1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, where $b_n = |a_n|$, converges if $0 < b_{n+1} \le b_n$ for all n and $\lim_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. (a) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

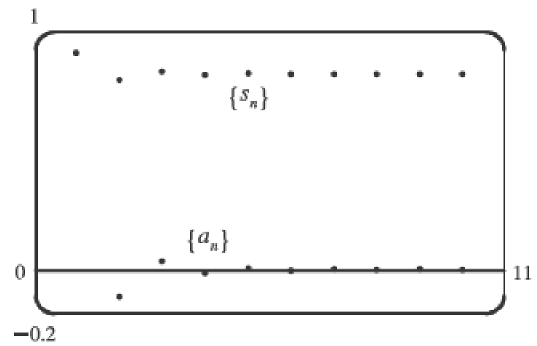
(c) Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

7. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3 - 1/n}{2 + 1/n} = \frac{3}{2} \ne 0$. Since $\lim_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

10. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

12.

n	a_n	s_n
1	1	1
2	−0.125	0.875
3	0.03704	0.91204
4	−0.01563	0.89641
5	0.008	0.90441
6	−0.00463	0.89978
7	0.00292	0.90270
8	−0.00195	0.90074
9	0.00137	0.90212
10	−0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \approx 0.90112$ is $|s - s_{10}| \le b_{11} = 1/11^3 \approx 0.0007513$.

13. If $p > 0$, $\dfrac{1}{(n+1)^p} \le \dfrac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$, so the series converges by the Alternating Series Test.

If $p \le 0$, $\lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{n^p}$

converges $\Leftrightarrow$ $p > 0$.

15. The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^6}$ satisfies (i) of the Alternating Series Test because $\dfrac{1}{(n+1)^6} < \dfrac{1}{n^6}$ and (ii) $\lim\limits_{n\to\infty} \dfrac{1}{n^6} = 0$, so the series is convergent. Now $b_5 = \dfrac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \dfrac{1}{6^6} \approx 0.00002 < 0.00005$, so by the Alternating Series Estimation Theorem, $n = 5$. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

21. Using the Ratio Test, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-3)^{n+1}/(n+1)^3}{(-3)^n/n^3}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(-3)n^3}{(n+1)^3}\right| = 3 \lim\limits_{n\to\infty} \left(\dfrac{n}{n+1}\right)^3 = 3 > 1$,

so the series $\sum\limits_{n=1}^{\infty} \dfrac{(-3)^n}{n^3}$ diverges.

24. $\sum\limits_{n=1}^{\infty} \left|(-1)^{n-1} \dfrac{\sqrt{n}}{n+1}\right| = \sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n}}{n+1} = \sum\limits_{n=1}^{\infty} a_n$. If $b_n = \dfrac{1}{\sqrt{n}}$, then $\sum\limits_{n=1}^{\infty} b_n$ is a divergent p-series $\left(p = \frac{1}{2} \le 1\right)$. Applying the Limit Comparison Test, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\sqrt{n}/(n+1)}{1/\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1 > 0$, so both series diverge and the given series is *not* absolutely convergent.

25. $\lim\limits_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim\limits_{k\to\infty} \left[\dfrac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right] = \lim\limits_{k\to\infty} \dfrac{k+1}{k}\left(\dfrac{2}{3}\right)^1 = \dfrac{2}{3} \lim\limits_{k\to\infty} \left(1 + \dfrac{1}{k}\right) = \frac{2}{3}(1) = \frac{2}{3} < 1$, so the series

$\sum\limits_{n=1}^{\infty} k\left(\frac{2}{3}\right)^k$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

29.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)4^{2(n+1)+1}} \cdot \frac{(n+1)4^{2n+1}}{10^n}\right] = \lim_{n\to\infty} \left[\frac{10^{n+1}}{(n+2)4^{2n+3}} \cdot \frac{(n+1)4^{2n+1}}{10^n}\right]$$

$$= \lim_{n\to\infty} \left(\frac{10}{4^2} \cdot \frac{n+1}{n+2}\right) = \frac{5}{8} < 1,$$

so the series $\sum\limits_{n=1}^{\infty} \dfrac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

38. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty} \frac{2^5 n^5}{(n+1)^5} = 32 \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^5} = 32 \lim_{n\to\infty} \frac{1}{(1+1/n)^5}$

$= 32(1) = 32 > 1,$

so the series $\sum_{n=2}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$ diverges by the Root Test.

