Lab 2.3: The harmonic oscillator with modified damping.

Differential Equations, Fall 2002, TESC
Text = Differential Equations (2002, ed.2) by Blanchard, Devaney, and Hall (pp.219-220)
3.Dec. 2002 - E.J. Zita

Overview:

We investigate the dynamics of harmonic oscillators for four cases:

(1) Simple harmonic oscillator (SHO with no damping): $$m\frac{d^2y}{dt^2}+ky=0$$

(2) Harmonic oscillator with damping: $$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=0$$

(3) Harmonic oscillator with nonlinear damping: $$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}\left|\frac{dy}{dt}\right|+ky=0$$

(4) Nonlinear second-order equation: $$m\frac{d^2y}{dt^2}+\left(y^2-\alpha\right)\frac{dy}{dt}+ky=0$$

For each case, we investigate three sets of intial conditions (y_0,v_0), and
two sets of parameters (mass *m*, damping constants *b* and α, and spring constant *k*).
We chose parameter sets #1 and #6 on p.220.

Methods:
We write each second-order differential equation as two first-order equations,
dy/dt = v and v = f(y,t). We then let v=x in the "HPG System Solver" software on the DETools disk, and approximately solve each system numerically and plot timeseries and phase plots.

Short answers:

(1) The simple harmonic oscillators have sinusoidal solutions of frequency $\omega=\sqrt{k/m}$ with constant amplitude, and the phase plot is a limit cycle, as expected.

(2) The (undriven) damped harmonic oscillator has sinusoidal oscillations with a lower frequency and exponentially decaying amplitude. The plase plot spirals in to zero.

(3) The harmonic oscillator with nonlinear damping has less perfectly sinusoidal oscillations, and the phase plot is similarly asymmetric. Otherwise, this looks much like case 2.

(4) The nonlinear second order equation has markedly nonsinusoidal, semiperiodic oscillations and a markedly asymmetric phase plot. Oscillations approach a limit cycle, whether the initial conditions (IC) are inside or ourside the limit cycle. For IC near the limit cycle (e.g. (8,0)), solutions advance numerically very sluggishly or even stall.

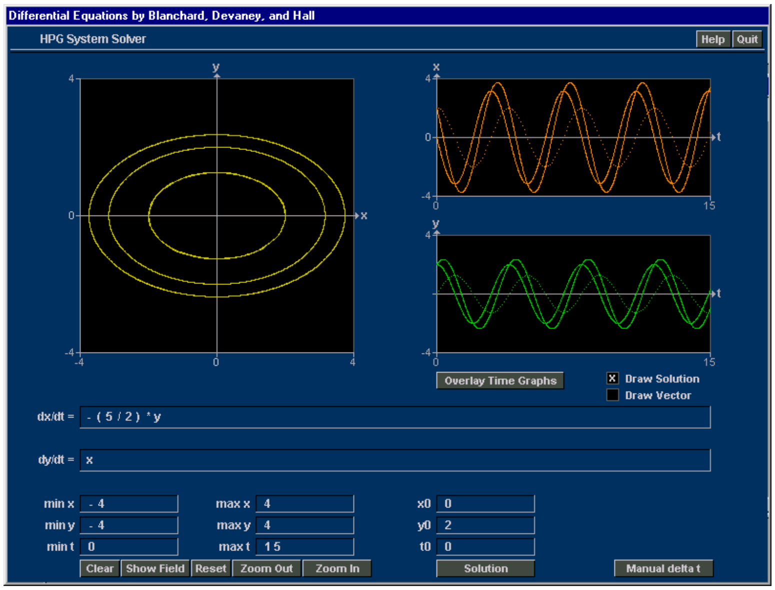
Key points: Only simple harmonic oscillators are truly sinusoidal ad periodic. Damping retains sinusoidal oscillations, but they are not truly periodic. Nonlinearly damped oscillations are not sinusoidal. The nonlinear second order equation in (4) has a limit cycle which depends on the parameter set chosen.

