CHAPTER THREE

Solutions for Section 3.1

Exercises

1. The derivative, $f'(x)$, is defined as
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
If $f(x) = 7$, then
$$f'(x) = \lim_{h\to 0} \frac{7-7}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

2. The definition of the derivative says that
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$
Therefore,
$$f'(x) = \lim_{h\to 0} \frac{[17(x+h)+11] - [17x+11]}{h} = \lim_{h\to 0} \frac{17h}{h} = 17.$$

3. $y' = 11x^{10}$.
4. $y' = 12x^{11}$.
5. $y' = 11x^{-12}$.
6. $y' = 3.2x^{2.2}$.
7. $y' = -12x^{-13}$.
8. $y' = \frac{4}{3}x^{1/3}$.
9. $y' = \frac{3}{4}x^{-1/4}$.
10. $y' = -\frac{3}{4}x^{-7/4}$.
11. $f'(x) = -4x^{-5}$.
12. $f'(x) = \frac{1}{4}x^{-3/4}$.
13. $f'(x) = ex^{e-1}$.
14. $y' = 6x^{1/2} - \frac{5}{2}x^{-1/2}$.
15. $f'(t) = 6t - 4$.
16. $y' = 17 + 12x^{-1/2}$.
17. Dividing gives $g(t) = t^2 + k/t$ so $g'(t) = 2t - \dfrac{k}{t^2}$.
18. The power rule gives $f'(x) = 20x^3 - \dfrac{2}{x^3}$.
19. $h'(w) = 6w^{-4} + \dfrac{3}{2}w^{-1/2}$
20. $y' = 18x^2 + 8x - 2$.
21. $y' = 15t^4 - \frac{5}{2}t^{-1/2} - \frac{7}{t^2}$.
22. $y' = 6t - \frac{6}{t^{3/2}} + \frac{2}{t^3}$.
23. $y' = 2z - \frac{1}{2z^2}$.
24. $y = x + \frac{1}{x}$, so $y' = 1 - \frac{1}{x^2}$.
25. $f(z) = \dfrac{z}{3} + \dfrac{1}{3}z^{-1} = \dfrac{1}{3}\left(z + z^{-1}\right)$, so $f'(z) = \dfrac{1}{3}\left(1 - z^{-2}\right) = \dfrac{1}{3}\left(\dfrac{z^2-1}{z^2}\right)$.

26. $f(t) = \frac{1}{t^2} + \frac{1}{t} - \frac{1}{t^4} = t^{-2} + t^{-1} - t^{-4}$
$f'(t) = -2t^{-3} - t^{-2} + 4t^{-5}$.

27. $y = \frac{\theta}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta}} = \sqrt{\theta} - \frac{1}{\sqrt{\theta}}$
$y' = \frac{1}{2\sqrt{\theta}} + \frac{1}{2\theta^{3/2}}$.

28. $j'(x) = \frac{3x^2}{a} + \frac{2ax}{b} - c$

29. Since $4/3$, π, and b are all constants, we have

$$\frac{dV}{dr} = \frac{4}{3}\pi(2r)b = \frac{8}{3}\pi rb.$$

30. Since w is a constant times q, we have $dw/dq = 3ab^2$.

31. Since a, b, and c are all constants, we have

$$\frac{dy}{dx} = a(2x) + b(1) + 0 = 2ax + b.$$

32. Since a and b are constants, we have

$$\frac{dP}{dt} = 0 + b\frac{1}{2}t^{-1/2} = \frac{b}{2\sqrt{t}}.$$

33. $g'(x) = -\frac{1}{2}(5x^4 + 2)$.

34. $y' = -12x^3 - 12x^2 - 6$.

35. $g(z) = z^5 + 5z^4 - z$
$g'(z) = 5z^4 + 20z^3 - 1$.

Problems

36. So far, we can only take the derivative of powers of x and the sums of constant multiples of powers of x. Since we cannot write $\sqrt{x+3}$ in this form, we cannot yet take its derivative.

37. The x is in the exponent and we haven't learned how to handle that yet.

38. $g'(x) = \pi x^{(\pi-1)} + \pi x^{-(\pi+1)}$, by the power and sum rules.

39. $y' = 6x$. (power rule and sum rule)

40. We cannot write $\frac{1}{3x^2+4}$ as the sum of powers of x multiplied by constants.

41. $y' = -2/3z^3$. (power rule and sum rule)

42. $f'(t) = 6t^2 - 8t + 3$ and $f''(t) = 12t - 8$.

43.

$$\begin{aligned} f'(x) = 12x^2 + 12x - 23 &\geq 1 \\ 12x^2 + 12x - 24 &\geq 0 \\ 12(x^2 + x - 2) &\geq 0 \\ 12(x+2)(x-1) &\geq 0. \end{aligned}$$

Hence $x \geq 1$ or $x \leq -2$.

44. Decreasing means $f'(x) < 0$:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3),$$

so $f'(x) < 0$ when $x < 3$ and $x \neq 0$. Concave up means $f''(x) > 0$:

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

so $f''(x) > 0$ when

$$12x(x-2) > 0$$
$$x < 0 \quad \text{or} \quad x > 2.$$

So, both conditions hold for $x < 0$ or $2 < x < 3$.

45. The graph increases when $dy/dx > 0$:

$$\frac{dy}{dx} = 5x^4 - 5 > 0$$
$$5(x^4-1) > 0 \quad \text{so} \quad x^4 > 1 \quad \text{so} \quad x > 1 \text{ or } x < -1.$$

The graph is concave up when $d^2y/dx^2 > 0$:

$$\frac{d^2y}{dx^2} = 20x^3 > 0 \quad \text{so} \quad x > 0.$$

We need values of x where $\{x > 1$ or $x < -1\}$ AND $\{x > 0\}$, which implies $x > 1$. Thus, both conditions hold for all values of x larger than 1.

46. Since $f(x) = x^3 - 6x^2 - 15x + 20$, we have $f'(x) = 3x^2 - 12x - 15$. To find the points at which $f'(x) = 0$, we solve

$$3x^2 - 12x - 15 = 0$$
$$3(x^2 - 4x - 5) = 0$$
$$3(x+1)(x-5) = 0.$$

We see that $f'(x) = 0$ at $x = -1$ and at $x = 5$. The graph of $f(x)$ in Figure 3.1 appears to be horizontal at $x = -1$ and at $x = 5$, confirming what we found analytically.

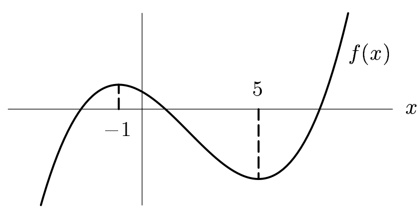


Figure 3.1

47.

$$f'(x) = -8 + 2\sqrt{2}x$$
$$f'(r) = -8 + 2\sqrt{2}r = 4$$
$$r = \frac{12}{2\sqrt{2}} = 3\sqrt{2}.$$

48. **(a)** Since the power of x will go down by one every time you take a derivative (until the exponent is zero after which the derivative will be zero), we can see immediately that $f^{(8)}(x) = 0$.

(b) $f^{(7)}(x) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot x^0 = 5040$.

49. Differentiating gives

$$f'(x) = 6x^2 - 4x \quad \text{so} \quad f'(1) = 6 - 4 = 2.$$

Thus the equation of the tangent line is $(y-1) = 2(x-1)$ or $y = 2x - 1$.