4. If $a_n = \frac{(-1)^n x^n}{n+1}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{1+1/(n+1)} = |x|$.

By the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$ converges when $|x| < 1$, so $R = 1$. When $x = -1$, the series diverges because it is the harmonic series; when $x = 1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-1, 1]$.

7. If $a_n = \frac{x^n}{n!}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x| \cdot 0 = 0 < 1$ for *all* real x.

So, by the Ratio Test, $R = \infty$ and $I = (-\infty, \infty)$.

10. $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0 < 1$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

13. If $a_n = \frac{(x-2)^n}{n^2+1}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n}\right| = |x-2| \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = |x-2|$. By the Ratio Test, the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$ converges when $|x-2| < 1$ $[R = 1]$ $\Leftrightarrow$ $-1 < x-2 < 1$ $\Leftrightarrow$ $1 < x < 3$. When $x = 1$, the series $\sum_{n=0}^{\infty} (-1)^n \frac{1}{n^2+1}$ converges by the Alternating Series Test; when $x = 3$, the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by comparison with the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $[p = 2 > 1]$. Thus, the interval of convergence is $I = [1, 3]$.

15. If $a_n = \frac{3^n (x+4)^n}{\sqrt{n}}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{3^n (x+4)^n}\right| = 3\,|x+4| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = 3\,|x+4|$.

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{3^n (x+4)^n}{\sqrt{n}}$ converges when $3\,|x+4| < 1$ $\Leftrightarrow$ $|x+4| < \frac{1}{3}$ $[R = \frac{1}{3}]$ $\Leftrightarrow$ $-\frac{1}{3} < x+4 < \frac{1}{3}$ $\Leftrightarrow$ $-\frac{13}{3} < x < -\frac{11}{3}$. When $x = -\frac{13}{3}$, the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ converges by the Alternating Series Test; when $x = -\frac{11}{3}$, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges $\left[p = \frac{1}{2} \le 1\right]$. Thus, the interval of convergence is $I = \left[-\frac{13}{3}, -\frac{11}{3}\right)$.

19. If $a_n = n!\,(2x-1)^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)\,|2x-1| \to \infty$ as $n\to\infty$

for all $x \neq \frac{1}{2}$. Since the series diverges for all $x \neq \frac{1}{2}$, $R = 0$ and $I = \left\{\frac{1}{2}\right\}$.

22. If $a_n = \frac{x^{2n}}{n\,(\ln n)^2}$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(n+1)[\ln(n+1)]^2}\cdot\frac{n\,(\ln n)^2}{x^{2n}}\right| = |x^2|\lim_{n\to\infty}\frac{n\,(\ln n)^2}{(n+1)[\ln(n+1)]^2} = x^2.$

By the Ratio Test, the series $\sum_{n=2}^{\infty}\frac{x^{2n}}{n\,(\ln n)^2}$ converges when $x^2 < 1 \iff |x| < 1$, so $R = 1$. When $x = \pm 1$, $x^{2n} = 1$,

and we get the series $\sum_{n=2}^{\infty}\frac{1}{n\,(\ln n)^2}$. Since the function $f(x) = \frac{1}{x(\ln x)^2}$ is continuous, positive, and decreasing on $[2,\infty)$,

the Integral Test applies.

$$\int_2^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\left[\frac{-1}{\ln x}\right]_2^t \quad \text{[by substitution with } u = \ln x\text{]}$$

$$= -\lim_{t\to\infty}\left(\frac{1}{\ln t} - \frac{1}{\ln 2}\right) = \frac{1}{\ln 2},$$

so the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges. Thus, the interval of convergence is $I = [-1, 1]$.

25. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = 4$. So by Theorem 3, it must converge for at least $-4 < x \leq 4$. In particular, it converges when $x = -2$; that is, $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n = (-1)^n/(n4^n)$.]

26. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \leq x < 4$ and diverges for at least $x \geq 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n(-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

33. For $2 < x < 3$, $\sum c_n x^n$ diverges and $\sum d_n x^n$ converges. By Exercise 8.2.61, $\sum(c_n + d_n)\,x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum(c_n + d_n)\,x^n$ is 2.

35. No. If a power series is centered at a, its interval of convergence is symmetric about a. If a power series has an infinite radius of convergence, then its interval of convergence must be $(-\infty,\infty)$, not $[0,\infty)$.