OUTLINE for QM week 3 — 20 Jan 03

King quote - MLK day

questions on HW - #2.6

quiz

Harmonic oscillator - review & continue
2.13 and setup 2.14 (worksheets from last week)

Start Ch 3

Ch 3 HW #9, 10, 18, 22, ~~23 (a, b, first part of c)~~, 36 (a)

ANSWERS:

3.9 $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & (3-2i) & 4 \end{pmatrix}$ $\begin{pmatrix} -3 & (1+3i) & 3i \\ (4+3i) & 9 & 6-2i \\ 6i & (6-2i) & 6 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 3 \\ 6+3i & -3i & 12 \end{pmatrix}$

$\begin{pmatrix} -3 & 1+3i & 3i \\ 2+3i & 9 & 3-2i \\ -6+3i & 6+i & -6 \end{pmatrix}$ $\begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & 2i \\ -i & 3 & 2 \end{pmatrix}$

$Tr = 5$ $\det = 3$ $B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -3i & i \\ 0 & 3 & 0 \\ -i & -6 & 2 \end{pmatrix}$

3.10 $\begin{pmatrix} 3i \\ 6+2i \\ 6 \end{pmatrix}$ $-2-4i$ $8+4i$ $\begin{pmatrix} 2i & -(1+i) & 0 \\ 4i & (-2+2i) & 0 \\ 4 & (2+2i) & 0 \end{pmatrix}$

3.18 $\lambda = 1$, $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ 3.22: $\lambda = -1, 2$, $a = \frac{1}{\sqrt{6}}\begin{pmatrix} i-1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$

3.23 $\det = 0$, $Tr = 6$ $\lambda = 0, 3, 3$ $a = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}$, $S = \frac{1}{\sqrt{6}}\begin{pmatrix} \sqrt{2} & -\sqrt{2}i & \sqrt{2} \\ \sqrt{3} & \sqrt{3}i & 0 \\ 1 & -i & 2 \end{pmatrix}$

QM Ch 3 - New notations

WAVEFUNCTIONS as Complex VECTORS, OPERATORS as Complex MATRICES or TENSORS

$\psi \to |\psi\rangle \qquad \hat{p} \to |p|$

3.1.1 VECTORS $|v\rangle = V_x \hat{i} + V_y \hat{j} + V_z \hat{k} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \vec{V}$

p.83 $\vec{u}^* = u^\dagger = \langle u| = (u_x^* \quad u_y^* \quad u_z^*)$

$\langle u|v\rangle = \vec{u}^\dagger \cdot \vec{V} = u_x^* V_x + u_y^* V_y + u_z^* V_z = \int u^*(x) V(x)\, dx$

These complex vectors obey the same identities as usual vectors.

TRANSPOSE : column → row : $\tilde{V} = (V_x \quad V_y \quad V_z)$

CONJUGATE : $i \to -i$; $\vec{a} = \begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}$, $\vec{a}^* = \begin{pmatrix} 1 \\ -i \\ 2 \end{pmatrix}$

ADJOINT = transposed conjugate : $a^\dagger = \tilde{a}^* = (1 \quad +i \quad 2)$

$|a^\dagger\rangle = \langle a|$

→ Practice: Ex 3.9, 10 p. 86

p.77 "A SET OF LINEARLY INDEPENDENT VECTORS THAT SPANS A SPACE IS CALLED A BASIS."

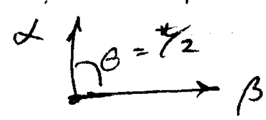
SPAN : any vector in the space can be written as a combination of the basis vectors

DIMENSION of space = number of BASIS vectors that span it.