50. **(a)** We have $f(2) = 8$, so a point on the tangent line is $(2, 8)$. Since $f'(x) = 3x^2$, the slope of the tangent is given by

$$m = f'(2) = 3(2)^2 = 12.$$

Thus, the equation is

$$y - 8 = 12(x-2) \quad \text{or} \quad y = 12x - 16.$$

(b) See Figure 3.2. The tangent line lies below the function $f(x) = x^3$, so estimates made using the tangent line are underestimates.

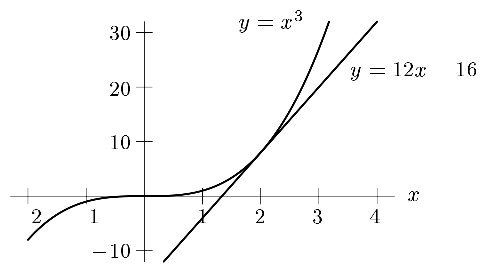


Figure 3.2

51. The slopes of the tangent lines to $y = x^2 - 2x + 4$ are given by $y' = 2x - 2$. A line through the origin has equation $y = mx$. So, at the tangent point, $x^2 - 2x + 4 = mx$ where $m = y' = 2x - 2$.

$$\begin{aligned} x^2 - 2x + 4 &= (2x - 2)x \\ x^2 - 2x + 4 &= 2x^2 - 2x \\ -x^2 + 4 &= 0 \\ -(x+2)(x-2) &= 0 \\ x &= 2, -2. \end{aligned}$$

Thus, the points of tangency are $(2, 4)$ and $(-2, 12)$. The lines through these points and the origin are $y = 2x$ and $y = -6x$, respectively. Graphically, this can be seen in Figure 3.3:

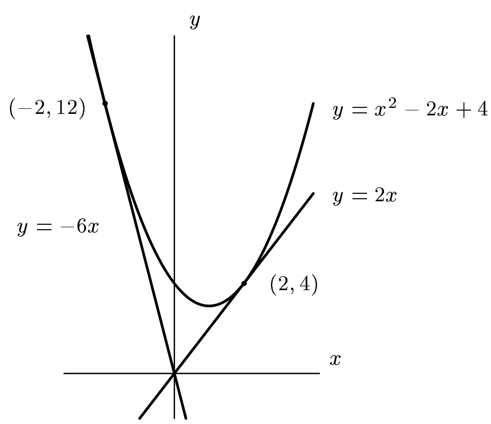


Figure 3.3

52. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. This means $f'(1) = n \cdot 1^{n-1} = n \cdot 1 = n$, because any power of 1 equals 1.

53. Since $f(x) = ax^n$, $f'(x) = anx^{n-1}$. We know that $f'(2) = (an)2^{n-1} = 3$, and $f'(4) = (an)4^{n-1} = 24$. Therefore,

$$\begin{aligned} \frac{f'(4)}{f'(2)} &= \frac{24}{3} \\ \frac{(an)4^{n-1}}{(an)2^{n-1}} &= \left(\frac{4}{2}\right)^{n-1} = 8 \\ 2^{n-1} &= 8, \text{ and thus } n = 4. \end{aligned}$$

Substituting $n = 4$ into the expression for $f'(2)$, we get $3 = a(4)(8)$, or $a = 3/32$.

54. Yes. To see why, we substitute $y = x^n$ into the equation $13x\dfrac{dy}{dx} = y$. We first calculate $\dfrac{dy}{dx} = \dfrac{d}{dx}(x^n) = nx^{n-1}$. The differential equation becomes

$$13x(nx^{n-1}) = x^n$$

But $13x(nx^{n-1}) = 13n(x \cdot x^{n-1}) = 13nx^n$, so we have

$$13n(x^n) = x^n$$

This equality must hold for all x, so we get $13n = 1$, so $n = 1/13$. Thus, $y = x^{1/13}$ is a solution.

55. Since $f(t) = 700 - 3t^2$, we have $f(5) = 700 - 3(25) = 625$ cm. Since $f'(t) = -6t$, we have $f'(5) = -30$ cm/year. In the year 2000, the sand dune was 625 cm high and it was eroding at a rate of 30 centimeters per year.

56. **(a)** Velocity $v(t) = \frac{dy}{dt} = \frac{d}{dt}(1250 - 16t^2) = -32t$.
Since $t \geq 0$, the ball's velocity is negative. This is reasonable, since its height y is decreasing.

(b) Acceleration $a(t) = \frac{dv}{dt} = \frac{d}{dt}(-32t) = -32$.
So its acceleration is the negative constant -32.

(c) The ball hits the ground when its height $y = 0$. This gives

$$1250 - 16t^2 = 0$$
$$t = \pm 8.84 \text{ seconds}$$

We discard $t = -8.84$ because time t is nonnegative. So the ball hits the ground 8.84 seconds after its release, at which time its velocity is

$$v(8.84) = -32(8.84) = -282.88 \text{ feet/sec} = -192.84 \text{ mph.}$$

57. **(a)** The average velocity between $t = 0$ and $t = 2$ is given by

$$\text{Average velocity} = \frac{f(2) - f(0)}{2 - 0} = \frac{-4.9(2^2) + 25(2) + 3 - 3}{2 - 0} = \frac{33.4 - 3}{2} = 15.2 \text{ m/sec.}$$

(b) Since $f'(t) = -9.8t + 25$, we have

$$\text{Instantaneous velocity} = f'(2) = -9.8(2) + 25 = 5.4 \text{ m/sec.}$$

(c) Acceleration is given $f''(t) = -9.8$. The acceleration at $t = 2$ (and all other times) is the acceleration due to gravity, which is -9.8 m/sec^2.

(d) We can use a graph of height against time to estimate the maximum height of the tomato. See Figure 3.4. Alternately, we can find the answer analytically. The maximum height occurs when the velocity is zero and $v(t) = -9.8t + 25 = 0$ when $t = 2.6$ sec. At this time the tomato is at a height of $f(2.6) = 34.9$. The maximum height is 34.9 meters.

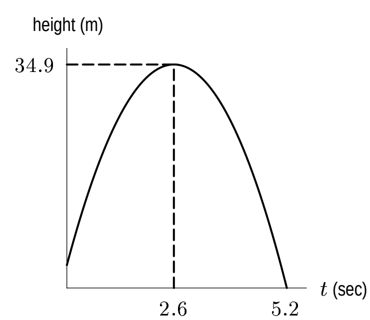


Figure 3.4

(e) We see in Figure 3.4 that the tomato hits ground at about $t = 5.2$ seconds. Alternately, we can find the answer analytically. The tomato hits the ground when

$$f(t) = -4.9t^2 + 25t + 3 = 0.$$

We solve for t using the quadratic formula:

$$t = \frac{-25 \pm \sqrt{(25)^2 - 4(-4.9)(3)}}{2(-4.9)}$$
$$t = \frac{-25 \pm \sqrt{683.8}}{-9.8}$$
$$t = -0.12 \quad \text{and} \quad t = 5.2.$$

We use the positive values, so the tomato hits the ground at $t = 5.2$ seconds.

58. $\frac{dF}{dr} = -\frac{2GMm}{r^3}$.

