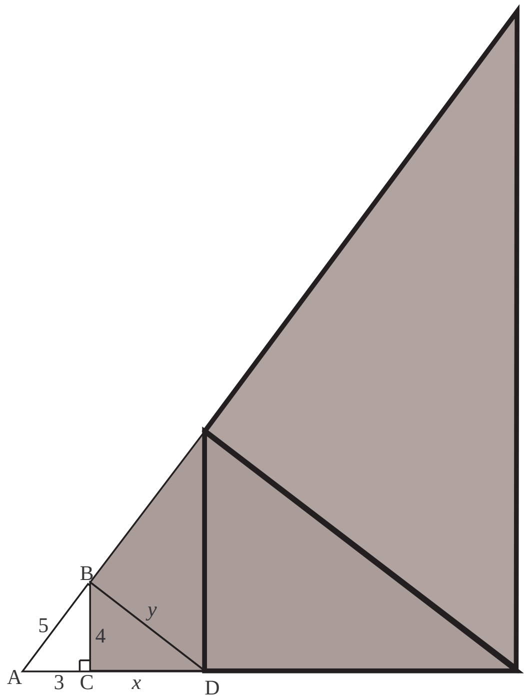
1. (a) Consider the 3-4-5 triangle on the right. Find x and y so that the shaded figure is a gnomon, ie so that the large triangle ABD is geometrically similar to the original triangle ABC.

Since triangle ABC is similar to triangle ADB the following ratios hold:

$$\frac{y}{4} = \frac{5}{3} \Rightarrow y = \frac{20}{3} = 4(\frac{5}{3})$$
$$\frac{3+x}{5} = \frac{5}{3} \Rightarrow x+3 = \frac{25}{3} \Rightarrow x = \frac{25}{3} - 3 = \frac{16}{3}$$



(b) Draw in the next three gnomon on the diagram above.

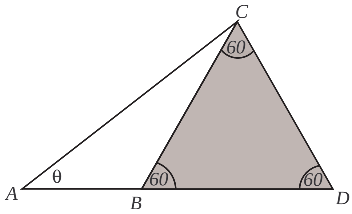
(c) Calculate the growth factor for the area of this shape and hence find the area of the figure you have drawn.

The area of ABC is $A_1 = \frac{1}{2}(3)(4) = 6$ the area of the new figure ADB after the first gnomon is added is $A_2 = \frac{1}{2}(5y) = \frac{50}{3} = A_1(\frac{25}{9})$. So the growth factor is $\frac{25}{9}$. Therefore, after four gnomon are added the total area is $6(\frac{25}{9})^4 = 357$. Note: you might consider adding up the areas of each gnomon. The first gnomon has area 32/3. The others are scaled up by a factor of (25/9) (Careful: it is not (16/9) as some assumed in the workshop). The area of the gnomon form a geometric series which you can add using the formula. You can check that this sum plus the area of the initial triangle gives the same answer.

2. You may have observed in the previous question that the gnomon is geometrically similar to the original triangle. In fact the gnomon for all right angle triangles is geometrically similar

to the original triangle. Using this fact, or otherwise, find a triangle which is its own gnomon.
A triangle which is its own gnomon is a right angled isosceles triangle – ie a square cut along its diagonal.

3. We have discussed the golden rectangle which is the unique rectangle which had a square as a gnomon. The analogous situation for triangles might be to consider a triangle which had an equilateral triangle as a gnomon. However, it turns out that an equilateral triangle cannot be a gnomon of another triangle. Show this geometrically.
If an equilateral triangle were a gnomon, as shown in the diagram below, then the angle θ would have to be 60^o. But then ABD would be a triangle whose internal angles add up to more than 180^o which is impossible.

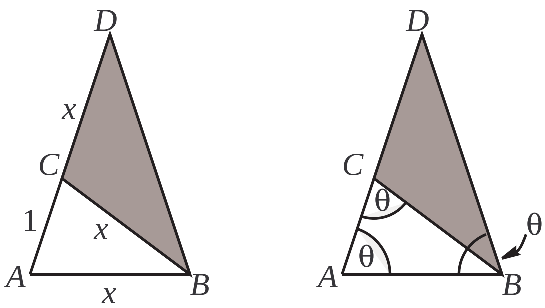


Instead let's consider isosceles triangles. There is a unique isosceles triangle that has as its gnomon another isosceles triangle.