Case (1) Simple harmonic oscillator (SHO with no damping): $$m\frac{d^2y}{dt^2}+ky=0$$

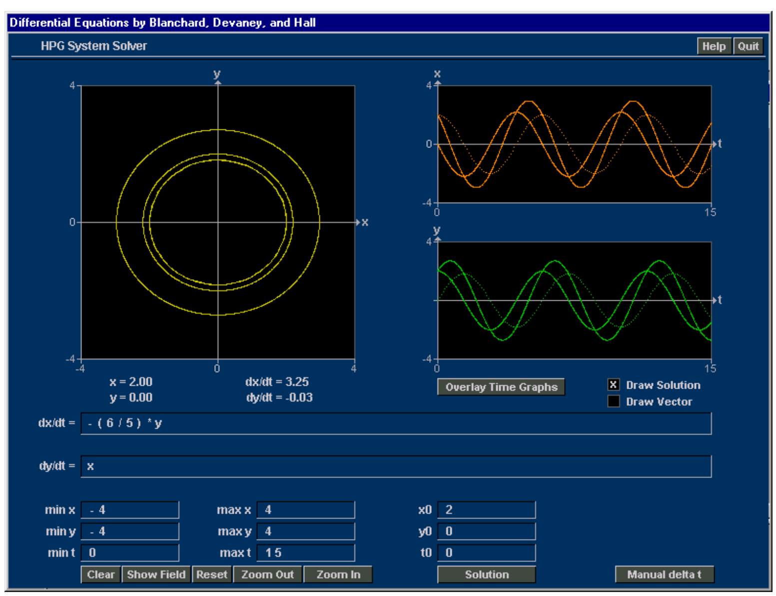
$\dfrac{dy}{dt}\equiv v\,,\quad \dfrac{dv}{dt}=\dfrac{-k}{m}y\,.$ (Let x=v in HPGSystemSolver.)

These simple harmonic oscillators have sinusoidal solutions of frequency $\omega = \sqrt{k/m}$ with constant amplitude, and the phase plot is a limit cycle, as expected. If it looks like the frequency depends on the initial conditions (for given m and k), note that this is an optical illusion - the initial conditions just determine the amplitude and the *phase* of oscillations.

Parameter set #1: m=2 k/m=5/2



Parameter set #6: m=5 k/m=6/5

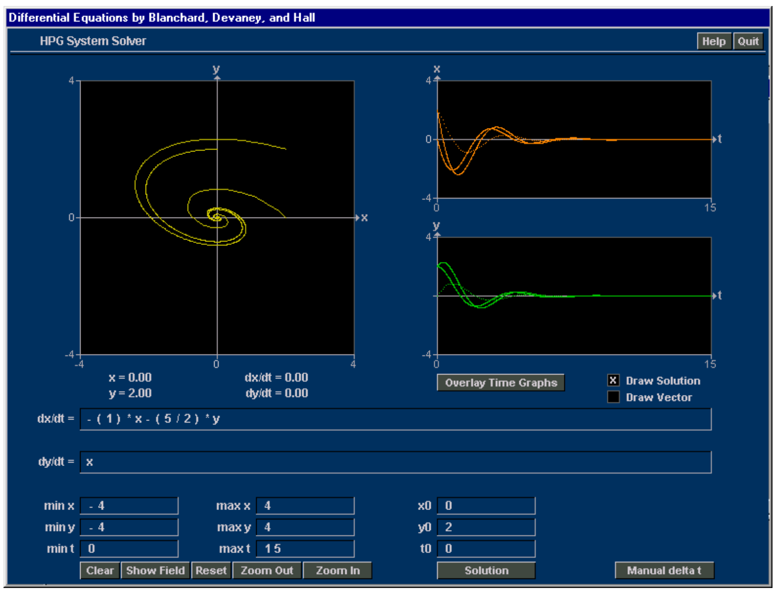


Case (2) Harmonic oscillator with damping: $$m\frac{d^2y}{dt^2}+b\frac{dy}{dt}+ky=0$$

