Physical Systems
Thus 31 Oct. 2002

fall Mechanics midterm
~~Electrostatics~~ oscillations

X Zita - Solutions

I affirm that I have worked this exam independently, without texts, outside help, integral tables, calculator, solutions, or software. (Please sign legibly.)

Read this cover page completely before you start.

Read through the whole exam before you start, and do the easiest problems first.
Don't stay on any one problem for more than half an hour.
Move on and come back to a problem later if necessary.

You may use your HOMEWORK ONLY for reference on this exam.
If you feel you need more information, don't hesitate to ask Zita.
There are no trick questions.

This is designed as a one-hour exam. You have two hours to do it.
Solutions will be made available after all exams are turned in.

Please show your work, and explain your methods concisely, for full credit.
Do the problems in the simplest meaningful way you know.

Leave expressions in their simplest exact form (e.g. $2\pi/\sqrt{3}$). If a numerical answer is required, feel free to make reasonable simplifying approximations such as $g \sim 10\ m/s^2$ or $\pi = 3$.

Always include units where appropriate, label axes, and so forth.

Please indicate your answer clearly, e.g. by boxing it.
If you need more space for your work, feel free to attach additional sheets or use the back.
Direct me to look on back if necessary, or staple any extra sheet(s) in order neatly.

A meaningful method and demonstrated understanding of concepts is more important than memorized equations or a numerical answer.

******************** Leave the space below for Zita to make notes************************

oscillators – equilibrium – force – energy – freq – TP

force (t) → V

force (x) → V

damping & driving

Sketches & physical interpretation

fundamental relationships & definitions

A. **Qualitative questions about MECHANICS:**

1. **Sketch** the potential energy for any oscillating system.

2. What can you say about each of these at a **stable equilibrium point** of an oscillator?

	Max	min	zero	constant	variable
(a) Potential energy		✓			
(b) Kinetic energy	✓				
(c) Force			✓		
(d) Displacement			✓		
(e) Velocity	✓				
(f) Acceleration			✓		
(g) Total energy				✓	

3(a). What **form of solution** would you guess for the differential equation $\frac{d^2x}{dt^2} = -bx$?

$x = X_0 e^{i\omega t}$ or $x = A\cos\omega t + B\sin\omega t$

or $x = \alpha\cos(\omega t + \varphi)$ or $x = \beta\sin(\omega t + \varphi)$

(b) What is the **angular frequency** of the oscillator, in terms of parameters in the diffeq above?

$\frac{dx}{dt} = i\omega X_0 e^{i\omega t} = i\omega x$

$\frac{d^2x}{dt^2} = i\omega(i\omega x) = -\omega^2 x = -bx \longrightarrow \omega = \pm\sqrt{b}$

(c) How would the **frequency change** if you added some **damping** to the system?
Increase? $\boxed{\text{Decrease?}}$ Stay the same? (No calculations necessary!)

(d) If you wanted to **drive the system at resonance**, what driving frequency would you choose?
Higher than the frequency in part (c).
$\boxed{\text{Lower than the frequency in part (c).}}$
The same as frequency as in part (c).
The same frequency as in part (b).

B. Basic relations and motion

1. For the force $\vec{F}(t) = -\hat{j}\, F_0 \cos \omega t$ (where *t* is the only variable) $F = ma = m\frac{dv}{dt}$

(a) Find the **velocity** *v(t)*. Show your work!

$v = \int \frac{dv}{dt}\,dt = \int \frac{F}{m}\,dt = -\frac{F_0}{m}\int \cos\omega t\,dt = -\frac{F_0}{m\omega}\sin\omega t + c$

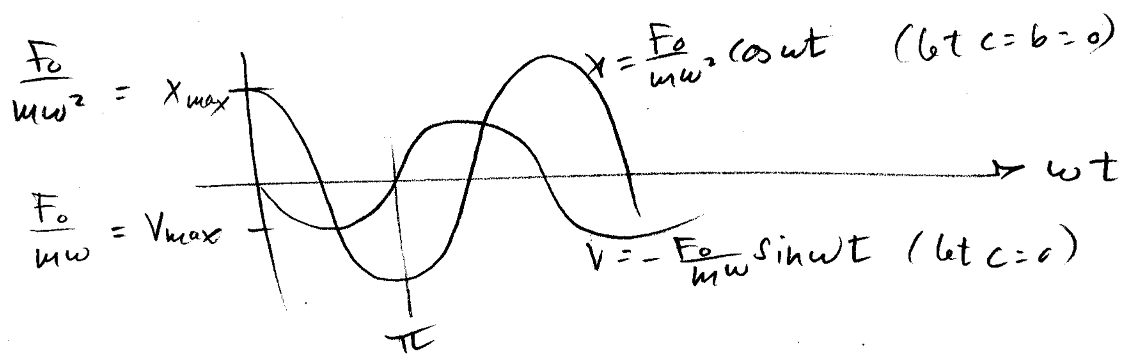
$\vec{v} = \left(-\frac{F_0}{m\omega}\sin\omega t + c\right)\hat{j}$

