.66581
10	8.64639



The series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$ diverges, since its terms do not approach 0.

9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (8.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence.

11. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$ is a geometric series with ratio $r = -\frac{4}{3}$. Since $|r| = \frac{4}{3} > 1$, the series diverges.

16. $\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}} = \sum_{n=1}^{\infty} \frac{10(10)^{n-1}}{(-9)^{n-1}} = 10 \sum_{n=1}^{\infty} \left(-\frac{10}{9}\right)^{n-1}$. The latter series is geometric with $a = 10$ and ratio $r = -\frac{10}{9}$.

Since $|r| = \frac{10}{9} > 1$, the series diverges.

21. $\sum_{k=2}^{\infty} \frac{k^2}{k^2-1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2-1} = 1 \neq 0$.

25. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0.$

30. $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$ diverges because $\sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. (If it converged, then $\frac{1}{2} \cdot 2 \sum_{n=1}^{\infty} \frac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.) If the given series converges, then the difference $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right) - \sum_{n=1}^{\infty} \frac{3}{5^n}$ must converge (since $\sum_{n=1}^{\infty} \frac{3}{5^n}$ is a convergent geometric series) and equal $\sum_{n=1}^{\infty} \frac{2}{n}$, but we have just seen that $\sum_{n=1}^{\infty} \frac{2}{n}$ diverges, so the given series must also diverge.

31. Using partial fractions, the partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1} \right)$$
$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

47. For $n = 1$, $a_1 = 0$ since $s_1 = 0$. For $n > 1$,

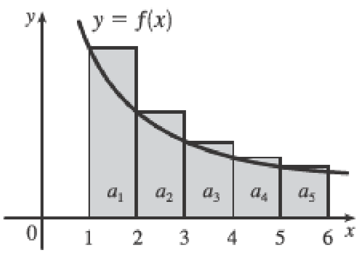
$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n - (n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

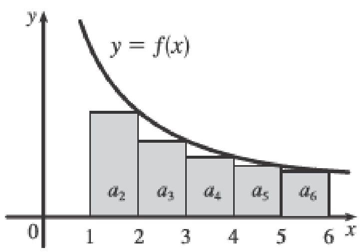
Also, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - 1/n}{1 + 1/n} = 1$.

59. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

61. Suppose on the contrary that $\sum (a_n + b_n)$ converges. Then $\sum (a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8(iii), $\sum [(a_n + b_n) - a_n]$ would also be convergent. But $\sum [(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

2. From the first figure, we see that $\int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





3. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

5. $\sum_{n=1}^{\infty} n^b$ is a p-series with $p=-b$. $\sum_{n=1}^{\infty} b^n$ is a geometric series. By (1), the p-series is convergent if $p>1$. In this case, $\sum_{n=1}^{\infty} n^b = \sum_{n=1}^{\infty} (1/n^{-b})$, so $-b>1 \quad\Leftrightarrow\quad b<-1$ are the values for which the series converge. A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r|<1$, so $\sum_{n=1}^{\infty} b^n$ converges if $|b|<1 \quad\Leftrightarrow\quad -1<b<1$.

7. The function $f(x)=1/\sqrt[5]{x}=x^{-1/5}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\int_1^{\infty} x^{-1/5}\,dx = \lim_{t\to\infty}\int_1^t x^{-1/5}\,dx = \lim_{t\to\infty}\left[\frac{5}{4}x^{4/5}\right]_1^t = \lim_{t\to\infty}\left(\frac{5}{4}t^{4/5}-\frac{5}{4}\right)=\infty$, so $\sum_{n=1}^{\infty} 1/\sqrt[5]{n}$ diverges.

10. $\dfrac{n^3}{n^4-1} > \dfrac{n^3}{n^4} = \dfrac{1}{n}$ for all $n\geq 2$, so $\sum_{n=2}^{\infty}\dfrac{n^3}{n^4-1}$ diverges by comparison with $\sum_{n=2}^{\infty}\dfrac{1}{n}$, which diverges because it is a p-series with $p=1\leq 1$ (the harmonic series).

13. $1+\dfrac{1}{8}+\dfrac{1}{27}+\dfrac{1}{64}+\dfrac{1}{125}+\cdots=\sum_{n=1}^{\infty}\dfrac{1}{n^3}$. This is a p-series with $p=3>1$, so it converges by (1).

16. $f(x)=\dfrac{x^2}{x^3+1}$ is continuous and positive on $[2,\infty)$, and also decreasing since $f'(x)=\dfrac{x(2-x^3)}{(x^3+1)^2}<0$ for $x\geq 2$, so we can use the Integral Test [note that f is *not* decreasing on $[1,\infty)$].

$\displaystyle\int_2^{\infty}\frac{x^2}{x^3+1}\,dx=\lim_{t\to\infty}\left[\tfrac{1}{3}\ln(x^3+1)\right]_2^t=\tfrac{1}{3}\lim_{t\to\infty}\left[\ln(t^3+1)-\ln 9\right]=\infty$, so the series $\sum_{n=2}^{\infty}\dfrac{n^2}{n^3+1}$ diverges, and so does the given series, $\sum_{n=1}^{\infty}\dfrac{n^2}{n^3+1}$.

20. $\dfrac{n^2-1}{3n^4+1}<\dfrac{n^2}{3n^4+1}<\dfrac{n^2}{3n^4}=\dfrac{1}{3}\dfrac{1}{n^2}$. $\sum_{n=1}^{\infty}\dfrac{n^2-1}{3n^4+1}$ converges by comparison with $\sum_{n=1}^{\infty}\dfrac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series [$p=2>1$]. The terms of the given series are positive for $n>1$, which is good enough.

29. Use the Limit Comparison Test with $a_n=\sin\left(\dfrac{1}{n}\right)$ and $b_n=\dfrac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and

$\lim_{n\to\infty}\dfrac{a_n}{b_n}=\lim_{n\to\infty}\dfrac{\sin(1/n)}{1/n}=\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1>0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty}\sin(1/n)$ also diverges. [Note that we could also use l'Hospital's Rule to evaluate the limit:

$\lim_{x\to\infty}\dfrac{\sin(1/x)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\dfrac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2}=\lim_{x\to\infty}\cos\dfrac{1}{x}=\cos 0=1.$]

42. First we observe that, by l'Hospital's Rule, $\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{1+x} = 1$. Also, if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ by Theorem 8.2.6. Therefore, $\lim_{n\to\infty} \frac{\ln(1+a_n)}{a_n} = \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 > 0$. We are given that $\sum a_n$ is convergent and $a_n > 0$. Thus, $\sum \ln(1+a_n)$ is convergent by the Limit Comparison Test.