In Class Portion

1. Given the function

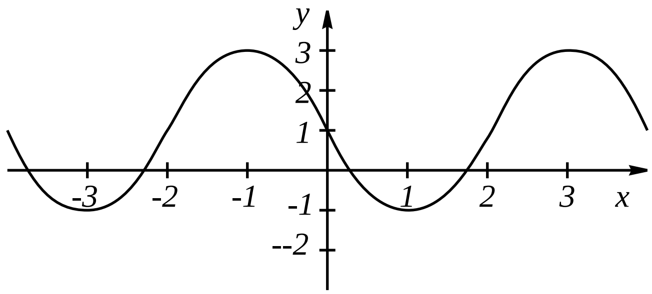
$$z = x^3 - x^2 - y^2 + 2xy - 3x$$

   (a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$

   (b) Find $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$

   (c) Find all the critical points of the function and use the second partial derivatives to establish whether they are maxima, minima or saddle points.

2. The equation of the graph below can be expressed in the form $y = a\cos(bx + \phi) + c$.



(a) Find the value of a, b, c and ϕ.

(b) Find the slope of the graph at $x = 0$.

3. A dynamical system in two variables x and y satisfies the following equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y\sin x \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \cos x \end{aligned} \tag{1}$$

(a) Find the equilibrium point(s) of this dynamical system.

(b) Show that the function $E = \dfrac{e^{y^2/2}}{\sin x}$ is a first integral of this system. (That is, show that E is constant).

4. A function $z = f(x, y)$ passes through the point $P = (1, 2, 8)$ and has first partial derivatives $z_x = 3$ and $z_y = 1$ at P.

   (a) Write down the equation of a linear function that has those properties.

   (b) Draw a contour plot of the linear function showing contours for z=0,2,4 and 8.

   (c) At the point P in what direction is the slope of $z = f(x, y)$ steepest and what is the slope in that direction?

Take Home Portion

Please complete all questions and show your work. This test should be completed individually. You may use your textbook and notes, but you must not collaborate with other people.

1. By finding the appropriate partial derivatives show that the function $h(x, y) = e^{-y/b^2} \sin b\,x$ satisfies the following partial differential equation.

$$\frac{\partial h}{\partial y} = \frac{\partial^2 h}{\partial x^2}$$

2. Well respected biologists have shown that the human body surface area A depends on the persons weight W in pounds and their height H in feet according to the power law

$$A = W^{0.4} H^{0.8}$$

where A is measured in square feet.

(a) Write down the microscope equation expressing ΔA in terms of ΔW and ΔH.

(b) A particular teenage boy is 6 ft and weighs 170 pounds and he is still growing in height at a rate of 0.2 ft a year and gaining weight at a rate of 5 pounds a year. Find the approximate rate at which his surface area is changing.

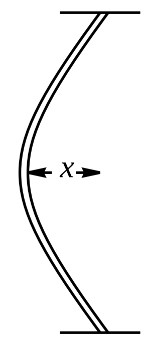
3. Given the function $z = 12x - 3y - x^3 + y^3$

(a) Find all the critical points of the the function and indicate whether they are maxima, minima or saddle points?

(b) Use Mathematica to plot a coutour plot of the function in the region $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. Locate the all the maxima, minima of this region.

(c) Find the maximum and minimum values of z along the curve $y = x$ and locate these on your contour plot?

4. A thin flexible vertical rod is compressed until it bows outward (i.e. it buckles). The following pair of differential equations describes the dynamics of the motion when the rod is then disturbed from this position. x represents the horizontal displacement of the center of the rod from the vertical axis and v the velocity of that point.



$$\frac{dv}{dt} = -ax - bx^3 - cv$$
$$\frac{dx}{dt} = v$$

For the following questions use the parameters values $a = -2$, $b = 1$, and $c = 2/10$

(a) Where are the equilibrium points for this system?

(b) From a starting state of $x = 1, v = 1$, estimate the state of the system 0.1 seconds later using a single Euler step.

(c) Given the initial conditions $x = 1, v = 1$, which equilibrium point will the system be attracted to? (You can use Euler's method on the computer to answer this question.)

(d) Using an initial displacement of $x = 1$, draw a phase portrait for the system showing trajectories for 10 different starting v values ranging from $v = 0$ to $v = 3$. Identify which initial conditions are attracted to which equilibrium point. Print your program and your graph.

(e) Set the parameter c to zero and plot the phase portrait using the same initial conditions used in part 4. Describe how the trajectories change when c is set to 0. What do you think c represents physically in the system.

(f) Using the "Taxonomy of Equilibrium Points" in Callahan Section 8.3, classify each equilibrium point in this system for both the $c = 0.2$ case and the $c = 0$ case.