7. Oct. 02
ZK

PS fall EM QUIZ #1 - Ch 1+2 - SOLUTIONS

(1) $\bar{v} = x^2\,\hat{i} + 2yz\,\hat{j} + 3xz^2\,\hat{k}$

$$\bar{\nabla}\cdot\bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$
$$= \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}2yz + \frac{\partial}{\partial z}3xz^2$$
$$\underline{\bar{\nabla}\cdot\bar{v} = 2x + 2z + 6xz} = \text{divergence}$$

$$\bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 2yz & 3xz^2 \end{vmatrix} = \begin{array}{l} \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}2yz\right) \\ -\hat{j}\left(\frac{\partial}{\partial x}3xz^2 - \frac{\partial}{\partial z}x^2\right) \\ +\hat{k}\left(\frac{\partial}{\partial x}2yz - \frac{\partial}{\partial y}x^2\right) \end{array}$$

$$\text{curl} = \underline{\bar{\nabla}\times\bar{v}} = \hat{i}(0-2y) - \hat{j}(3z^2-0) + \hat{k}(0-0) = \underline{-2y\,\hat{i} - 3z^2\,\hat{j}}$$

(2) First, find the charge $q(r)$ inside a radius $(r<R)$.

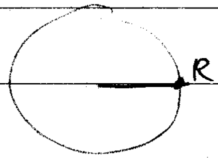
$$\rho = kr = \frac{dq}{d\tau} \quad \text{so} \quad q(r) = \int dq = \int kr\,d\tau$$
$$= \int kr\; r^2 \sin\theta\,dr\,d\theta\,d\varphi$$
$$= k\int_0^r r^3 dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\varphi$$
$$q(r) = k\frac{r^4}{4}\left[-\cos\theta\Big|_0^\pi\right]2\pi = \frac{-k\pi r^4}{2}\left((-1)-(1)\right) = k\pi r^4$$

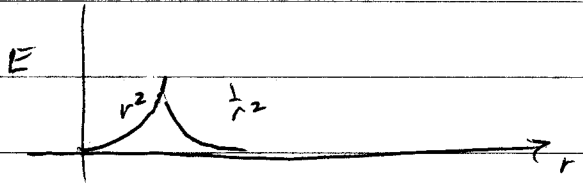
$$\frac{q(r)}{\epsilon_0} = \oint \bar{E}\cdot d\bar{A} = E(r)4\pi r^2 = \frac{k\pi r^4}{\epsilon_0} \rightarrow \underline{\vec{E}(r<R)} = \frac{k\pi r^4}{4\pi\epsilon_0 r^2} = \underline{\frac{kr^2}{4\epsilon_0}\,\hat{r}}$$

$$q_{tot} = q(R) = Q = k\pi R^4 \rightarrow E(r\geq R) = \frac{Q}{4\pi\epsilon_0 r^2} = \frac{kR^4}{4\epsilon_0 r^2}$$

Extra credit: find the potential everywhere if

$$E(r<R) = \frac{Cr^2}{4\epsilon_0}, \quad E(r>R) = \frac{CR^4}{4\epsilon_0 r^2}$$





Let $V(\infty) = 0$. Then OUTSIDE: $V(r>R) = -\int_\infty^r E(r>R)\,dr = V_{out}$

$$V_{out} = V(r>R) = -\frac{CR^4}{4\epsilon_0}\int \frac{dr}{r^2} = -\frac{CR^4}{4\epsilon_0}\left(-\frac{1}{r}\right)\Big|_\infty^r = \frac{CR^4}{4\epsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \underline{\frac{CR^4}{4\epsilon_0 r}}$$

At the surface, $V(R) = \frac{CR^4}{4\epsilon_0 R} = \underline{\frac{CR^3}{4\epsilon_0}}$

$$V_{in} - V(R) = -\int_R^r E_{in}\,dr = -\int \frac{Cr^2}{4\epsilon_0}\,dr = -\frac{C}{4\epsilon_0}\frac{r^3}{3}\Big|_R^r$$
$$= +\frac{C}{4\epsilon_0 3}\left(-r^3 + R^3\right)$$

$$V_{in} = V(R) + \frac{CR^3}{4\epsilon_0 3} - \frac{Cr^3}{4\epsilon_0 3}$$
$$= \frac{C}{12\epsilon_0}\left(3R^3 + R^3 - r^3\right) = \underline{\frac{C}{12\epsilon_0}\left(4R^3 - r^3\right)}$$

Check: $-\frac{\partial V_{in}}{\partial r} = \frac{+\partial}{\partial r}\frac{C}{12\epsilon_0}r^3 = \frac{C}{12\epsilon_0}3r^2 = \frac{Cr^2}{4\epsilon_0} = E_{in}$ ✓