Part I: In Class Test

1. Differentiate with respect to x
   (a) $5x^3 + 4$
   (b) $e^x \cos x$
   (c) $\ln(1 + x^2)$
2. Integrate with respect to x
   (a) $\int (e^x + e^{-x})\,\mathrm{d}x$
   (b) $\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$ (use substitution)
   (c) $\int x \ln x\,\mathrm{d}x$ (use integration by parts)

3. Find an expression in terms of a for the area enclosed by the curve with equation $y = x(a^2 - x^2)$ and the x-axis between $x = 0$ and $x = a$.

4. The normal distribution (or bell curve) has probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

The area under this curve between $x = -1$ and $x = 1$ represents the probability that measured data values will occur within one standard deviation of the mean. To calculate this area we will use a Taylor approximation.

(a) First write down the sixth order Taylor polynomial approximation of f(x). (Hint: It is easiest to start with the Taylor polynomial for e^u).

(b) Use this Taylor polynomial to find the approximate value of the integral $\int_{-1}^{1} f(x)\, dx$.

(c) How does your value compare with the often quoted value of 70%?

5. Evaluate the following integral, using either integration by parts or trig substitution.

$$\int_0^{\pi/2} \sin^2 x \, dx$$

6. Given the function $w = x^3y^2 - 2xy^4$ find

(a) $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$

(b) $\dfrac{\partial^2 w}{\partial x^2}$ and $\dfrac{\partial^2 w}{\partial y^2}$

(c) $\dfrac{\partial^2 w}{\partial x \partial y}$

7. A function $z = f(x, y)$ passes through the point $P = (1, 2, 4)$ and has first partial derivatives $z_x = -2$ and $z_y = 1$ at P.

   (a) Write down the equation of a linear function that has those properties.

   (b) Draw a contour plot of the linear function showing contours for $z = 0, 2, 4$ and 6.

   (c) At the point P in what direction is the slope of $z = f(x, y)$ steepest and what is the slope in that direction?

Part II: Take Home Test

Please complete all questions and show your work. This test should be completed individually. You may use your textbook and notes, but you must not collaborate with other people. The test is due on Tuesday, June 1st at 9:00 am.

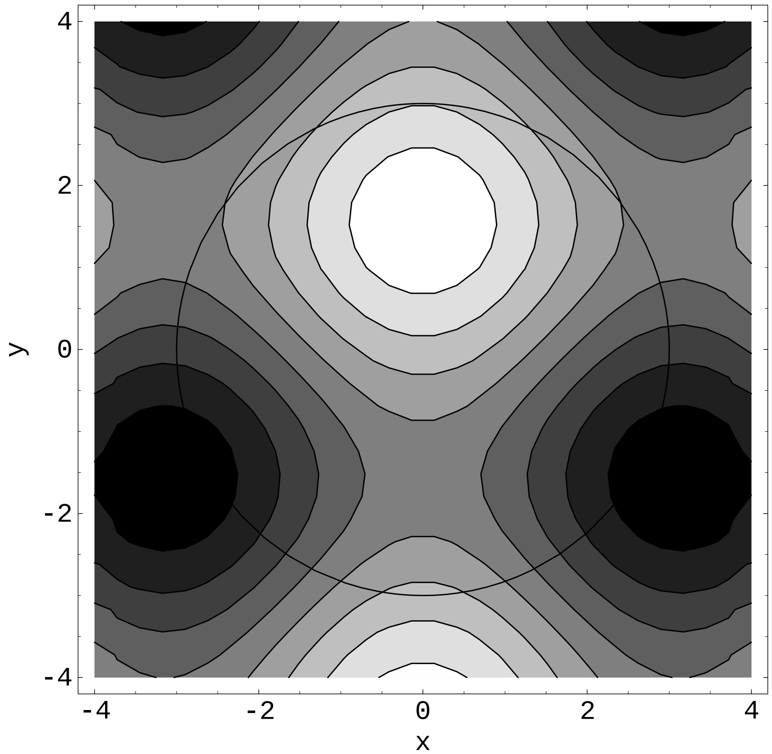
1. Given that

$$y = \frac{\sin x}{x},$$

find y' and y'' and hence show that $xy'' + 2y' + xy = 0$.

