PS+ Diffeq Oct 2002 Zita

LAB 1.4 Exponential and Logistic Population Models

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1.1

In the text, we modeled the U.S. population over the last 210 years using both an exponential growth model and a logistic growth model. For this lab project, we ask that you model the population growth of a particular state. Population data for several states are given in Table 1.11. (Your instructor will assign the state(s) you should consider.)

We have also discussed three general approaches that can be employed to study a differential equation: numerical techniques yield graphs of approximate solutions, geometric/qualitative techniques provide predictions of the long-term behavior of the solution, and in special cases analytic techniques provide explicit formulas for the solution. In your report, you should use as many of these techniques as is appropriate to help understand the models.

Table 1.11
Population (in thousands) of selected states (see www.census.gov)

Year	Massachusetts	New York	North Carolina	Alabama	Florida	California	Montana	Hawaii
1790	379	340	394					
1800	423	589	478	1				
1810	472	959	556	9				
1820	523	1373	638	127				
1830	610	1919	738	309	35			
1840	738	2429	753	591	54			
1850	995	3097	869	772	87	93		
1860	1231	3881	993	964	140	380		
1870	1457	4383	1071	996	188	560	20	
1880	1783	5083	1399	1262	269	865	39	
1890	2239	6003	1618	1513	391	1213	143	
1900	2805	7269	1893	1829	529	1485	243	154
1910	3366	9114	2206	2138	753	2378	376	192
1920	3852	10385	2559	2348	968	3427	549	256
1930	4250	12588	3170	2646	1468	5677	538	368
1940	4317	13479	3571	2832	1897	6907	559	423
1950	4691	14830	4061	3062	2771	10586	591	500
1960	5149	16782	4556	3267	4952	15717	675	633
1970	5689	18241	5084	3444	6791	19971	694	770
1980	5737	17558	5880	3894	9747	23668	787	965
1990	6016	17990	6628	4040	12938	29760	799	1108
2000	6349	18976	8049	4447	15982	33871	902	1212

Your report should address the following items:

1. Using an exponential growth model, determine as accurate a prediction as possible for the population of your state in the year 2010. How much does your prediction differ from the prediction that comes from linear extrapolation using the populations in 1990 and 2000? To what extent do solutions of your model agree with the historical data? See 1.1 #4 (p. 14)

2. Produce a logistic growth model for the population of your state. What is the carrying capacity for your model? Using Euler's method, predict the population in the years 2010 and 2050. Using analytic techniques, obtain a formula for the population function $P(t)$ that satisfies your model. To what extent do solutions of your model agree with the historical data? See 1.1 # 13

3. Comment on how much confidence you have in your predictions of the future populations. Discuss which model, exponential or logistic growth, is better for your data and why (and if neither is very good, suggest alternatives).

Your report: The body of your report should address all three items, one at a time, in the form of a short essay. For each model, you must choose specific values for certain parameters (the growth-rate parameter and the carrying capacity). Be sure to give a complete justification of why you made the choices that you did. You should include pictures and graphs of data and of solutions of your models *as appropriate*. (Remember that one carefully chosen picture can be worth a thousand words, but a thousand pictures aren't worth anything.)

I'll choose Montana, since that is closest to my State.

(PART 2-3)

Use my analytic solution to the logistic model to predict the population in 2010 and 2050:
Based on the shape of my curve, I expect both of these to be close to the carrying capacity of N=1200. My solution is
P(t) = Nb/(b+e^-kt) where b = 20/1080 = 0.019

year	t	P(t)
2000	130	1174
2010	140	1186
2050	180	1199

(Notice that my exact analytic solution for P(year=2000) is different than the approximate Euler solution above.)

I could make a better model of Montana's growth with an exponential curve for the fast growth period and a logistic curve for the recent, slower growth period, but the carrying capacity is always a guess, and I'll still miss other dynamics such as California economics and weather changes.