SCHWARZ INEQUALITY:

$|\langle\alpha|\beta\rangle|^2 \le \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \qquad \cos\theta = \sqrt{\dfrac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}}$

ORTHOGONAL: perpendicular : $\langle\alpha|\beta\rangle = \vec{\alpha}\cdot\vec{\beta} = \alpha\beta\cos\theta = 0$



NORMALIZED: $\overset{\text{UNIT}}{\text{vector}}$ has length 1 : $|e_i\rangle = \dfrac{|e_1\rangle}{\|e_1\|}$

NORM = length of vector : $\|e_1\| = \sqrt{\langle e_1|e_1\rangle} = \sqrt{\vec{e}_1\cdot\vec{e}_1} = \sqrt{e_1^2}$

ORTHONORMAL: set of orthogonal unit vectors

p. 79 GRAM-SCHMIDT procedure lets you construct an ORTHONORMAL BASIS from a set of linearly independent vectors.

(i) normalize one vector → $|e_1'\rangle$

(ii) subtract the projection of the second vector : $|e_2\rangle - \langle e_1'|e_2\rangle|e_1'\rangle$

(iii) normalize this perpendicular vector → $|e_2'\rangle$

(iv) repeat for the other vectors. DRAW IF POSSIBLE.

→ Practice: three-space basis above Prob. 3.5 p. 80

TRIANGLE INEQUALITY: $\|\,|\alpha\rangle + |\beta\rangle\,\| \leq \|\alpha\| + \|\beta\|$

$\overleftrightarrow{T}$ MATRICES transform VECTORS.

$\hat{T}$ OPERATORS transform WAVEFUNCTIONS.

p. 81 MATRIX $\overleftrightarrow{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{32} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$ vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

operation = transformation : $\vec{b} = \overleftrightarrow{T}\vec{a} = \begin{pmatrix} T_{11}a_1 + T_{12}a_2 + T_{13}a_3 \\ T_{21}a_1 + T_{22}a_2 + T_{32}a_3 \\ T_{31}a_1 + T_{32}a_2 + T_{33}a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

matrices add, etc.

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

WAYS TO CHANGE MATRICES:

<u>TRANSPOSE</u> : switch columns with rows : $\tilde{T} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

SYMMETRIC if $T = \tilde{T}$ $(c = b)$

ANTI-SYMMETRIC if $T = -\tilde{T}$ $(c = -b,\ a = d = 0)$

<u>CONJUGATE</u>: switch signs of imaginary parts : $T^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$

REAL if $T^* = T$, IMAGINARY if $T^* = -T$.

<u>ADJOINT</u> = transpose conjugate : $T^\dagger = \tilde{T}^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$
= hermitian conjugate

HERMITIAN = SELF-ADJOINT if $T^\dagger = T$ → REAL EIGENVALUES

SKEW-HERMITIAN if $T^\dagger = -T$

MATRIX MULTIPLICATION is NOT COMMUTATIVE for incommensurable operators

$\hat{x}\hat{p} \neq \hat{p}\hat{x}$, since $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$xp - px \equiv [x, p]$ = COMMUTATOR

TRANSPOSE of product $(\widetilde{xp}) = \tilde{p}\,\tilde{x}$

UNIT MATRIX = IDENTITY MATRIX : $1 = \overleftrightarrow{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & 1 & \vdots \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}$

$\overleftrightarrow{I}\overleftrightarrow{T} = \overleftrightarrow{T}$, $a\overleftrightarrow{I} = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & a \end{pmatrix}$ $\quad I = T^{-1}T = TT^{-1}$

INVERSE T^{-1} times T yields I. SINGULAR matrix has no INVERSE.

<u>UNITARY</u> : $U^\dagger = U^{-1}$: adjoint = inverse. (Do Prob 3.9, 10 p. 86)

Ex: $-i\hbar \frac{\partial}{\partial x} \psi = p \psi$ or $\hat{D} f = \frac{\partial f}{\partial x} = \lambda f$

p. 88 OPERATOR · wave function = OBSERVABLE · wave function
= measurement · state

$\overleftrightarrow{T} \; \vec{a} = \lambda \; \vec{a}$

eigenvector = eigenfunction
eigenvalue

QM p. 88 and DiffEq Ch3: To find eigenvalues λ,

First: Solve $\det(\overleftrightarrow{T} - \lambda \overleftrightarrow{I}) = \begin{vmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{vmatrix} = 0$

Practice: example on p. 88, Probs. 3.18 (p. 91), 3.22, 3.23 (p. 94)

Alternate method: diagonalize T, and its elements are the λ!

next: For each eigenvalue λ_i, find its eigenvector $\vec{a}^{(i)}$.