(b) Find the **displacement** *x(t)* and **describe** the motion. $v = \frac{dx}{dt}$

$x = \int \frac{dx}{dt}\,dt = \int v\,dt = \int\left(-\frac{F_0}{m\omega}\sin\omega t + c\right)dt$

$\vec{x} = \left(+\frac{F_0}{m\omega^2}\cos\omega t + ct + b\right)\hat{j}$

(c) **Sketch** *v(t)* and *x(t)* on the same plot. **Label** key points on your axes, including extrema.



2. For the force $\vec{F}(x) = \hat{i}\, F_0\, e^{-2kx}$ (where *x* is the only variable)

(a) Find the **velocity** *v(x)* and **describe** the motion. Show your work! $F = m\frac{dv}{dt} = m\frac{dx}{dt}\frac{dv}{dx} = mv\frac{dv}{dx}$

$\int v\,dv = \frac{v^2}{2} = \int \frac{F}{m}\,dx = \int \frac{F_0}{m}e^{-2kx}\,dx = -\frac{F_0}{2km}e^{-2kx} + c'$

$\vec{v} = \pm\left[-\frac{F_0}{km}e^{-2kx} + c\right]^{1/2}\hat{i}$

(b) Find the **potential energy** *U(x)*.

$U = -\int \vec{F}\,d\vec{x} = +\frac{F_0}{2k}e^{-2kx} + U_0$

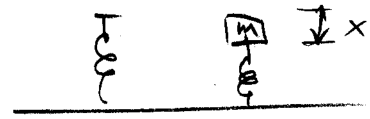
from Giancoli Ch 8 #3

C. **Oscillators**: a simple example and a general case

1. **When a 100 kg person climbs into a 1000 kg car, the car's springs compress vertically by 2 cm.**

$=m$ (100 kg), $=M$ (1000 kg), $=x$ (2 cm)

(a) What will be the frequency of vibration when the car hits a bump?



$\Sigma F = 0$

$mg = kx$

$k = \frac{mg}{x} = \frac{100\,kg \times 10\,\frac{m}{s^2}}{2\times10^{-2}\,m}$

$k = 5\times10^4 \left(\frac{kg}{s^2} = \frac{N}{m}\right)$

$\omega^2 = \frac{k}{m_{tot}} = \frac{k}{m+M} = \frac{5\times10^4\,\frac{kg}{s^2}}{1.1\times10^3\,kg}$

$\boxed{\omega = \sqrt{\frac{50}{1.1}}\ \frac{rad}{s}}$

(b) If there are six people in the car, will the frequency of ~~vibration increase, decrease~~ ase, or stay the same? Why? (No calculation necessary.)

→ greater mass, same k ⟶ lower frequency

(c) If the car hits a bump 5 cm high, write an equation that describes the oscillation. Label all your quantities and include units where appropriate.

$x(t) = A\cos(\omega t)$

t = time from bump

$\omega = \sqrt{\frac{k}{m_{tot}}}$

$A = 5$ cm

2. **If you know the total force** $\vec{F}(r) = F(r)\,\hat{r}$ **for an oscillating system:** (short answers - no calculations)

(a) How can you find the equilibrium displacement r_0?

It's where $F(r) = 0$

(b) How can you find the potential energy U(r)?

$U = -\int F\,dr$

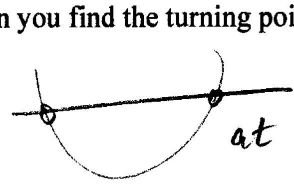
(c) If you sketched F and U, which would vanish at equilibrium? Which would be a miminum?

$F = 0$ $U = min$

(d) If you know the kinetic energy T(0) = T(t=0) for a given initial displacement r(0) = r(t=0), how can you find the total energy E?

$E = T(0) + U(0)$ where $U(0) = -\int F\,dr$ (at $t=0$)

(e) How can you find the turning points?



at TP, $U = E$ and $T = 0$.

D. **Diatomic molecules** From Giancoli Ch 14 # 70

The potential energy of the two atoms in a diatomic (two-atom) molecule can be written $U = -\frac{a}{r^6} + \frac{b}{r^{12}}$, where r is the separation between the two atoms and a and b are positive constants.

