1.

Thus 4 Feb 98

LAST WEEK I derived Schrödinger eqn

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi.}$$

from Energy Conservation:

$$\hat{E}\Psi = \frac{\hat{p}^2}{2m}\Psi + \hat{V}\Psi$$

. Now derive momentum operator $\hat{p}$!

Now the Schrödinger equation says that

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi, \qquad [1.23]$$

and hence also (taking the complex conjugate of Equation 1.23)

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*, \qquad [1.24]$$

$$\frac{\partial}{\partial t}|\Psi^*\Psi| = \Psi^*\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial t}\Psi$$

$$= \left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}\Psi^* V\Psi}\right) + \left(\frac{i\hbar}{2m}\right)\Psi\frac{\partial^2\Psi^*}{\partial x^2} + \cancel{\frac{i}{\hbar}\Psi V\Psi^*}$$

$$\frac{\partial}{\partial t}\Psi^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \Psi\frac{\partial^2\Psi^*}{\partial x^2}\right)$$

so

$$\frac{\partial}{\partial t}|\Psi|^2 = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\right]. \qquad \boxed{[1.25]}$$

The integral (Equation 1.21) can now be evaluated explicitly:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\bigg|_{-\infty}^{+\infty}. \qquad [1.26]$$

But $\Psi(x,t)$ must go to zero as x goes to ($\pm$) infinity—otherwise the wave function would not be normalizable. It follows that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}|\Psi(x,t)|^2\,dx = 0, \qquad [1.27]$$

and hence that the integral on the left is *constant* (independent of time); if Ψ is normalized at $t = 0$, it *stays* normalized for all future time. QED

2 1

state Ψ, the expectation value of x is

$$\boxed{\langle x \rangle = \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2\,dx.} \qquad [1.28]$$

might be interested in knowing how fast it moves. Referring to Equations 1 1.28, we see that[9] 1,25 f

$$\frac{d\langle x\rangle}{dt} = \int x\frac{\partial}{\partial t}|\Psi|^2\,dx = \frac{i\hbar}{2m}\int x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx. \qquad [1.29]$$

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g,$$

IBP

$$\int_a^b f\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}g\,dx + fg\Big|_a^b.$$

ssion can be simplified using integration by parts[10]:

$$\int \underset{f}{x}\,\frac{\partial}{\partial x}\underbrace{\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)}_{g}dx = \int f\frac{\partial g}{dx}\,dx$$

$$= -\int\frac{dx}{dx}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)dx$$

$$= -\int\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)dx \quad - \cancel{fg} \to 0$$

$$\frac{d\langle x\rangle}{dt} = \frac{i\hbar}{2m}\left[-\int\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)dx\right]$$

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{2m}\int\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx. \qquad [1.30]$$

the fact that $\partial x/\partial x = 1$, and threw away the boundary term, on the ground goes to zero at ($\pm$) infinity.] Performing another integration by parts on the term, we conclude that

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int \Psi^*\frac{\partial\Psi}{\partial x}\,dx.$$

$$\int\frac{\partial\psi^*}{\partial x}\psi\,dx = \int\psi\frac{\partial\psi^*}{\partial x}\,dx =$$

$$= -\int\frac{\partial\psi}{\partial x}\psi^*\,dx + \cancel{\psi\psi^*}\Big|_{-\infty}^{\infty} \to 0$$

$$\int\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right)dx$$

$$= \int\left(\psi^*\frac{\partial\psi}{\partial x} + \psi^*\frac{\partial\psi}{\partial x}\right)dx = 2\int\psi^*\frac{\partial\psi}{\partial x}\,dx$$

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{2m}\,2\int\psi^*\frac{\partial\psi}{\partial x}\,dx = -\frac{i\hbar}{m}\int\psi^*\frac{\partial\psi}{\partial x}\,dx$$

3

$$\langle v \rangle = \frac{d\langle x \rangle}{dt}. \qquad [1.32]$$

ells us, then, how to calculate $\langle v \rangle$ directly from Ψ.
it is customary to work with momentum ($p = mv$), rather than ve-

$$\boxed{\langle p \rangle = m\frac{d\langle x \rangle}{dt} = -i\hbar \int \left(\Psi^* \frac{\partial \Psi}{\partial x} \right) dx.} \qquad [1.33]$$