The set of eigenvectors $\vec{a}$ spans $\overleftrightarrow{T}$'s space.

p. 97 LINEAR TRANSFORMATIONS keep ψ in the original space.

Surprisingly, $\hat{p}$ is a linear transformation but $\hat{x}$ is not,
(in space of P(N) = Nth order polynomials, $x\psi \to$ higher order polynomials)

HERMITIAN operators yield REAL observables. (Ex: $i\hat{D}$ but not $\hat{D}$)
$T = \tilde{T}^* = T^\dagger$

p. 99 Set of eigenvectors = complete basis ONLY in finite-dim. spaces
p. 101 HILBERT SPACE = complete basis.

p.104 $\underset{104}{\underline{3.3}}$ FUNDAMENTAL PRINCIPLES OF QM

1. State of particle = normalized $|\Psi\rangle$ wavefunction in the Hilbert space L_2 = set of all square-integrable fns

2. OBSERVABLES $\hat{Q}$ = Hermitian operators: $\hat{Q}|\Psi\rangle = \langle Q\rangle|\Psi\rangle$

EXPECTATION VALUES $\langle Q\rangle = \langle\Psi|Q\Psi\rangle = \int \Psi^* Q \Psi\, dx$
determinate states = eigenvalues

3. Measurement of observable Q is CERTAIN to yield eigenvalue λ ONLY if $|\Psi\rangle$ = eigenvector.

p.105 Otherwise, for $|\Psi\rangle$ = mixture of eigenvectors $= \sum_{n=1}^{\infty} c_n|e_n\rangle$, probability of measuring a given eigenvalue λ_n is
$P(\lambda_n) = |c_n|^2 = |\langle e_n|\Psi\rangle|^2$

p.107 Ex: Momentum eigenfunctions $|e_p(x)\rangle = Ae^{ikx}$ where $A = \frac{1}{\sqrt{2\pi\hbar}}$
x-space and $p = \hbar k$

Momentum-space wavefunction, on the other hand, is

[3,132] $\Phi(p,t) = \langle e_p|\Psi\rangle = A\int_{-\infty}^{\infty} e^{-ipx/\hbar}\,\Psi(x,t)\,dx$ = p-component of $\Psi(x,t)$

→ PRACTICE: $\underset{108}{\underline{\#3.36}}$ (uses $\underset{29}{\underline{2.\cancel{5}}}$)

$\frac{3.4}{109}$ UNCERTAINTY PRINCIPLE $\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2$

Recall that uncertainty$^2 = \sigma^2 = \langle j^2\rangle - \langle j\rangle^2$ p. 8

and Commutator $[\hat{A},\hat{O}] = AB - BA$ p. 83

Ex: Let $\hat{A} = x$ and $\hat{B} = \vec{p} = -i\hbar\frac{\partial}{\partial x}$.

Then $[\hat{x},\hat{p}]f(x) = xp\,f(x) - px\,f(x)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

So $\sigma_x^2 \sigma_p^2 =$

UNCERTAINTY: Cannot measure both A and B precisely at once.

A and B are "incommensurable" or "incompatible" observables.

The more precisely you measure one, the less precisely can you measure the other.

p. 111 Uncertainty is minimized ($\sigma_x \sigma_p = \frac{\hbar}{2}$) when $\psi \sim e^{-ax^2}$ = Gaussian.