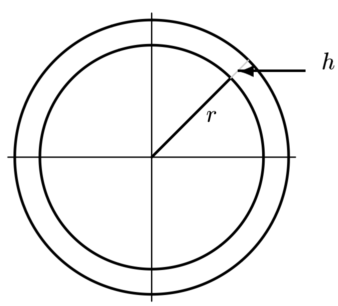
59. **(a)** $T = 2\pi\sqrt{\frac{l}{g}} = \frac{2\pi}{\sqrt{g}}\left(l^{\frac{1}{2}}\right)$, so $\frac{dT}{dl} = \frac{2\pi}{\sqrt{g}}\left(\frac{1}{2}l^{-\frac{1}{2}}\right) = \frac{\pi}{\sqrt{gl}}$.

(b) Since $\frac{dT}{dl}$ is positive, the period T increases as the length l increases.

60. **(a)** $A = \pi r^2$
$\frac{dA}{dr} = 2\pi r$.

(b) This is the formula for the circumference of a circle.

(c) $A'(r) \approx \frac{A(r+h)-A(r)}{h}$ for small h. When $h > 0$, the numerator of the difference quotient denotes the area of the region contained between the inner circle (radius r) and the outer circle (radius $r + h$). See figure below. As h approaches 0, this area can be approximated by the product of the circumference of the inner circle and the "width" of the region, i.e., h. Dividing this by the denominator, h, we get A' = the circumference of the circle with radius r.



We can also think about the derivative of A as the rate of change of area for a small change in radius. If the radius increases by a tiny amount, the area will increase by a thin ring whose area is simply the circumference at that radius times the small amount. To get the rate of change, we divide by the small amount and obtain the circumference.

61. $V = \frac{4}{3}\pi r^3$. Differentiating gives $\frac{dV}{dr} = 4\pi r^2$ = surface area of a sphere.

The difference quotient $\frac{V(r+h)-V(r)}{h}$ is the volume between two spheres divided by the change in radius. Furthermore, when h is very small, the difference between volumes, $V(r + h) - V(r)$, is like a coating of paint of depth h applied to the surface of the sphere. The volume of the paint is about $h \cdot$ (Surface Area) for small h: dividing by h gives back the surface area.

Thinking about the derivative as the rate of change of the function for a small change in the variable gives another way of seeing the result. If you increase the radius of a sphere a small amount, the volume increases by a very thin layer whose volume is the surface area at that radius multiplied by that small amount.

62. **(a)**

$$\frac{d(x^{-1})}{dx} = \lim_{h\to 0}\frac{(x+h)^{-1} - x^{-1}}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{x(x+h)}\right]$$
$$= \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x^2} = -1x^{-2}.$$

$$\begin{aligned}\frac{d(x^{-3})}{dx} &= \lim_{h\to 0}\frac{(x+h)^{-3}-x^{-3}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^3}-\frac{1}{x^3}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^3-(x+h)^3}{x^3(x+h)^3}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^3-(x^3+3hx^2+3h^2x+h^3)}{x^3(x+h)^3}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{-3hx^2-3xh^2-h^3}{x^3(x+h)^3}\right]\\ &= \lim_{h\to 0}\frac{-3x^2-3xh-h^2}{x^3(x+h)^3}\\ &= \frac{-3x^2}{x^6} = -3x^{-4}.\end{aligned}$$

(b) For clarity, let $n = -k$, where k is a positive integer. So $x^n = x^{-k}$.

$$\begin{aligned}\frac{d(x^{-k})}{dx} &= \lim_{h\to 0}\frac{(x+h)^{-k}-x^{-k}}{h}\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{(x+h)^k}-\frac{1}{x^k}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^k-(x+h)^k}{x^k(x+h)^k}\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^k-x^k-khx^{k-1}-\overbrace{\ldots-h^k}^{\text{terms involving } h^2 \text{ and higher powers of } h}}{x^k(x+h)^k}\right]\\ &= \frac{-kx^{k-1}}{x^k(x)^k} = \frac{-k}{x^{k+1}} = -kx^{-(k+1)} = -kx^{-k-1}.\end{aligned}$$

Solutions for Section 3.2

Exercises

1. $f'(x) = 2e^x + 2x.$

2. $y' = 10t + 4e^t.$

3. $y' = (\ln 5)5^x.$

4. $f'(x) = (\ln 2)2^x + 2(\ln 3)3^x.$

5. $y' = 10x + (\ln 2)2^x.$

6. $f'(x) = 12e^x + (\ln 11)11^x.$

7. $\dfrac{dy}{dx} = 4(\ln 10)10^x - 3x^2.$

8. $\dfrac{dy}{dx} = 3 - 2(\ln 4)4^x.$

9. $\dfrac{dy}{dx} = \dfrac{1}{3}(\ln 3)3^x - \dfrac{33}{2}(x^{-\frac{3}{2}}).$

10. $f'(x) = ex^{e-1}$.

11. $f(x) = e^{1+x} = e^1 \cdot e^x$. Then, since e^1 is just a constant,
$f'(x) = e \cdot e^x = e^{1+x}$.

12. $f(t) = e^t \cdot e^2$. Then, since e^2 is just a constant, $f'(t) = \frac{d}{dt}(e^t e^2) = e^2 \frac{d}{dt} e^t = e^2 e^t = e^{t+2}$.

13. $y = e^\theta e^{-1} \quad y' = \dfrac{d}{d\theta}(e^\theta e^{-1}) = e^{-1}\dfrac{d}{d\theta}e^\theta = e^\theta e^{-1} = e^{\theta-1}$.

14. $z' = (\ln 4)e^x$.

15. $z' = (\ln 4)^2 4^x$.

16. $f'(t) = (\ln(\ln 3))(\ln 3)^t$.

17. $f'(x) = 3x^2 + 3^x \ln 3$

18. $\dfrac{dy}{dx} = 5 \cdot 5^t \ln 5 + 6 \cdot 6^t \ln 6$

19. $\dfrac{dy}{dx} = \pi^x \ln \pi$

20. $h'(z) = (\ln(\ln 2))(\ln 2)^z$.

21. $f'(x) = (\ln \pi)\pi^x$.

22. This is the sum of an exponential function and a power function, so $f'(x) = \ln(\pi)\pi^x + \pi x^{\pi-1}$.

23. $y'(x) = a^x \ln a + ax^{a-1}$.

24. $f'(x) = \pi^2 x^{(\pi^2-1)} + (\pi^2)^x \ln(\pi^2)$

25. $f'(z) = (2\ln 3)z + (\ln 4)e^z$.

26. $g'(x) = \dfrac{d}{dx}(2x - x^{-1/3} + 3^x - e) = 2 + \dfrac{1}{3x^{\frac{4}{3}}} + 3^x \ln 3$.

27. $y' = 2x + (\ln 2)2^x$.

28. $y' = \frac{1}{2}x^{-\frac{1}{2}} - \ln\frac{1}{2}(\frac{1}{2})^x = \frac{1}{2\sqrt{x}} + \ln 2(\frac{1}{2})^x$.

29. We can take the derivative of the sum $x^2 + 2^x$, but not the product.

30. Once again, this is a product of two functions, 2^x and $\frac{1}{x}$, each of which we can take the derivative of; but we don't know how to take the derivative of the product.

