$\#_1 3$
142

(a) Find $\langle r \rangle$ and $\langle r^2 \rangle$ for e^- in H in ground state.

$$\psi_{100} = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

$\frac{[4,80]}{138}$

$$\langle r \rangle = \langle \psi | r \psi \rangle = \frac{1}{\pi a^3} \int r e^{-2r/a} \, d\text{volume} = \frac{4\pi}{\pi a^3} \int r e^{-2r/a} r^2 dr$$

DWIGHT
860,07
230

$$\langle r \rangle = \frac{4}{a^3} \int_0^\infty r^3 e^{br} \, dr \qquad b = \frac{-2}{a}$$

DEFINITE INTEGRAL : $\int_0^\infty x^n e^{-bx} dx = \frac{n!}{b^{n+1}}$

$$\int_0^\infty r^3 e^{-br} dr =$$

$\langle r \rangle =$

$$\langle r^2 \rangle = \langle \psi | r^2 \psi \rangle = \frac{4\pi}{\pi a^3} \int r^2 e^{-2r/a} r^2 dr =$$

$=$

(b) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for e^- in ground state H.

$r^2 = x^2 + y^2 + z^2 = 3x^2$ by symmetry

(4,13)

$$\psi_{nlm} = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right) Y_l^m(\theta, \phi). \quad [4.89]$$

$\psi_{211} =$

$L_0^3\left(\frac{r}{a}\right)$ Y_1^1

p. 128

6 $\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{i\phi}$

p. 140

$\langle x^2 \rangle$

$=$

$=$

$= 12a^2$

4.14 What is the probability that e^- in ψ_{100} will be found
142 INSIDE THE NUCLEUS?

(a) EXACTLY: $P = \int \psi_{100}^2 \, d\text{volume} = \frac{4\pi}{\pi a^3} \int_0^b e^{-2r/a} r^2 dr$

DWIGHT $\frac{567.2}{134}$ $\int r^2 e^{kr} dr = e^{kr}\left[\frac{r^2}{k} - \frac{2r}{k^2} + \frac{2}{k^3}\right]$ $k = -2/a$

$$P = +1 - 2e^{-2b/a}\left(\frac{b^2}{a^2} + \frac{b}{a} + \frac{1}{2}\right)$$

(b) Expand result as a power series in $\varepsilon = \frac{2b}{a}$ and show that the lowest-order term is $P \approx \frac{4}{3}\left(\frac{b}{a}\right)^3$

$$P = 1 - e^{-\varepsilon}\left[\frac{1}{2}\left(\frac{2b}{a}\right)^2 + \frac{2b}{a} + 1\right] = 1 - e^{-\varepsilon}\left[1 + \varepsilon + \frac{1}{2}\varepsilon^2\right]$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

(4.14)

(c) If $\psi \approx$ constant over the tiny nucleus, $P \approx \frac{4}{3}\pi b^3 |\psi(0)|^2$

Check: $|\psi(0)|^2 =$

$P \approx$

(d) Use $b \simeq 10^{-15}$ m and $a \simeq 0.5 \cdot 10^{-10}$ m to evaluate P.

$$P = \frac{4}{3}\left(\frac{10^{-15}}{\frac{1}{2} \cdot 10^{-10}}\right)^3 = \frac{4}{3} \cdot 8 \, (10^{-5})^3 \simeq 10 \cdot 10^{-15} \simeq \underline{10^{-14}}$$

Roughly speaking, this represents the "fraction of time the e^- spends inside the nucleus" - VIRTUALLY NONE!

Problem 4.16 Consider the earth-sun system as a gravitational analog to the hydrogen atom.

(a) What is the potential energy function (replacing Equation 4.52)? (Let m be the mass of the earth and M the mass of the sun.) $V = -\frac{GmM}{r}$

(b) What is the "Bohr radius" for this system? Work out the actual numerical value.

(c) Write down the gravitational "Bohr formula", and, by equating E_n to the classical energy of a planet in a circular orbit of radius r_o, show that $n = \sqrt{r_o/a}$. From this, estimate the quantum number n of the earth. $r_0 = 93 \times 10^6$ miles $\cdot$ 150×10^6 km $\simeq 10^{2+6+3}$ m

(d) Suppose the earth made a transition to the next lower level $(n-1)$. How much energy (in Joules) would be released? What would the wavelength of the emitted photon (or, more likely, graviton) be?

$$F = ma \qquad\qquad L = mvr = n\hbar$$

$$\frac{GmM}{r^2} = \frac{mv^2}{r} \qquad\qquad v = n\hbar/mr$$

$$v^2 = \frac{GM}{r} \qquad = \qquad v^2 = \frac{n^2\hbar^2}{m^2r^2}$$

(b)

(c) $E = \frac{V}{2} = -$

Classically,

$n = \sqrt{\frac{r_0}{a}} \simeq \sqrt{}$

***Problem 4.20**

(a) Starting with the canonical commutation relations for position and momentum, Equation 4.10, work out the following commutators:

$$[L_z, x] = i\hbar y, \quad [L_z, y] = -i\hbar x, \quad [L_z, z] = 0$$
$$[L_z, p_x] = i\hbar p_y, \quad [L_z, p_y] = -i\hbar p_x, \quad [L_z, p_z] = 0. \qquad [4.122]$$

(b) Use these results to rederive Equation 4.98 directly from Equation 4.96.

[4.10] $[r_i, p_j] = -[p_i, r_j] = i\hbar\,\delta_{ij}$,

$[r_i, r_j] = [p_i, p_j] = 0$

[4.96] $\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cdot & \cdot & \cdot \\ p_x & p_y & p_z \end{vmatrix} = \begin{matrix} \hat{i}\,(y p_z - z p_y) \\ -\hat{j}\,(x p_z - z p_x) \\ +\hat{k}\,(x p_y - y p_x) \end{matrix} = \begin{matrix} \hat{i}\,L_x \\ +\hat{j}\,L_y \\ +\hat{k}\,L_z \end{matrix}$

$[L_z, x] = [(x p_y - y p_x), x] = [x p_y, x] - [y p_x, x]$

$= x[p_y, x] - y[p_x, x]$ since x & y are just scalar operators

$= x \cdot 0 - y(-i\hbar)$ by [4.10]

$[L_z, x] = i\hbar y$