PART 1

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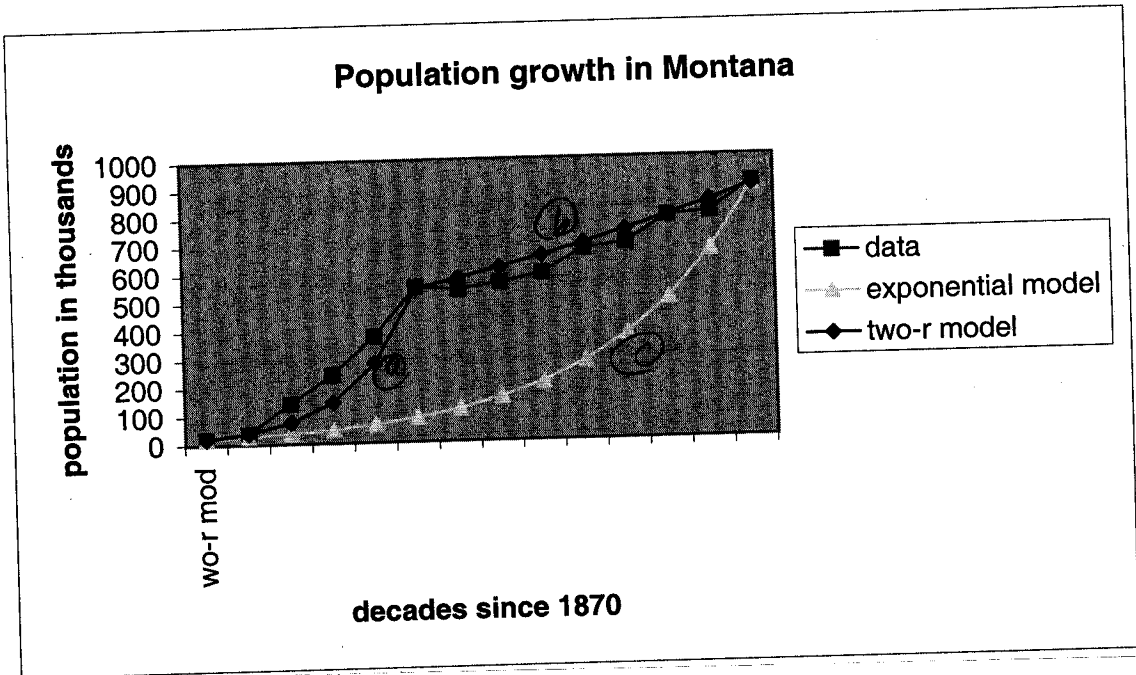
Modeling population growth in Montana
Exponential growth models:

				r	c 0.029		a 0.0660	b 0.0063	
					predictions		fast	slow	**two-r model**
		data: P=							
year	t	pop/1000	P/P0	ln(P/P0)	P(t)	t-50	P(0-50)	P(50-130)	
1870	0	20	1.00	0.00	20		20		20
1880	10	39	1.95	0.67	27		39		39
1890	20	143	7.15	1.97	36		75		75
1900	30	243	12.15	2.50	48		145		145
1910	40	376	18.80	2.93	64		280		280
1920	50	549	27.45	3.31	86	0	542	542	542
1930	60	538	26.90	3.29	116	10		577	577
1940	70	559	27.95	3.33	155	20		614	614
1950	80	591	29.55	3.39	207	30		654	654
1960	90	675	33.75	3.52	278	40		696	696
1970	100	694	34.70	3.55	372	50		741	741
1980	110	787	39.35	3.67	498	60		789	789
1990	120	799	39.95	3.69	667	70		840	840
2000	130	902	45.10	3.81	894	80		894	894



(c) A single exponential curve does NOT fit the data well.

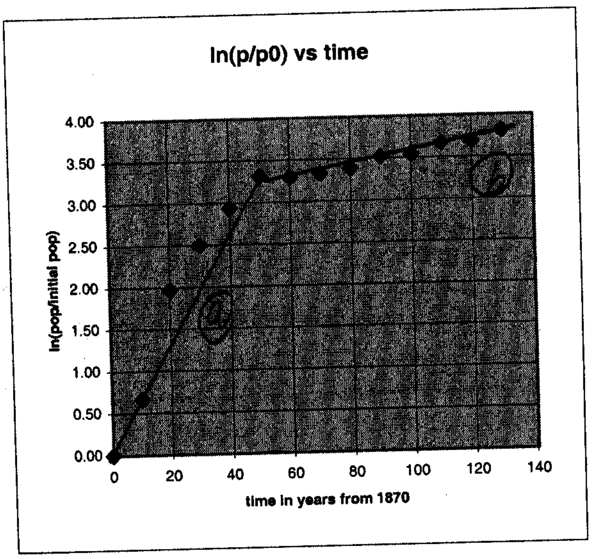
A fast (a) period of exponential growth followed by a slower (b) period of exponential growth fits better.

I figured out growth rates for $\frac{P}{P_0} = e^{rt}$
by plotting this <u>linear</u> relationship: $\ln \frac{P}{P_0} = rt$
and finding the slope r!

$y = mx$

slope

Growth rate=slope of ln/t curve~3.8/130 = 0.029 (1/ r = rate



ould get a better fit with two exponential growth periods (the "two-r model" above):

fast growth for the first 50 years at a rate of 3.3/50= 0.0660 ra (1/years)
slow growth for the next 80 years at rate (3.8-3.3)/(130-50)= 0.0063 rb (1/years)

n P(0-50) = P0 e^(ra t) peaks at P(50), and
0-130) = P(50 e^(rb t)

Lab 1.4 PART②

2. Logistic growth model $\frac{dP}{dt} = k(1-\frac{P}{N})P$

k = growth rate = r ; N = carrying capacity

From fig 1, I estimate the carrying capacity to be about

$N = 1200$ (in thousands of people). Try this with my original exponential growth rate of $k = 0.029$ ($\frac{1}{\text{years}}$). That gave a poor match so I ADJUSTED k to 0.065 for a better fit (this is phenomenology). Clearly my guess of $N = 1200$ could be low especially if Californians decide privacy is more valuable than warmth. So I have low confidence in my predictions, because more factors are at work than exponential growth + carrying capacity.