31. Since $y = e^5 e^x$, $y' = e^5 e^x = e^{x+5}$.

32. $y = e^{5x} = (e^5)^x$, so $y' = \ln(e^5) \cdot (e^5)^x = 5e^{5x}$.

33. The exponent is x^2, and we haven't learned what to do about that yet.

34. $f'(z) = (\ln\sqrt{4})(\sqrt{4})^z = (\ln 2)2^z$.

35. We can't use our rules if the exponent is $\sqrt{\theta}$.

Problems

36.

$$\frac{dP}{dt} = 35{,}000 \cdot (\ln 0.98)(0.98^t).$$

At $t = 23$, this is $35{,}000(\ln 0.98)(0.98^{23}) \approx -444.3\frac{\text{people}}{\text{year}}$. (Note: the negative sign indicates that the population is decreasing.)

37. Since $P = 1 \cdot (1.05)^t$, $\frac{dP}{dt} = \ln(1.05)1.05^t$. When $t = 10$,

$$\frac{dP}{dt} = (\ln 1.05)(1.05)^{10} \approx \$0.07947/\text{year} \approx 7.95¢/\text{year}.$$

38. We have $f(t) = 5.3(1.018)^t$ so $f'(t) = 5.3(\ln 1.018)(1.018)^t = 0.095(1.018)^t$. Therefore

$$f(0) = 5.3 \text{ billion people}$$

and

$$f'(0) = 0.095 \text{ billion people per year.}$$

In 1990, the population of the world was 5.3 billion people and was increasing at a rate of 0.095 billion people per year.

We also have

$$f(30) = 5.3(1.018)^{30} = 9.1 \text{ billion people,}$$

and

$$f'(30) = 0.095(1.018)^{30} = 0.16 \text{ billion people per year.}$$

In the year 2020, this model predicts that the population of the world will be 9.1 billion people and will be increasing at a rate of 0.16 billion people per year.

39. $\dfrac{dV}{dt} = 75(1.35)^t \ln 1.35 \approx 22.5(1.35)^t$.

40. **(a)** $V(4) = 25(0.85)^4 = 25(0.522) = 13{,}050$. Thus the value of the car after 4 years is $\$13{,}050$.

(b) We have a function of the form $f(t) = Ca^t$. We know that such functions have a derivative of the form $(C \ln a) \cdot a^t$. Thus, $V'(t) = 25(0.85)^t \cdot \ln 0.85 = -4.063(0.85)^t$. The units would be the change in value (in thousands of dollars) with respect to time (in years), or thousands of dollars/year.

(c) $V'(4) = -4.063(0.85)^4 = -4.063(0.522) = -2.121$. This means that at the end of the fourth year, the value of the car is decreasing by $\$2121$ per year.

(d) $V(t)$ is a positive decreasing function, so that the value of the automobile is positive and decreasing. $V'(t)$ is a negative function whose magnitude is decreasing, meaning the value of the automobile is always dropping, but the yearly loss of value is less as time goes on. The graphs of $V(t)$ and $V'(t)$ confirm that the value of the car decreases with time. What they do not take into account are the *costs* associated with owning the vehicle. At some time, t, it is likely that the costs of owning the vehicle will outweigh its value. At that time, it may no longer be worthwhile to keep the car.

41. **(a)** The rate of change of the population is $P'(t)$. If $P'(t)$ is proportional to $P(t)$, we have

$$P'(t) = kP(t).$$

(b) If $P(t) = Ae^{kt}$, then $P'(t) = kAe^{kt} = kP(t)$.

42. **(a)** $f(x) = 1 - e^x$ crosses the x-axis where $0 = 1 - e^x$, which happens when $e^x = 1$, so $x = 0$. Since $f'(x) = -e^x$, $f'(0) = -e^0 = -1$.

(b) $y = -x$

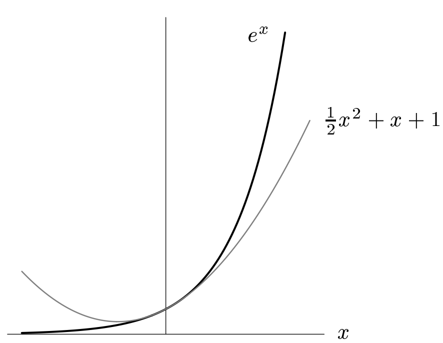
(c) The negative of the reciprocal of -1 is 1, so the equation of the normal line is $y = x$.

43. Since $y = 2^x$, $y' = (\ln 2)2^x$. At $(0, 1)$, the tangent line has slope $\ln 2$ so its equation is $y = (\ln 2)x + 1$. At c, $y = 0$, so $0 = (\ln 2)c + 1$, thus $c = -\frac{1}{\ln 2}$.

44.

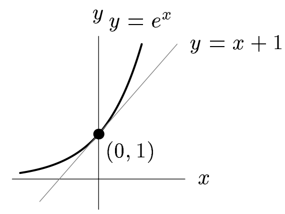
$$\begin{aligned} g(x) &= ax^2 + bx + c & f(x) &= e^x \\ g'(x) &= 2ax + b & f'(x) &= e^x \\ g''(x) &= 2a & f''(x) &= e^x \end{aligned}$$

So, using $g''(0) = f''(0)$, etc., we have $2a = 1$, $b = 1$, and $c = 1$, and thus $g(x) = \frac{1}{2}x^2 + x + 1$, as shown in the figure below.



The two functions do look very much alike near $x = 0$. They both increase for large values of x, but e^x increases much more quickly. For very negative values of x, the quadratic goes to ∞ whereas the exponential goes to 0. By choosing a function whose first few derivatives agreed with the exponential when $x = 0$, we got a function which looks like the exponential for x-values near 0.

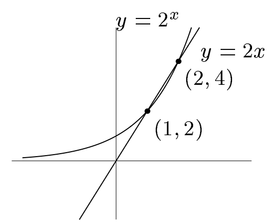
45. The derivative of e^x is $\frac{d}{dx}(e^x) = e^x$. Thus the tangent line at $x = 0$, has slope $e^0 = 1$, and the tangent line is $y = x + 1$. A function which is always concave up will always stay above any of its tangent lines. Thus $e^x \geq x + 1$ for all x, as shown in the figure below.



46. The equation $2^x = 2x$ has solutions $x = 1$ and $x = 2$. (Check this by substituting these values into the equation). The graph below suggests that these are the only solutions, but how can we be sure?

Let's look at the slope of the curve $f(x) = 2^x$, which is $f'(x) = (\ln 2)2^x \approx (0.693)2^x$, and the slope of the line $g(x) = 2x$ which is 2. At $x = 1$, the slope of $f(x)$ is less than 2; at $x = 2$, the slope of $f(x)$ is more than 2. Since the slope of $f(x)$ is always increasing, there can be no other point of intersection. (If there were another point of intersection, the graph f would have to "turn around".)

Here's another way of seeing this. Suppose $g(x)$ represents the position of a car going a steady 2 mph, while $f(x)$ represents a car which starts ahead of g (because the graph of f is above g) and is initially going slower than g. The car f is first overtaken by g. All the while, however, f is speeding up until eventually it overtakes g again. Notice that the two cars will only meet twice (corresponding to the two intersections of the curve): once when g overtakes f and once when f overtakes g.



47. For $x = 0$, we have $y = a^0 = 1$ and $y = 1 + 0 = 1$, so both curves go through the point $(0, 1)$ for all values of a.

