QM HW Ch 4 @ - Physical Systems
winter week 8

due 4 Mar 03
E/I

Ch 4 # 1, 2, 3, 4, 10, 13, 14, 16
122 124 128 140 142 144

***Problem 4.1**

(a) Work out all of the **canonical commutation relations** for components of the operators **r** and **p**: $[x, y]$, $[x, p_y]$, $[x, p_x]$, $[p_y, p_z]$, and so on. *Answer:*

$$[r_i, p_j] = -[p_i, r_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [p_i, p_j] = 0. \qquad [4.10]$$

DEFINITION: $\delta_{ij} = \begin{cases} 1 \text{ if } i=j, \\ 0, \text{ if } i\neq j \end{cases}$

$p_j = -i\hbar \frac{\partial}{\partial r_j}$

(b) Show that

$$\frac{d}{dt}\langle \mathbf{r}\rangle = \frac{1}{m}\langle \mathbf{p}\rangle, \quad \text{and} \quad \frac{d}{dt}\langle \mathbf{p}\rangle = \langle -\nabla V\rangle. \qquad [4.11]$$

(Each of these, of course, stands for *three* equations—one for each component.) *Hint:* Note that Equation 3.148 is valid in three dimensions.

$[x, y] = xy - yx = 0 = [y, x]$

$$[x, p_x]\psi = xp_x\psi - p_x x\psi = x\left(-i\hbar\frac{\partial}{\partial x}\psi\right) - \left(-i\hbar\frac{\partial}{\partial x}\right)x\psi$$
$$= -i\hbar\left[x\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(x\psi)\right]$$
$$= -i\hbar\left[x\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial x} - \psi\frac{\partial x}{\partial x}\right] = +i\hbar\psi$$

So $[x, p_x] = i\hbar$ ...

***Problem 4.10** Work out the radial wave functions R_{30}, R_{31}, and R_{32}, using the recursion formula (Equation 4.76). Don't bother to normalize them.

$v(\rho)$ is a polynomial of degree $j_{max} = n - l - 1$

$$R_{10}(r) = \frac{a_0}{a} e^{-r/a}.$$

$$v = \sum a_j \rho^j \qquad a_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} a_j.$$

$$R = \frac{A}{r} \rho^{l+1} v(\rho) e^{-\rho} \quad \text{where } \rho = \kappa r = \frac{r}{an} \quad (a = \text{Bohr radius})$$

$$v(\rho) = L^{2l+1}_{n-l-1}(2\rho) e^{-\rho} \quad \text{and Laguerre polynomials are in Table 4.5 p.140}$$

$\boxed{R_{30}}$ $n=3,\ l=0,\ j_{max} = 3-0-1 = 2$ (a_0 = Some constant to be absorbed into Normalization A)

$$j=0: \quad a_1 = \frac{2(0+0+1-3)}{(0+1)(0+2\cdot 0+2)} a_0 = -\frac{4}{2} = -2a_0$$

$$j=1: \quad a_2 = \frac{2(1+0+1-3)}{(1+1)(1+0+2)} a_1 = \frac{2(-1)}{2\cdot 3} = -\frac{1}{3} a_1 = \frac{2}{3} a_0$$

$$j=2: \quad a_3 = \frac{2(2+0+1-3)}{(3)(2+0+2)} a_1 = 0$$

$$v_{30} = \sum a_j \rho^j = a_0 \rho^0 + a_1 \rho^1 + a_2 \rho^2 + 0\rho^3$$
$$= a_0 - 2a_0 \rho + \frac{2}{3} a_0 \rho^2 = a_0 \left(1 - 2\rho + \frac{2}{3}\rho^2\right)$$

$$\rho_{30} = \frac{r}{3a} \quad \text{so} \quad v_{30} = a_0 \left(1 - \frac{2r}{3a} + \frac{2}{3}\frac{r^2}{9a^2}\right)$$

Check this against tabulated Laguerre polynomials:

$$L^{2l+1}_{n-l-1}(x) = L^{0+1}_{3-1}(x) = L^{1}_{2}(x) = 18 - 18x + 3x^2 = 1 - x + \frac{1}{9}x^2$$

$$L(2\rho) = 1 - 2\rho + \frac{1}{9}4\rho^2 = 1 - 2\left(\frac{r}{3a}\right) + \frac{4}{9}\left(\frac{r}{3a}\right)^2 \quad \text{close} \ldots$$

.10
2

$$R_{30} = \frac{A}{r} \rho^{0+1} v_{30} e^{-\rho} = \underline{\frac{A}{r}\left(\frac{r}{3a}\right)\left[1 - \frac{2r}{3a} + \frac{2}{27}\left(\frac{r}{a}\right)^2\right] e^{-r/3a}}$$

This matches tabulated R_{30} in Table 4.6 p.141 to within a normalization const.