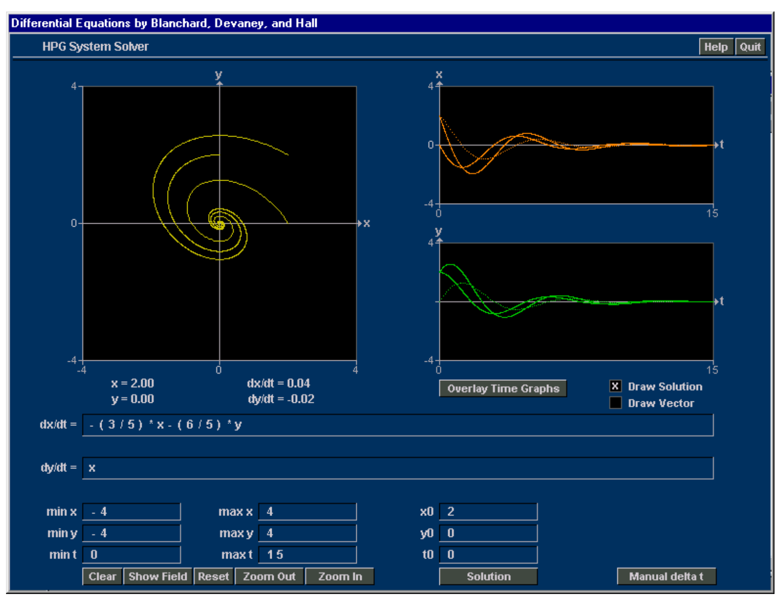
$$\frac{dy}{dt}\equiv v\,,\quad \frac{dv}{dt}=-\frac{b}{m}v-\frac{k}{m}y\quad.$$ (Let x=v in HPGSystemSolver.)

These damped harmonic oscillator have sinusoidal oscillations with a lower frequency and exponentially decaying amplitude. The plase plot spirals in to zero.

Parameter set #1: m=2 k/m=5/2 b/m=1



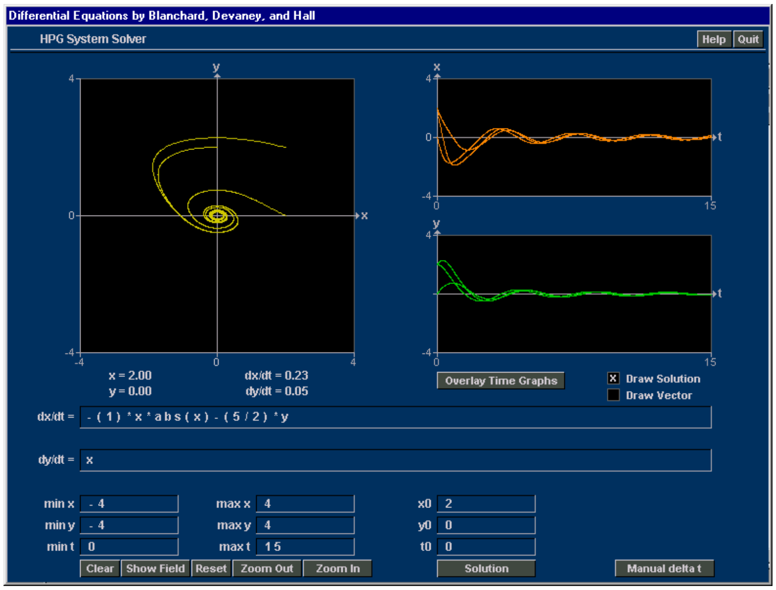
Parameter set #6: m=5 k/m=6/5 b/m=3/5



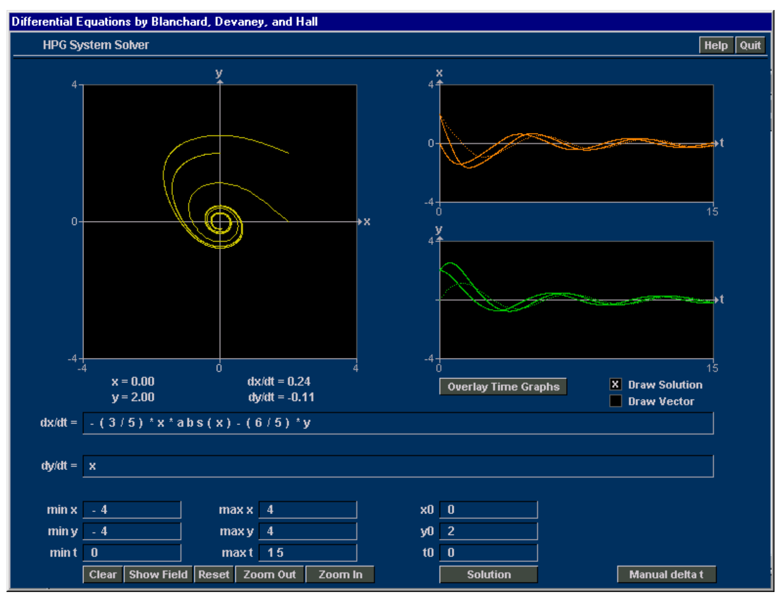
Case (3) Harmonic oscillator with nonlinear damping: $m\dfrac{d^2y}{dt^2}+b\dfrac{dy}{dt}\left|\dfrac{dy}{dt}\right|+ky=0$