Differentiating gives

$$\left.\frac{d(a^x)}{dx}\right|_{x=0} = \left. a^x \ln a\right|_{x=0} = a^0 \ln a = \ln a$$

$$\left.\frac{d(1+x)}{dx}\right|_{x=0} = 1.$$

The graphs are tangent at $x = 0$ if

$$\ln a = 1 \quad \text{so} \quad a = e.$$

Solutions for Section 3.3

Exercises

1. By the product rule, $f'(x) = 2x(x^3 + 5) + x^2(3x^2) = 2x^4 + 3x^4 + 10x = 5x^4 + 10x$. Alternatively, $f'(x) = (x^5 + 5x^2)' = 5x^4 + 10x$. The two answers should, and do, match.

2. Using the product rule,

$$f'(x) = (\ln 2)2^x 3^x + (\ln 3)2^x 3^x = (\ln 2 + \ln 3)(2^x \cdot 3^x) = \ln(2 \cdot 3)(2 \cdot 3)^x = (\ln 6)6^x$$

or, since $2^x \cdot 3^x = (2 \cdot 3)^x = 6^x$,

$$f'(x) = (6^x)' = (\ln 6)(6^x).$$

The two answers should, and do, match.

3. $f'(x) = x \cdot e^x + e^x \cdot 1 = e^x(x + 1)$.

4. $y' = 2^x + x(\ln 2)2^x = 2^x(1 + x \ln 2)$.

5. $y' = \frac{1}{2\sqrt{x}}2^x + \sqrt{x}(\ln 2)2^x$.

6. $f'(x) = (x^2 - x^{\frac{1}{2}}) \cdot 3^x(\ln 3) + 3^x\left(2x - \frac{1}{2}x^{-\frac{1}{2}}\right) = 3^x\left[(\ln 3)(x^2 - x^{\frac{1}{2}}) + \left(2x - \frac{1}{2\sqrt{x}}\right)\right]$.

7. It is easier to do this by multiplying it out first, rather than using the product rule first: $z = s^4 - s, \quad z' = 4s^3 - 1$.

8. $\frac{dy}{dt} = 2te^t + (t^2+3)e^t = e^t(t^2+2t+3)$.

9. $y' = (3t^2 - 14t)e^t + (t^3 - 7t^2 + 1)e^t = (t^3 - 4t^2 - 14t + 1)e^t$.

10. $f'(x) = \frac{e^x \cdot 1 - x \cdot e^x}{(e^x)^2} = \frac{e^x(1-x)}{(e^x)^2} = \frac{1-x}{e^x}$.

11. $g'(x) = \frac{50xe^x - 25x^2e^x}{e^{2x}} = \frac{50x - 25x^2}{e^x}$.

12. $g'(w) = \frac{3.2w^{2.2}(5^w) - (\ln 5)(w^{3.2})5^w}{5^{2w}} = \frac{3.2w^{2.2} - w^{3.2}(\ln 5)}{5^w}$.

13. $q'(r) = \frac{3(5r+2) - 3r(5)}{(5r+2)^2} = \frac{15r + 6 - 15r}{(5r+2)^2} = \frac{6}{(5r+2)^2}$

14. $g'(t) = \frac{(t+4)-(t-4)}{(t+4)^2} = \frac{8}{(t+4)^2}$.

15. $\frac{dz}{dt} = \frac{3(5t+2) - (3t+1)5}{(5t+2)^2} = \frac{15t + 6 - 15t - 5}{(5t+2)^2} = \frac{1}{(5t+2)^2}$.

16. $z' = \frac{(2t+5)(t+3) - (t^2+5t+2)}{(t+3)^2} = \frac{t^2+6t+13}{(t+3)^2}$.

17. Using the quotient rule gives $\frac{dz}{dt} = \frac{(2t+3)(t+1) - (t^2+3t+1)}{(t+1)^2}$ or $\frac{dz}{dt} = \frac{t^2+2t+2}{(t+1)^2}$.

18. Divide and then differentiate

$$f(x) = x + \frac{3}{x}$$
$$f'(x) = 1 - \frac{3}{x^2}.$$

19. $w = y^2 - 6y + 7. \quad w' = 2y - 6, y \neq 0$.

20. $y' = \frac{\frac{1}{2\sqrt{t}}(t^2+1) - \sqrt{t}(2t)}{(t^2+1)^2}$.

21. $\frac{d}{dz}\left(\frac{z^2+1}{\sqrt{z}}\right) = \frac{d}{dz}(z^{\frac{3}{2}} + z^{-\frac{1}{2}}) = \frac{3}{2}z^{\frac{1}{2}} - \frac{1}{2}z^{-\frac{3}{2}} = \frac{\sqrt{z}}{2}(3 - z^{-2})$.

22. $g'(t) = -4(3+\sqrt{t})^{-2}\left(\frac{1}{2}t^{-1/2}\right) = \frac{-2}{\sqrt{t}(3+\sqrt{t})^2}$

23. $h'(r) = \frac{d}{dr}\left(\frac{r^2}{2r+1}\right) = \frac{(2r)(2r+1) - 2r^2}{(2r+1)^2} = \frac{2r(r+1)}{(2r+1)^2}$.

24. Notice that you can cancel a z out of the numerator and denominator to get

$$f(z) = \frac{3z}{5z+7}, \qquad z \neq 0$$

Then

$$\begin{aligned} f'(z) &= \frac{(5z+7)3 - 3z(5)}{(5z+7)^2} \\ &= \frac{15z + 21 - 15z}{(5z+7)^2} \\ &= \frac{21}{(5z+7)^2}, z \neq 0. \end{aligned}$$

[If you used the quotient rule correctly without canceling the z out first, your answer should simplify to this one, but it is usually a good idea to simplify as much as possible before differentiating.]

25. $w'(x) = \dfrac{17e^x(2^x) - (\ln 2)(17e^x)2^x}{2^{2x}} = \dfrac{17e^x(2^x)(1-\ln 2)}{2^{2x}} = \dfrac{17e^x(1-\ln 2)}{2^x}$.

26. $h'(p) = \dfrac{2p(3+2p^2) - 4p(1+p^2)}{(3+2p^2)^2} = \dfrac{6p+4p^3-4p-4p^3}{(3+2p^2)^2} = \dfrac{2p}{(3+2p^2)^2}$.

27.

$$\begin{aligned} f'(x) &= \frac{(2+3x+4x^2)(1) - (1+x)(3+8x)}{(2+3x+4x^2)^2} \\ &= \frac{2+3x+4x^2-3-11x-8x^2}{(2+3x+4x^2)^2} \\ &= \frac{-4x^2-8x-1}{(2+3x+4x^2)^2}. \end{aligned}$$

28. We use the quotient rule. We have

$$f'(x) = \frac{(cx+k)(a) - (ax+b)(c)}{(cx+k)^2} = \frac{acx+ak-acx-bc}{(cx+k)^2} = \frac{ak-bc}{(cx+k)^2}.$$

29. $w' = (3t^2+5)(t^2-7t+2) + (t^3+5t)(2t-7)$.

