Part I

1. A particle executes simple harmonic motion. When the velocity of the particle is a maximum which one of the following gives the correct values of potential energy and acceleration of the particle.

   (a) potential energy is maximum and acceleration is maximum.

   (b) potential energy is maximum and acceleration is zero.

   (c) potential energy is minimum and acceleration is maximum.

   (d) potential energy is minimum and acceleration is zero.

   Answer (d). When velocity is maximum displacement is zero so potential energy and acceleration are both zero.

2. A mass vibrates on the end of the spring. The mass is replaced with another mass and the frequency of oscillation doubles. The mass was changed by a factor of

   (a) 1/4 (b) 1/2 (c) 2 (d) 4

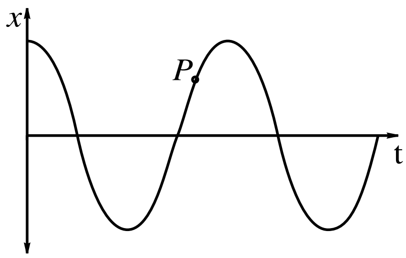
   Answer (a). Since the frequency has increased the mass must have decreased. frequency is inversely proportional to the square root of mass, so to double frequency the mass must change by a factor of 1/4.

3. A mass vibrates on the end of the spring. The mass is replaced with another mass and the frequency of oscillation doubles. The maximum acceleration of the mass

   (a) remains the same. (b) is halved. (c) is doubled. (d) is quadrupled.

   Answer (d). Acceleration is proportional to frequency squared. If frequency is doubled than acceleration is quadrupled.

4. A particle oscillates on the end of a spring and its position as a function of time is shown below.



   At the moment when the mass is at the point P it has

   (a) positive velocity and positive acceleration

   (b) positive velocity and negative acceleration

   (c) negative velocity and negative acceleration

   (d) negative velocity and positive acceleration

   Answer (b). The slope is positive so velocity is positive. Since the slope is getting smaller with time the acceleration is negative.

Part II

1. A clock maker wants to design a grandfather clock which keeps time from a 1.0 kg mass which vibrates on the end of a spring.

   (a) What should the spring constant be if the mass is designed to oscillate with a period of 1 second?

   $T = 2\pi\sqrt{m/k} \;\Rightarrow\; 1.0 = 2\pi\sqrt{1/k} \;\Rightarrow\; \sqrt{k} = 2\pi \;\Rightarrow\; k = 4\pi^2 = 39.5$ N/m.

   (b) After constructing the clock she notices that on a particularly hot day the clock does not keep the correct time. Explain what might be happening to cause this?

   As the spring heats up it lengthens and so the spring constant decreases. So we would expect the period to increase.

   (c) After careful observation she determines that the clock is losing 1 second every minute. What is the actual period and what is the actual spring constant.

   The clock ticks 59 times in 60 seconds so the frequency is 59/60 ticks per second and the period is 60/59=1.02 seconds.

   Solving the equation $T = 2\pi\sqrt{m/k}$ for the spring constant with this period gives $1.02 = 2\pi\sqrt{1/k} \;\Rightarrow\; \sqrt{k} = (2\pi)/1.02 \;\Rightarrow\; k = 38.2$ N/m.

   (d) To compensate for this problem she decides to replace the 1.0 kg mass on the end of the spring with a different one. What should the new mass be?

   With a spring constant $k = 38.2$ N/m and a desired period of $T = 1.0$ seconds we solve the equation $T = 2\pi\sqrt{m/k}$ for m now.

   $T^2 = 4\pi^2 m/k \;\Rightarrow\; m = T^2 k/(4\pi^2) = 1.0^2(38.2)/(4\pi^2) = 0.967$ kg.

2. A small mass rests on a horizontal platform which vibrates vertically in simple harmonic motion with period 0.50 s.

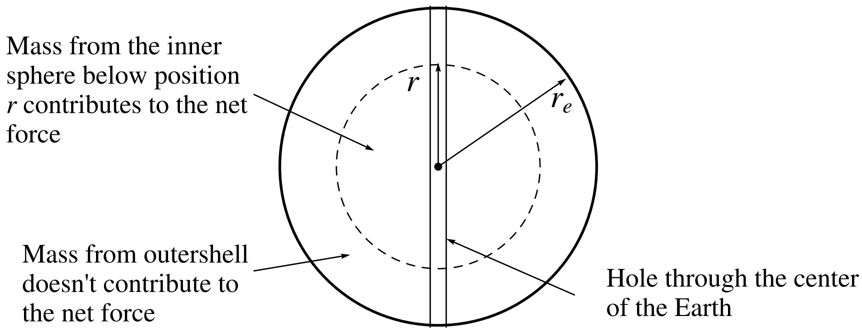
   (a) Find the maximum amplitude of the motion which will allow the mass to stay in contact with the platform throughout the motion.
   The maximum acceleration that will allow the object to remain in contact with the platform at all times is when $a_{\text{max}} = g = 9.81$ m/s^2.
   But $a_{\text{max}} = \omega^2 A = (2\pi/T)^2 A \Rightarrow 9.81 = (2\pi/0.5)^2 A = 158A \Rightarrow A = 0.062$ m