$\dfrac{dy}{dt}\equiv v\,,\quad \dfrac{dv}{dt}=-\dfrac{b}{m}v\left|v\right|-\dfrac{k}{m}y$. (Let x=v in HPGSystemSolver.)
These damped harmonic oscillators look like case(2) above except less symmetrical.

Parameter set #1: m=2 k/m=5/2 b/m=1



Parameter set #6: m=5 k/m=6/5 b/m=3/5



Case (4) Nonlinear second-order equation: $$m\frac{d^2y}{dt^2}+\left(y^2-\alpha\right)\frac{dy}{dt}+ky=0$$

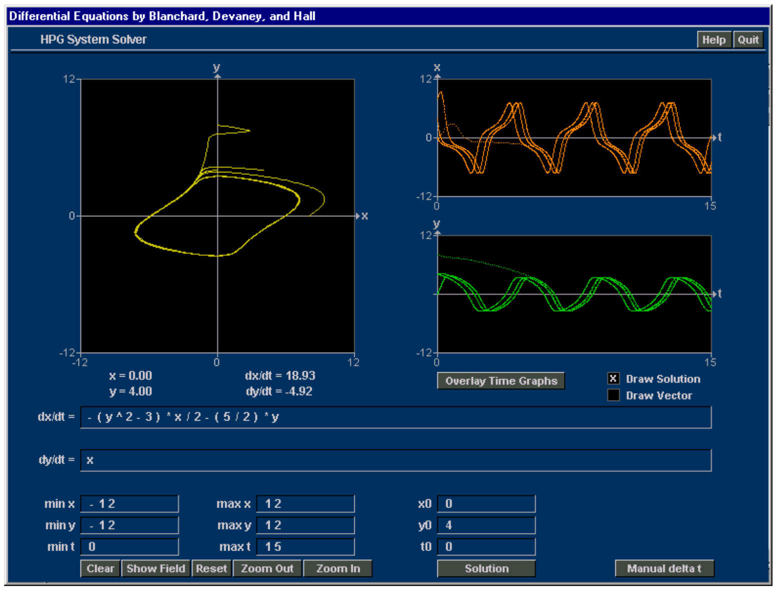
$\dfrac{dy}{dt}\equiv v\,,\quad \dfrac{dv}{dt}=-\dfrac{\left(y^2-\alpha\right)}{m}v-\dfrac{k}{m}y$. (Let x=v in HPGSystemSolver.)

Nonlinear, asymmetric, and solutions approach limit cycle whether IC are inside $[(y_0,v_0) = (0,4)$ or $(4,0)]$ outside

$[(y_0,v_0) = (8,0)$ or $(0,8)]$. Curiously, $(y_0,v_0) = (8,8)$ stalls after crossing y axis.

Note that scales are increased from $x_{max}=y_{max}=4$ in cases (1-3) to $x_{max}=y_{max}=12$ here.

Parameter set #1: m=2 k/m=5/2 α=3.



Parameter set #6: m=5 k/m=6/5 α=5

The timeseries are more nearly in phase for set 1. Phase information is lost in the phase plots, which look very similar for sets 1 and 6. In set 6, $(y_0,v_0) = (8,8)$ crosses the y-axis slowly (iterations must be very close), then picks up steam (bigger space steps for the same time steps)