Problems

30.

$$\begin{aligned} f'(x) &= 3(2x-5) + 2(3x+8) = 12x+1 \\ f''(x) &= 12. \end{aligned}$$

31. Using the product rule, we have

$$\begin{aligned} f'(x) &= e^{-x} - xe^{-x} \\ f''(x) &= -e^{-x} - e^{-x} + xe^{-x} = e^{-x}(x-2). \end{aligned}$$

Since $e^{-x} > 0$, for all x, we have $f''(x) < 0$ if $x - 2 < 0$, that is, $x < 2$.

32. Using the quotient rule, we have

$$\begin{aligned} g'(x) &= \frac{0 - 1(2x)}{(x^2+1)^2} = \frac{-2x}{(x^2+1)^2} \\ g''(x) &= \frac{-2(x^2+1)^2 + 2x(4x^3+4x)}{(x^2+1)^4} \\ &= \frac{-2(x^2+1)^2 + 8x^2(x^2+1)}{(x^2+1)^4} \\ &= \frac{-2(x^2+1) + 8x^2}{(x^2+1)^3} \\ &= \frac{2(3x^2-1)}{(x^2+1)^3}. \end{aligned}$$

Since $(x^2+1)^3 > 0$ for all x, we have $g''(x) < 0$ if $(3x^2-1) < 0$, or when

$$\begin{gathered} 3x^2 < 1 \\ -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}. \end{gathered}$$

33. Since $f(0) = -5/1 = -5$, the tangent line passes through the point $(0, -5)$, so its vertical intercept is -5. To find the slope of the tangent line, we find the derivative of $f(x)$ using the quotient rule:

$$f'(x) = \frac{(x+1)\cdot 2 - (2x-5)\cdot 1}{(x+1)^2} = \frac{7}{(x+1)^2}.$$

At $x = 0$, the slope of the tangent line is $m = f'(0) = 7$. The equation of the tangent line is $y = 7x - 5$.

34.

$$\begin{aligned} f(t) &= \frac{1}{e^t} \\ f'(t) &= \frac{e^t \cdot 0 - e^t \cdot 1}{(e^t)^2} \\ &= \frac{-1}{e^t} = -e^{-t}. \end{aligned}$$

35. $f(x) = e^x \cdot e^x$
$f'(x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$.

36.

$$\begin{aligned} f(x) &= e^x e^{2x} \\ f'(x) &= e^x (e^{2x})' + (e^x)' e^{2x} \\ &= 2e^x e^{2x} + e^x e^{2x} \text{ (from Problem 35)} \\ &= 3e^{3x}. \end{aligned}$$

37. Since $\frac{d}{dx} e^{2x} = 2e^{2x}$ and $\frac{d}{dx} e^{3x} = 3e^{3x}$, we might guess that $\frac{d}{dx} e^{4x} = 4e^{4x}$.

38. (a) Although the answer you would get by using the quotient rule is equivalent, the answer looks simpler in this case if you just use the product rule:

$$\begin{aligned} \frac{d}{dx}\left(\frac{e^x}{x}\right) &= \frac{d}{dx}\left(e^x \cdot \frac{1}{x}\right) = \frac{e^x}{x} - \frac{e^x}{x^2} \\ \frac{d}{dx}\left(\frac{e^x}{x^2}\right) &= \frac{d}{dx}\left(e^x \cdot \frac{1}{x^2}\right) = \frac{e^x}{x^2} - \frac{2e^x}{x^3} \\ \frac{d}{dx}\left(\frac{e^x}{x^3}\right) &= \frac{d}{dx}\left(e^x \cdot \frac{1}{x^3}\right) = \frac{e^x}{x^3} - \frac{3e^x}{x^4}. \end{aligned}$$

(b) $\frac{d}{dx} \frac{e^x}{x^n} = \frac{e^x}{x^n} - \frac{ne^x}{x^{n+1}}.$

39.

$$\begin{aligned} \frac{d(x^2)}{dx} &= \frac{d}{dx}(x \cdot x) \\ &= x\frac{d(x)}{dx} + x\frac{d(x)}{dx} \\ &= 2x. \end{aligned}$$

$$\begin{aligned} \frac{d(x^3)}{dx} &= \frac{d}{dx}(x^2 \cdot x) \\ &= x^2\frac{d(x)}{dx} + x\frac{d(x^2)}{dx} \\ &= x^2\frac{d(x)}{dx} + x\left[x\frac{d(x)}{dx} + x\frac{d(x)}{dx}\right] \\ &= x^2\frac{d(x)}{dx} + x^2\frac{d(x)}{dx} + x^2\frac{d(x)}{dx} \\ &= 3x^2. \end{aligned}$$

40. Since

$$x^{1/2} \cdot x^{1/2} = x,$$

we differentiate to obtain

$$\frac{d}{dx}(x^{1/2}) \cdot x^{1/2} + x^{1/2} \cdot \frac{d}{dx}(x^{1/2}) = 1.$$

Now solve for $d(x^{1/2})/dx$:

$$\begin{aligned} 2x^{1/2}\frac{d}{dx}(x^{1/2}) &= 1 \\ \frac{d}{dx}(x^{1/2}) &= \frac{1}{2x^{1/2}}. \end{aligned}$$

41. **(a)** We have $h'(2) = f'(2) + g'(2) = 5 - 2 = 3$.
(b) We have $h'(2) = f'(2)g(2) + f(2)g'(2) = 5(4) + 3(-2) = 14$.
(c) We have $h'(2) = \dfrac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{5(4) - 3(-2)}{4^2} = \dfrac{26}{16} = \dfrac{13}{8}$.

42. **(a)** $G'(z) = F'(z)H(z) + H'(z)F(z)$, so
$G'(3) = F'(3)H(3) + H'(3)F(3) = 4 \cdot 1 + 3 \cdot 5 = 19$.
(b) $G'(w) = \dfrac{F'(w)H(w) - H'(w)F(w)}{[H(w)]^2}$, so $G'(3) = \dfrac{4(1) - 3(5)}{1^2} = -11$.

43. $f'(x) = 10x^9e^x + x^{10}e^x$ is of the form $g'h + h'g$, where

$$g(x) = x^{10},\ g'(x) = 10x^9$$

and

$$h(x) = e^x,\ h'(x) = e^x.$$

Therefore, using the product rule, let $f = g \cdot h$, with $g(x) = x^{10}$ and $h(x) = e^x$. Thus

$$f(x) = x^{10}e^x.$$

44. **(a)** $f(140) = 15{,}000$ says that $15{,}000$ skateboards are sold when the cost is \$140 per board.
$f'(140) = -100$ means that if the price is increased from \$140, roughly speaking, every dollar of increase will decrease the total sales by 100 boards.
(b) $\dfrac{dR}{dp} = \dfrac{d}{dp}(p \cdot q) = \dfrac{d}{dp}\,(p \cdot f(p)) = f(p) + pf'(p)$.
So,

$$\begin{aligned}\left.\frac{dR}{dp}\right|_{p=140} &= f(140) + 140f'(140)\\ &= 15{,}000 + 140(-100) = 1000.\end{aligned}$$

(c) From (b) we see that $\left.\dfrac{dR}{dp}\right|_{p=140} = 1000 > 0$. This means that the revenue will increase by about \$1000 if the price is raised by \$1.