2. A cylindrical can of radius r and height h is heated. The radius increases at a uniform rate of 0.0015 cm/min and the height increases at a rate of 0.0025 cm/min. Use the microscope equation expressing changes in volume as a function of changes in radius and height to find the rate of increase of the volume of the can when r=2 cm and h=6 cm. (You will need the formula for the volume of a cylinder as a function or radius and height in order to derive the microscope equation.)

3. The contour plot shown in the figure below is defined by the equation $z = \cos(x) + \sin(y)$ the points on this surface with x and y coordinates falling on a circle of radius 3 centered on the origin are also shown



(a) Locate and label all the maxima, minima and saddle points on the plot, making sure to distinguish which is which. Use the equation and the graph to determine the value the coordinates of each point. You should not need to take partial derivatives. Also, remember, that points have three coordinates.

(b) Determine the coordinates of the points along the curve that are local maxima.

(c) Sketch gradient vectors on the contour plot at the points (-2,0) and (0,0).

4. (a) Find the first three terms in the Taylor expansion of $\dfrac{1}{\sqrt{1-u}}$ centered at $u = 0$.

(b) Using the first two terms of this expansion show that

$$\frac{1}{\sqrt{1 - k^2 \sin^2 x}} \approx 1 + \frac{1}{2} k^2 \sin^2 x$$

if k is small.

(c) The period of a pendulum of length L with an initial amplitude θ_0 is

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 x}} \, \mathrm{d}x$$

where $k = \sin(\theta_0/2)$. Evaluate this integral using the approximation for the integrand evaluated in (b) to find an approximate expression for the period of a pendulum in terms of L, g and k.

5. Suppose the temperature in Fahrenheit on a 2 ft by 2 ft square concrete wall is given by the expression $T = 10x^2y + 40$, where x and y are horizontal and vertical coordinates measured in feet starting from the bottom left corner of the square. A small snail moves along the concrete in the direction of the temperature gradient $\vec{\nabla}T$.

   (a) If the snail starts off half way along the bottom edge at (1,0) find its initial direction of motion.

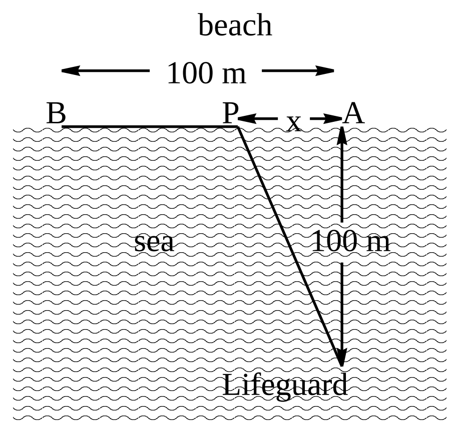
   (b) The snail travels along a path described by some function $y = f(x)$. Since the snail follows the temperature gradient the slope of this function is given by given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = T_y/T_x$. Hence show that the path of the snail satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2y} \,.$$

   (c) Use the method of separation of variables to solve the differential equation in (c) with initial position (1,0) and hence find the function $y = f(x)$ describing the path of the snail.

6. A lifeguard is in the sea 100 m opposite a point A on the shore. She wishes to arrive at point B, which is 100 m along the shore from A, as quickly as possible. She decides to swim towards a point P which is x metres along the shore from A and then run from P to B. If she swims at 1 m/s and runs at 10 m/s show that an expression for the time taken, T in seconds, is given by

$$T = \sqrt{10000 + x^2} + \frac{100 - x}{10}$$



Hence find the value of x for which the time T is minimum.

7. The problem of finding the location of the intersection between $y = e^{-x}$ and $y = \sin(x)$ as shown in the diagram below is equivalent to finding the zeros of the function $f(x) = \sin(x) - e^{-x}$. Unfortunately there is no analytical method for solving this equation. Instead use Newton's method to find the indicated point of intersection (x, y) accurate to 4 decimal places. Choose an initial guess of x=0. Make sure you show you method for the first two steps in Newton's method on the back of this page.