(a) At what value of r is $U(r)$ a minimum? A maximum? Show that the equilibrium separation $r_0 = \left(\frac{2b}{a}\right)^{1/6}$.

$\frac{\partial U}{\partial r} = +\frac{6a}{r^7} - \frac{12b}{r^{13}} = 0$ when $U =$ max or min. This happens for

$r \to \infty : U \to 0$ (local min) (also note that $\lim_{r \to 0} U = \infty$) ; global max

and for $6ar^6 = 12b$

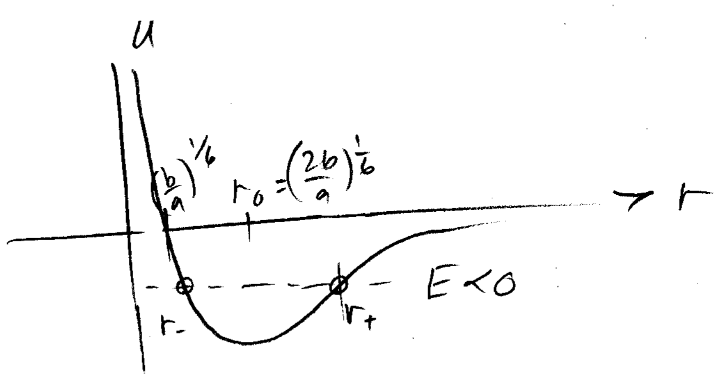
$r^6 = \frac{12b}{6a} = \frac{2b}{a} \to r_0 = \left(\frac{2b}{a}\right)^{1/6}$; global min

(b) At what value of r is $U(r) = 0$?

$U = 0$ when $\frac{a}{r^6} = \frac{b}{r^{12}}$

$ar^6 = b \to r = \left(\frac{b}{a}\right)^{1/6}$

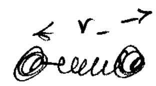
(c) SKETCH $U(r)$ as a function of r from $r = 0$ to several times r_0. (Do not spend a lot of time calculating points – just use what you found out above).

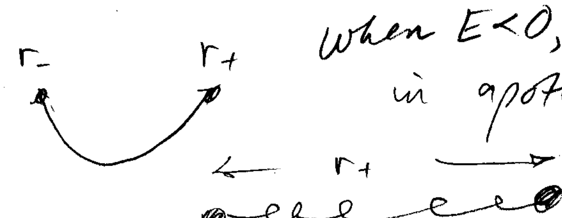


(d) Describe the motion of one atom with respect to the other when the total energy E<0.

when $E<0$, we have oscillations in a potential energy well

closest together at r_-

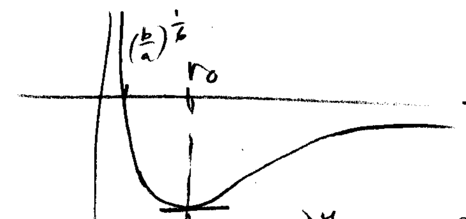




max separation between atoms at r_+

(e) Let F be the force each atom exerts on the other. For what values of r is $F>0$? $F<0$? $F=0$? When is the force attractive? Replusive?
(No new calculations needed! Look at your plot of U(r) and consider the relation between U and F)

$F = -\frac{\partial U}{\partial r}$: negative slope of U



$(r<r_0)$ $\frac{\partial u}{\partial r}<0$ repulsive $F>0$

$\frac{du}{dr}=0$ $F=0$

$\frac{\partial u}{\partial r}>0$ for $(r>r_0)$ $F<0$; attractive

(f) Calculate F as a function of r.

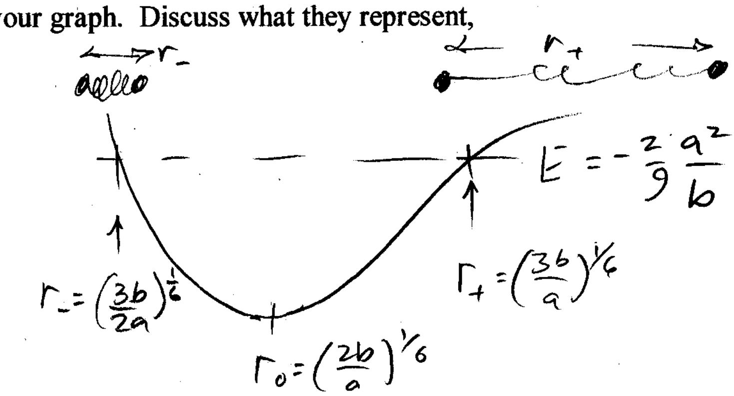
$$F = -\frac{\partial U}{\partial r} = -\frac{6a}{r^7} + \frac{12b}{r^{13}}$$

done in part (a)