45. We want dR/dr_1. Solving for R:

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_2 + r_1}{r_1 r_2},\ \text{ which gives }\ R = \frac{r_1 r_2}{r_2 + r_1}.$$

So, thinking of r_2 as a constant and using the quotient rule,

$$\frac{dR}{dr_1} = \frac{r_2(r_2 + r_1) - r_1 r_2(1)}{(r_2 + r_1)^2} = \frac{r_2^2}{(r_1 + r_2)^2}.$$

46. **(a)** If the museum sells the painting and invests the proceeds $P(t)$ at time t, then t years have elapsed since 2000, and the time span up to 2020 is $20 - t$. This is how long the proceeds $P(t)$ are earning interest in the bank. Each year the money is in the bank it earns 5% interest, which means the amount in the bank is multiplied by a factor of 1.05. So, at the end of $(20 - t)$ years, the balance is given by

$$B(t) = P(t)(1 + 0.05)^{20-t} = P(t)(1.05)^{20-t}.$$

(b)

$$B(t) = P(t)(1.05)^{20}(1.05)^{-t} = (1.05)^{20}\,\frac{P(t)}{(1.05)^t}.$$

(c) By the quotient rule,

$$B'(t) = (1.05)^{20}\left[\frac{P'(t)(1.05)^t - P(t)(1.05)^t \ln 1.05}{(1.05)^{2t}}\right].$$

So,

$$\begin{aligned} B'(10) &= (1.05)^{20}\left[\frac{5000(1.05)^{10} - 150{,}000(1.05)^{10}\ln 1.05}{(1.05)^{20}}\right] \\ &= (1.05)^{10}(5000 - 150{,}000\ln 1.05) \\ &\approx -3776.63. \end{aligned}$$

47. Note first that $f(v)$ is in $\frac{\text{liters}}{\text{km}}$, and v is in $\frac{\text{km}}{\text{hour}}$.

(a) $g(v) = \frac{1}{f(v)}$. (This is in $\frac{\text{km}}{\text{liter}}$.) Differentiating gives

$$g'(v) = \frac{-f'(v)}{(f(v))^2}.$$

So,

$$g(80) = \frac{1}{0.05} = 20\,\frac{\text{km}}{\text{liter}}.$$

$$g'(80) = \frac{-0.0005}{(0.05)^2} = -\frac{1}{5}\,\frac{\text{km}}{\text{liter}} \text{ for each } 1\,\frac{\text{km}}{\text{hr}} \text{ increase in speed.}$$

(b) $h(v) = v \cdot f(v)$. (This is in $\frac{\text{km}}{\text{hour}} \cdot \frac{\text{liters}}{\text{km}} = \frac{\text{liters}}{\text{hour}}$.) Differentiating gives

$$h'(v) = f(v) + v \cdot f'(v),$$

so

$$h(80) = 80(0.05) = 4\,\frac{\text{liters}}{\text{hr}}.$$

$$h'(80) = 0.05 + 80(0.0005) = 0.09\,\frac{\text{liters}}{\text{hr}} \text{ for each } 1\,\frac{\text{km}}{\text{hr}} \text{ increase in speed.}$$

(c) Part (a) tells us that at 80 km/hr, the car can go 20 km on 1 liter. Since the first derivative evaluated at this velocity is negative, this implies that as velocity increases, fuel efficiency decreases, i.e., at higher velocities the car will not go as far on 1 liter of gas. Part (b) tells us that at 80 km/hr, the car uses 4 liters in an hour. Since the first derivative evaluated at this velocity is positive, this means that at higher velocities, the car will use more gas per hour.

48. Assume for $g(x) \neq f(x)$, $g'(x) = g(x)$ and $g(0) = 1$. Then for

$$h(x) = \frac{g(x)}{e^x}$$

$$h'(x) = \frac{g'(x)e^x - g(x)e^x}{(e^x)^2} = \frac{e^x(g'(x) - g(x))}{(e^x)^2} = \frac{g'(x) - g(x)}{e^x}.$$

But, since $g(x) = g'(x)$, $h'(x) = 0$, so $h(x)$ is constant. Thus, the ratio of $g(x)$ to e^x is constant. Since $\dfrac{g(0)}{e^0} = \dfrac{1}{1} = 1$, $\dfrac{g(x)}{e^x}$ must equal 1 for all x. Thus $g(x) = e^x = f(x)$ for all x, so f and g are the same function.

49. **(a)** $f'(x) = (x-2) + (x-1)$.

(b) Think of f as the product of two factors, with the first as $(x-1)(x-2)$. (The reason for this is that we have already differentiated $(x-1)(x-2)$).

$$f(x) = [(x-1)(x-2)](x-3).$$

Now $f'(x) = [(x-1)(x-2)]'(x-3) + [(x-1)(x-2)](x-3)'$
Using the result of a):

$$\begin{aligned} f'(x) &= [(x-2) + (x-1)](x-3) + [(x-1)(x-2)] \cdot 1 \\ &= (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2). \end{aligned}$$

(c) Because we have already differentiated $(x-1)(x-2)(x-3)$, rewrite f as the product of two factors, the first being $(x-1)(x-2)(x-3)$:

$$f(x) = [(x-1)(x-2)(x-3)](x-4)$$

$$\text{Now } f'(x) = [(x-1)(x-2)(x-3)]'(x-4) + [(x-1)(x-2)(x-3)](x-4)'.$$

$$\begin{aligned}f'(x) &= [(x-2)(x-3)+(x-1)(x-3)+(x-1)(x-2)](x-4)\\ &\quad +[(x-1)(x-2)(x-3)]\cdot 1\\ &= (x-2)(x-3)(x-4)+(x-1)(x-3)(x-4)\\ &\quad +(x-1)(x-2)(x-4)+(x-1)(x-2)(x-3).\end{aligned}$$

From the solutions above, we can observe that when f is a product, its derivative is obtained by differentiating each factor in turn (leaving the other factors alone), and adding the results.

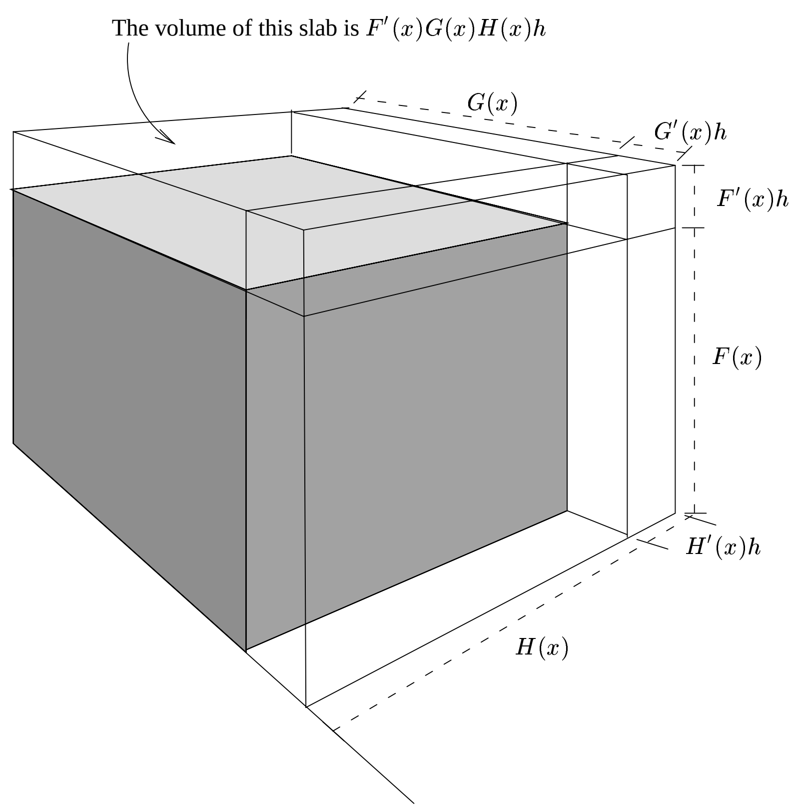
50. From the answer to Problem 49, we find that