$\boxed{R_{31}}$ $n=3, l=1$ $j_{max} = n-l-1 =$ ___ , $a_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} a_j$

$j=0$: $a_1 = \frac{2(0+1+1-3)}{(0+1)(0+2+2)} a_0 =$ ___ a_0

$j=1$: $a_2 =$ ______ $a_0 =$ ___ a_0

$j=2$; $a_3 =$ ___

$v(\rho) = \Sigma a_j \rho^j =$ ____________

$\rho = \frac{r}{an} =$ ______

$v(r) =$ ____________

$R_{31} = \frac{A}{r} \rho^{l+1} v e^{-\rho} =$

New Ch 4 HW: due Thur 6 Mar 03

4.18, 20, 28

149 150 159

Problem 4.18

(a) Prove that if f is simultaneously an eigenfunction of L^2 and of L_z (Equation 4.104), the square of the eigenvalue of L_z cannot exceed the eigenvalue of L^2. *Hint:* Examine the expectation value of L^2.

$$[L^2, \mathbf{L}] = 0. \qquad [4.103]$$

So L^2 is compatible with each component of $\mathbf{L}$, and we *can* hope to find simultaneous eigenstates of L^2 and (say) L_z:

$$L^2 f = \lambda f \quad \text{and} \quad L_z f = \mu f. \qquad [4.104]$$

(b) As it turns out (see Equations 4.118 and 4.119), the square of the eigenvalue of L_z never even *equals* the eigenvalue of L^2 (except in the special case $l = m = 0$). Comment on the implications of this result.

Evidently the eigenvalues of L_z are $m\hbar$, where m (the appropriateness of this letter will also be clear in a moment) goes from $-l$ to $+l$ in N integer steps. In particular, it follows that $l = -l + N$, and hence $l = N/2$, so l must be *an integer or a half-integer*. The eigenfunctions are characterized by the numbers l and m:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m; \quad L_z f_l^m = \hbar m f_l^m, \qquad [4.118]$$

where

$$l = 0, 1/2, 1, 3/2, \ldots; \quad m = -l, -l+1, \ldots, l-1, l. \qquad [4.119]$$

For a given value of l, there are $2l+1$ different values of m (i.e., $2l+1$ "rungs" on the "ladder").

uncertainty principle (Equation 4.100), and explain how the special case gets away with it.

$$\sigma_{L_x}^2 \sigma_{L_y}^2 \geq \left(\frac{1}{2i}\langle i\hbar L_z\rangle\right)^2 = \frac{\hbar^2}{4}\langle L_z\rangle^2, \qquad [L_x, L_y] = i\hbar L_z$$

$$\sigma_{L_x}\sigma_{L_y} \geq \frac{\hbar}{2}|\langle L_z\rangle|. \qquad [4.100]$$

***Problem 4.20**

(a) Starting with the canonical commutation relations for position and momentum, Equation 4.10, work out the following commutators:

$$[L_z, x] = i\hbar y, \qquad [L_z, y] = -i\hbar x, \qquad [L_z, z] = 0$$
$$[L_z, p_x] = i\hbar p_y, \qquad [L_z, p_y] = -i\hbar p_x, \qquad [L_z, p_z] = 0. \qquad [4.122]$$

(b) Use these results to rederive Equation 4.98 directly from Equation 4.96.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \qquad [4.95]$$

which is to say,

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad \text{and} \quad L_z = xp_y - yp_x. \qquad [4.96]$$

$$[L_x, L_y] = i\hbar L_z. \qquad [4.98]$$

***Problem 4.28** An electron is in the spin state

$$\chi = A\begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the "uncertainties" σ_{S_x}, σ_{S_y}, and σ_{S_z}.

(d) Confirm that your results are consistent with all three uncertainty principles (Equation 4.100 and its cyclic permutations—only with S in place of L, of course).