Analytic solution to $\frac{dP}{dt} = k(1-\frac{P}{N})P = \frac{k}{N}(N-P)P$

$$\frac{dP}{(N-P)P} = \frac{k}{N}dt$$

Partial fractions: $\frac{1}{P(N-P)} = \frac{A}{P} + \frac{B}{N-P} = \frac{A(N-P)+BP}{P(N-P)}$.

$1 = AN \therefore A = \frac{1}{N}$; $0 = P(B-A) \therefore B = A = \frac{1}{N}$

$$\frac{1}{P(N-P)} = \frac{1}{PN} + \frac{1}{N(N-P)}$$

DiffEq

Logistic

year
1870
1880
1890
1900
1910
1920
1930
1940
1950
1960
1970
1980
1990
2000

$\int \frac{k}{N}dt$

IC:

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dt= 10.00
k= 0.06
N= 1200

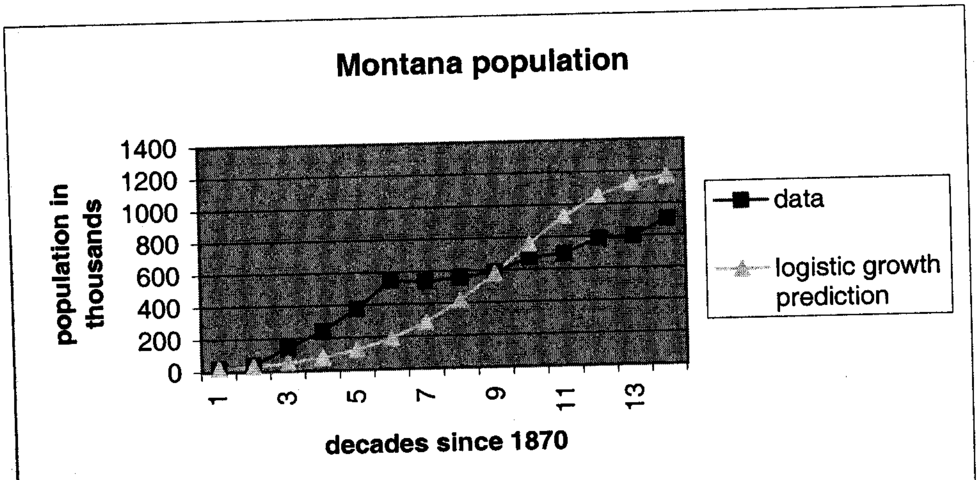
Logistic model for Montana population growth:

dP/dt = k(1-P/N)P   P(0)= 20

year	t	data: P= pop/1000
1870	0	20
1880	10	39
1890	20	143
1900	30	243
1910	40	376
1920	50	549
1930	60	538
1940	70	559
1950	80	591
1960	90	675
1970	100	694
1980	110	787
1990	120	799
2000	130	902

Euler method:

k	t(k)	P(k)	f = dP/dt	P(k+1)= P(k) + f dt
0	0	20	1.18	31.8
1	10.00	32	1.86	50
2	20.00	50	2.90	79
3	30.00	79	4.45	124
4	40.00	124	6.66	190
5	50.00	190	9.61	286
6	60.00	286	13.09	417
7	70.00	417	16.33	581
8	80.00	581	17.98	760
9	90.00	760	16.71	928
10	100.00	928	12.63	1054
11	110.00	1054	7.70	1131
12	120.00	1131	3.91	1170
13	130.00	1170	1.76	1188



ANALYTIC SOLUTION CONTINUED...

$$kdt = \int \frac{dP}{(N-P)P} = \int \frac{dP}{PN} + \int \frac{dP}{N(N-P)} \qquad \text{let } (N-P) = x,\ dP = -dx$$

$$\frac{k}{N}t + c' = \frac{1}{N}\int \frac{dP}{P} - \frac{1}{N}\int \frac{dx}{x}$$

$$kt + c = (\ln P - \ln x) = \ln \frac{P}{x} = \ln \frac{P}{N-P}$$

$$P = (N-P)\, b e^{kt}$$

$$P(1 + be^{kt}) = Nbe^{kt} \rightarrow P = Nb\frac{e^{kt}}{1+be^{kt}} = \frac{Nb}{b+e^{-kt}}$$

$$P(t=0) = P_0(1+b) = 20(1+b) = Nb \rightarrow 20 = b(N-20) \rightarrow b = \frac{20}{1080} = \underline{\qquad}$$

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