   (b) Assuming the mass oscillates at this amplitude what is the maximum speed of the mass?
   $v_{\text{max}} = \omega A = (2\pi/T)A = (2\pi/0.5)0.062 = 0.78$ m/s

   (c) Assuming the mass starts at equilibrium with the speed specified in (b) write down an expression for the position of the mass as a function of time.
   In simple harmonic motion the position is given by the expression $x = A\cos(\omega t + \phi)$. We have $A = 0.062$ m and $\omega = 2\pi/T = 12.5$ rad/s. Since this particle starts at equilibrium with an initial velocity we take $\phi = \pi/2$ if it starts moving down (in the negative direction) and $\phi = -\pi/2$ if it starts moving up (in the positive direction). So $x = 0.062\cos(12.5t \pm \pi/2)$

3. If a hole were drilled from one side of the Earth to the other it would be possible to jump from one side of the Earth to the other. Interestingly, the ride would end up being periodic – that is you would return repeatedly to your side of the Earth. Under certain assumptions the motion would be simple harmonic. In this question your task is to find the period of the motion and from this the maximum speed at the center of the earth.

   In order to find the period it is first necessary to set up the equation of motion from Newton's second Law. $F_{\text{net}} = ma$. The net force comes from Newton's Universal Law of Gravitation. $F = -GMm/r^2$. Here r is the distance of the jumping person from the center of the Earth at any given moment and M is the mass of that part of the Earth contained inside a sphere of radius r. (It turns out the mass outside this radius will pull on the person with equal force in all directions and hence will not contribute to the net force). See the diagram below.



In this model we will assume the density of the Earth is uniform (it is not!).

(a) Write down an expression for the mass M in terms of the mass of the earth M_e, the radius of the earth r_e and the distance from the center of the earth r. (Hint: you can save yourself a lot of algebra if you use the fact that mass is proportional to the cube of the radius when density is uniform.)
Since mass is proportional to radius cubed we have

$$\frac{M}{r^3} = \frac{M_\mathrm{e}}{r_\mathrm{e}^3} \ .$$

Therefore

$$M = M_\mathrm{e}\left(\frac{r^3}{r_\mathrm{e}^3}\right) \ .$$

(b) Show that with this expression for the mass M Newton's second law reduces to the form

$$a = -\omega^2 r \ .$$

That is acceleration is proportional to the distance from the center of the earth and the constant of proportionality is called ω^2. This relationship means the motion is simple harmonic. Write down an expression for ω^2.
Newton's second law implies

$$ma = -\frac{GMm}{r^2} \ \Rightarrow \ a = -\frac{GM}{r^2} = -\frac{GM_\mathrm{e}\left(\frac{r^3}{r_\mathrm{e}^3}\right)}{r^2} \ .$$

This reduces to

$$a = -\left(\frac{GM_\mathrm{e}}{r_\mathrm{e}^3}\right) r \ .$$

Thus the proportionality constant is $\omega^2 = GM_\mathrm{e}/r_\mathrm{e}^3$

(c) Recall that at the surface of the earth the acceleration due to gravity is $g = GM_e/r_e^2$. Write your expression for ω in terms of g and r_e. Evaluate ω and hence determine the period of the simple harmonic motion.

It follows from part (c) that $\omega^2 = g/r_e$. You may recognize this as the angular frequency of an earth surface orbit. This evaluates to $\omega^2 = 9.8/(6.37 \times 10^6) = 1.54 \times 10^{-6}$ so that $\omega = 1.24 \times 10^{-3}$ radians per second. Since $T = 2\pi/\omega$ we get $T = 2\pi/(1.24 \times 10^{-3}) = 5.1 \times 10^3$ seconds. This is 84 minutes. This turns out to be the same orbital period for an earth surface orbit.

(d) How long does it take to get to the other side of the earth if you jump into the hole with zero initial speed? How fast will you be traveling when you pass the center of the earth?

It takes half a period, which is 42 minutes. The speed at the center of the earth is given by $v_{\max} = A\omega$ where $A = r_e$ in this case. Thus $v = (6.37 \times 10^6)(1.24 \times 10^{-3} = 7900$ m/s.

(e) What is the maximum kinetic energy of a 70 kg person jumping in this hole? Hence or otherwise determine the the effective "spring constant" k for this person oscillating in this hole in the earth.

The maximum kinetic energy is $KE_{\max} = \frac{1}{2}mv^2 = \frac{1}{2}(70)(7900)^2 = 2.18$ GJ. This is a lot of energy! It comes from the initial potential energy of the person jumping. We now set this value for the maximum kinetic energy equal the to the expression for the maximum potential energy, which for simple harmonic motion is $\frac{1}{2}kA^2$. Hence $2.18 \times 10^9 = \frac{1}{2}k(6.37 \times 10^6)^2 \Rightarrow k = 1.1 \times 10^{-4}$ N/m. This is a pretty weak spring, but given that it acts for such a long distance it can get you going at a pretty good clip by the time you reach the center of the Earth.

An alternative method involves setting $F = kx$. With $F = mg$ and $x = A = r_e = 6.37 \times 10^6$. This gives $k = mg/r_e = (70)(9.8)/(6.37 \times 10^6) = 1.1 \times 10^{-4}$ N/m