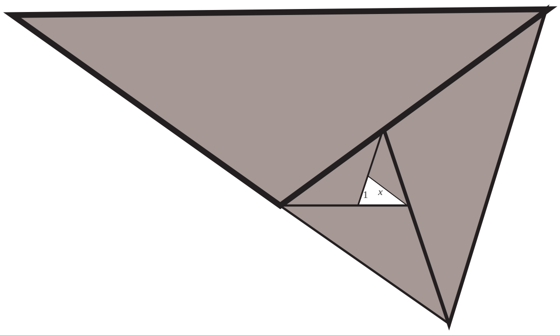
(a) Triangle ABC with unknown side x and angle θ is shown below with its gnomon BCD. Both are isosceles triangles. Find x and θ and hence determine all angles in the triangle. This triangle is called the golden triangle.
Similar triangles implies

$$\frac{AD}{AB} = \frac{AB}{AC} \Rightarrow \frac{1+x}{x} = \frac{x}{1} \Rightarrow x^2 = 1 + x \Rightarrow x^2 - x - 1 = 0$$

This quadratic equation should be familiar by now. The solutions is $x = \phi = \frac{1+\sqrt{5}}{2}$.
To find the angle θ recall that the sum of the angle in a triangle is 180^o. Also note that triangle DCB is isosceles. Call angle CDB β then angle DBC is also β. Angle DCB is therefore $180^o - 2\beta$. But DCB is also $180^o - \theta$, so $\theta = 2\beta$. Now, adding up the angles in ABD we get $2\theta + \beta = 180^0$. From this it follows that $5\beta = 180^o$ so that $\beta = 36^o$ and hence $\theta = 72^o$.



(b) Draw a small golden triangle below and add a sequence of 5 gnomon. What is the growth factor after each gnomon is added and what is the total area?



After a gnomon has been added each side of the triangle has grown by a factor ϕ so the area has grown by a factor ϕ^2. After five gnomon have been added the area has grown by a factor ϕ^{10}. The area of the first triangle is $A = \frac{1}{2}bh$ with $b = 1$. h must be obtained from Pythagoras' theorem, using half of the golden triangle with hypotenuse ϕ. Then we get

$$h^2 + (\tfrac{1}{2})^2 = \phi^2 \;\Rightarrow\; h^2 = \phi^2 - (\tfrac{1}{2})^2 = 2.368 \;\Rightarrow\; h = 1.538$$

So that the area of the first triangle is 0.769 and the total area after 5 gnomon are added is 94.6 square units.

The following assignment is due on October 25th at 9:30 am. Please show all your work on a separate piece of paper.

1. Let F_n be the nth Fibonacci number. That is $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Write out the 6 terms of the following sequences.

   (a) $u_n = 5F_n$
   5,5,10,15,25,40,65

   (b) $u_n = 2F_{n+1} - F_n$
   1,3,5,

   Show that both the sequences above are approximately geometric when n is large and evaluate the approximate growth factor.

2. Given that the golden ratio ϕ satisfies the equation $\phi^2 = 1 + \phi$ Show, by multiplying both sides of this equation by ϕ that:

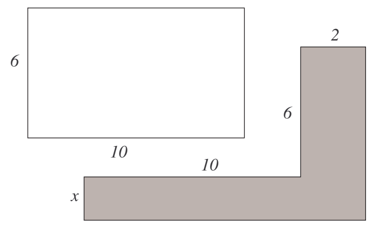
   (a) $\phi^3 = 2\phi + 1$

   (b) $\phi^4 = 3\phi + 2$

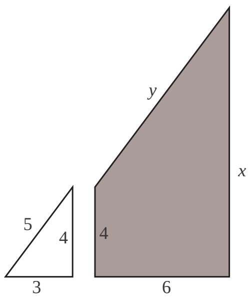
   (c) $\phi^5 = 5\phi + 3$

   Write down an expression for ϕ^n in terms of Fibonacci numbers.

3. Find the value of x so that the shaded area is a gnomon to the rectangle.



4. Find the values of x and y so that the shaded area is a gnomon to the white triangle.



5. Let ABCD be an arbitrary rectangle as shown in the figure on the right. Let AE be perpendicular to the diagonal BD and EF perpendicular to AB. Show that the rectangle BCEF is a gnomon to the rectangle ADEF.