$\frac{3.4.3}{112}$ ENERGY-TIME UNCERTAINTY: $\Delta E\,\Delta t \geq \frac{\hbar}{2}$

How fast does the system change? $\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$

$\therefore$ CONSERVED quantities COMMUTE with energy: $[\hat{H},\hat{Q}] = 0$

where $|f\rangle \equiv (\hat{A} - \langle A\rangle)|\Psi\rangle$. Likewise, for any *other* observable B,

$$\sigma_B^2 = \langle g|g\rangle, \quad \text{where } |g\rangle \equiv (\hat{B} - \langle B\rangle)|\Psi\rangle.$$

Therefore (invoking the Schwarz inequality, Equation 3.27),

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2. \qquad [3.135]$$

Now, for any complex number z,

$$|z|^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2 \geq (\text{Im}(z))^2 = [\frac{1}{2i}(z - z^*)]^2. \qquad [3.136]$$

Therefore, letting $z = \langle f|g\rangle$,

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2. \qquad [3.137]$$

But

$$\begin{aligned}
\langle f|g\rangle &= \langle(\hat{A} - \langle A\rangle)\Psi|(\hat{B} - \langle B\rangle)\Psi\rangle = \langle\Psi|(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\Psi\rangle \\
&= \langle\Psi|(\hat{A}\hat{B} - \hat{A}\langle B\rangle - \hat{B}\langle A\rangle + \langle A\rangle\langle B\rangle)\Psi\rangle \\
&= \langle\Psi|\hat{A}\hat{B}\Psi\rangle - \langle B\rangle\langle\Psi|\hat{A}\Psi\rangle - \langle A\rangle\langle\Psi|\hat{B}\Psi\rangle + \langle A\rangle\langle B\rangle\langle\Psi|\Psi\rangle \\
&= \langle\hat{A}\hat{B}\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle \\
&= \langle\hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle.
\end{aligned}$$

Similarly,

$$\langle g|f\rangle = \langle\hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle,$$

so

$$\langle f|g\rangle - \langle g|f\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle = \langle[\hat{A}, \hat{B}]\rangle,$$

where

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} \qquad [3.138]$$

is the commutator of the two operators. *Conclusion*:

$$\boxed{\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2.} \qquad [3.139]$$

This is the uncertainty principle in its most general form. (You might think the i makes it trivial—isn't the right side *negative*? No, for the commutator carries its own factor of i, and the two cancel out.)

For example, suppose the first observable is position ($\hat{A} = x$), and the second is momentum ($\hat{B} = (\hbar/i)d/dx$). To determine the commutator, we use an arbitrary "test function", $f(x)$:

$$[\hat{x}, \hat{p}]f(x) = x\frac{\hbar}{i}\frac{d}{dx}(f) - \frac{\hbar}{i}\frac{d}{dx}(xf) = \frac{\hbar}{i}\left[x\frac{df}{dx} - (f + x\frac{df}{dx})\right] = i\hbar f,$$

so

$$\boxed{[\hat{x}, \hat{p}] = i\hbar.} \qquad [3.140]$$

Accordingly,

$$\sigma_x^2\sigma_p^2 \geq \left(\frac{1}{2i}i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2,$$

or, since standard deviations are by their nature positive,

$$\sigma_x\sigma_p \geq \frac{\hbar}{2}. \qquad [3.141]$$

That proves the original Heisenberg uncertainty principle, but we now see that it is just one application of a far more general theorem: There will be an "uncertainty principle" for *any pair of observables whose corresponding operators do not commute*. We call them **incompatible observables**. Evidently, incompatible observables do not have shared eigenvectors—at least, they cannot have a *complete set* of common eigenvectors. Matrices representing incompatible observables cannot be simultaneously diagonalized (that is, they cannot both be brought to diagonal form by the *same* similarity transformation). On the other hand, *compatible* observables (whose operators *do* commute) share a complete set of eigenvectors, and the corresponding matrices *can* be simultaneously diagonalized (see Problem 3.40).