(g) If the kinetic energy is zero when the initial separation between the two molecules is $r_+ = \left(\frac{3b}{a}\right)^{1/6}$ (that is, $T(r_+) = 0$), what is the **total energy** of the system? Indicate it on your graph.

$$E = T(r_+) + U(r_+)$$
$$= 0 + U(r_+) = \frac{-a}{(3b/a)} + \frac{b}{(3b/a)^2}$$
$$= -\frac{a^2}{3b} + \frac{a^2 b}{9b^2} = \frac{a^2}{b}\left(-\frac{1}{3} + \frac{1}{9} = -\frac{3}{9} + \frac{1}{9} = -\frac{2}{9}\right)$$

(h) Find the **turning points** and indicate them on your graph. Discuss what they represent, physically. (see bottom of p. 5)



There is another place, r_-, the closest approach of the two atoms, where $T(r_-)=0$

and $E = U(r_-)$

$$-\frac{2}{9}\frac{a^2}{b} = -\frac{a}{r^6} + \frac{b}{r^{12}} \quad \text{multiply by } r^{12}$$

$$-\frac{2}{9}\frac{a^2}{b} r^{12} = -ar^6 + b \quad \text{let } r^6 = y$$

$$\frac{2}{9}\frac{a^2}{b} y^2 - ay + b = 0 \quad \text{multiply by } \frac{9}{2}\frac{b}{a^2}$$

$$y^2 - \frac{9}{2}\frac{b}{a} y + \frac{9}{2}\frac{b^2}{a^2} = 0 \quad \text{solve for } y$$

$$2y = +\frac{9}{2}\frac{b}{a} \pm \sqrt{\left(\frac{9}{2}\frac{b}{a}\right)^2 - 4\frac{9}{2}\frac{b^2}{a^2}} = \frac{9}{2}\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2}\left(\frac{9 \cdot 9}{4} - \frac{2 \cdot 9 \cdot 4}{4}\right)} = \frac{b}{a}\left(\frac{9}{2} \pm \sqrt{\frac{1 \cdot 9}{4}}\right) \to$$

over

Turning points where $r^6 = y$ and

$$2y = \frac{b}{a}\left(\frac{9}{2} \pm \frac{3}{2}\right) = \frac{b}{a}\left(\frac{12}{2} \text{ or } \frac{6}{2}\right)$$

$$= \frac{b}{a}(6 \text{ or } 3)$$

$$y = \frac{6}{2}\frac{b}{a} = r^6 \quad \text{or} \quad y = \frac{3}{2}\frac{b}{a} = r^6$$

$$\left(3\frac{b}{a}\right)^{1/6} = r_+ \qquad \boxed{\left(\frac{3}{2}\frac{b}{a}\right)^{\frac{1}{6}} = r_-}$$

we already know about this turning point

This is the second TP we were looking for.

Physical Systems
Thus 31 Oct. 2002

fall Mechanics midterm
Electrostatics

How do you feel about this exam, now that you have taken it?

ok! I had to be careful with masses in C.1
and with energies and algebra in D.4

What question(s) do you wish had been on this exam?

Maybe something quantitative on damped or
driven oscillators, or
an LC or RLC circuit

Challenge problem (only ifyou have extra time):

If the effective force constant of a system can be found from $k = \left.\frac{d^2U}{dx^2}\right|_{x_0}$, then what is the frequency of oscillation of either mass m in the diatomic molecule above, considering the other mass as fixed?

$$\frac{du}{dr} = \frac{6a}{r^7} - \frac{12b}{r^{13}}$$

$$\frac{d^2u}{dr^2} = \frac{-7 \cdot 6a}{r^8} - \frac{12(-13)b}{r^{14}} = \frac{6}{r^8}\left(-7a + \frac{2 \cdot 13 b}{r^6}\right)$$

$$\left.\frac{d^2u}{dr^2}\right|_{r_0 = \left(\frac{2b}{a}\right)^{1/6}} = \frac{6}{\left(\frac{2b}{a}\right)^{8/6}}\left(-7a + \frac{2 \cdot 13 b}{\left(\frac{2b}{a}\right)^{6/6}}\right) = \frac{6}{(2b/a)^{3/4}}\left(-7a + \frac{2 \cdot 13 b}{(2b/a)}\right)$$

$$k = 6\left(\frac{a}{2b}\right)^{3/4}(-7a + 13a) = 12a\left(\frac{a}{2b}\right)^{3/4}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{12a}{m}\left(\frac{a}{2b}\right)^{3/4}}$$