$$\begin{aligned}f'(x) &= (x-r_1)(x-r_2)\cdots(x-r_{n-1})\cdot 1\\ &\quad +(x-r_1)(x-r_2)\cdots(x-r_{n-2})\cdot 1\cdot(x-r_n)\\ &\quad +(x-r_1)(x-r_2)\cdots(x-r_{n-3})\cdot 1\cdot(x-r_{n-1})(x-r_n)\\ &\quad +\cdots+1\cdot(x-r_2)(x-r_3)\cdots(x-r_n)\\ &= f(x)\left(\frac{1}{x-r_1}+\frac{1}{x-r_2}+\cdots+\frac{1}{x-r_n}\right).\end{aligned}$$

51. (a) We can approximate $\frac{d}{dx}[F(x)G(x)H(x)]$ using the large rectangular solids by which our original cube is increased:

Volume of whole $-$ volume of original solid $=$ change in volume.

$F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x)$ = change in volume.



As in the book, we will ignore the smaller regions which are added (the long, thin rectangular boxes and the small cube in the corner.) This can be justified by recognizing that as $h \to 0$, these volumes will shrink much faster than the volumes of the big slabs and will therefore be insignificant. (Note that these smaller regions have an h^2 or h^3 in the formulas of their volumes.) Then we can approximate the change in volume above by:

$$\begin{aligned} F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x) \approx\ & F'(x)G(x)H(x)h \quad \text{(top slab)} \\ & + F(x)G'(x)H(x)h \quad \text{(front slab)} \\ & + F(x)G(x)H'(x)h \quad \text{(other slab).} \end{aligned}$$

Dividing by h gives

$$\frac{F(x+h)G(x+h)H(x+h) - F(x)G(x)H(x)}{h}$$
$$\approx F'(x)G(x)H(x) + F(x)G'(x)H(x) + F(x)G(x)H'(x).$$

Letting $h \to 0$

$$(FGH)' = F'GH + FG'H + FGH'.$$

(b) Verifying,

$$\begin{aligned} \frac{d}{dx}[(F(x)\cdot G(x))\cdot H(x)] &= (F\cdot G)'(H) + (F\cdot G)(H)' \\ &= [F'G + FG']H + FGH' \\ &= F'GH + FG'H + FGH' \end{aligned}$$

as before.

(c) From the answer to (b), we observe that the derivative of a product is obtained by differentiating each factor in turn (leaving the other factors alone), and adding the results. So, in general,

$$(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_n)' = f_1' f_2 f_3 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 \cdots f_{n-1} f_n'.$$

52. **(a)** Since $x = a$ is a double zero of a polynomial $P(x)$, we can write $P(x) = (x-a)^2 Q(x)$, so $P(a) = 0$. Using the product rule, we have

$$P'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x).$$

Substituting in $x = a$, we see $P'(a) = 0$ also.

(b) Since $P(a) = 0$, we know $x = a$ is a zero of P, so that $x - a$ is a factor of P and we can write

$$P(x) = (x-a)Q(x),$$

where Q is some polynomial. Differentiating this expression for P using the product rule, we get

$$P'(x) = Q(x) + (x-a)Q'(x).$$

Since we are told that $P'(a) = 0$, we have

$$P'(a) = Q(a) + (a-a)Q'(a) = 0$$

and so $Q(a) = 0$. Therefore $x = a$ is a zero of Q, so again we can write

$$Q(x) = (x-a)R(x),$$

where R is some other polynomial. As a result,

$$P(x) = (x-a)Q(x) = (x-a)^2 R(x),$$

so that $x = a$ is a double zero of P.

Solutions for Section 3.4

Exercises

1. $f'(x) = 99(x+1)^{98} \cdot 1 = 99(x+1)^{98}$.

2. $f'(x) = \dfrac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}$.

3. $w' = 100(t^2+1)^{99}(2t) = 200t(t^2+1)^{99}$.

4. $w' = 100(t^3+1)^{99}(3t^2) = 300t^2(t^3+1)^{99}$.

5. $w' = 100(\sqrt{t}+1)^{99}\left(\frac{1}{2\sqrt{t}}\right) = \frac{50}{\sqrt{t}}(\sqrt{t}+1)^{99}$.

6. $f'(t) = (e^{3t})(3) = 3e^{3t}$.

7. $h'(w) = 5(w^4-2w)^4(4w^3-2)$

8. We can write $w(r) = (r^4+1)^{1/2}$, so

$$w'(r) = \frac{1}{2}(r^4+1)^{-1/2}(4r^3) = \frac{2r^3}{\sqrt{r^4+1}}.$$

9. $g(x) = \pi e^{\pi x}$.

10. $f(\theta) = (2^{-1})^\theta = (\frac{1}{2})^\theta$ so $f'(\theta) = (\ln\frac{1}{2})2^{-\theta}$.

11. $y' = (\ln\pi)\pi^{(x+2)}$.

12. $g'(x) = 2(\ln 3)3^{(2x+7)}$.

13. $k'(x) = 4(x^3+e^x)^3(3x^2+e^x)$.

14. $f'(x) = 2e^{2x}[x^2+5^x] + e^{2x}[2x+(\ln 5)5^x] = e^{2x}[2x^2+2x+(\ln 5+2)5^x]$.

15. Using the product rule gives $v'(t) = 2te^{-ct} - ce^{-ct}t^2 = (2t-ct^2)e^{-ct}$.

16. $p'(t) = 4e^{4t+2}$.

17. $\dfrac{d}{dt}e^{(1+3t)^2} = e^{(1+3t)^2}\dfrac{d}{dt}(1+3t)^2 = e^{(1+3t)^2}\cdot 2(1+3t)\cdot 3 = 6(1+3t)e^{(1+3t)^2}$.

18. $z'(x) = \dfrac{(\ln 2)2^x}{3\sqrt[3]{(2^x+5)^2}}$.

19. $z' = 5\cdot\ln 2\cdot 2^{5t-3}$.

20. $w' = \dfrac{3}{2}\sqrt{x^2\cdot 5^x}[2x(5^x)+(\ln 5)(x^2)(5^x)] = \dfrac{3}{2}x^2\sqrt{5^{3x}}(2+x\ln 5)$.

21. $y' = \frac{3}{2}e^{\frac{3}{2}w}$.

22. $y' = -4e^{-4t}$.

23. $y' = \dfrac{3s^2}{2\sqrt{s^3+1}}$.

24. $w' = \dfrac{1}{2\sqrt{s}}e^{\sqrt{s}}$.

25. $y' = 1\cdot e^{-t^2} + te^{-t^2}(-2t)$

26. $f'(z) = \dfrac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}$.

27. We can write this as $f(z) = \sqrt{z}e^{-z}$, in which case it is the same as problem 26. So $f'(z) = \dfrac{1}{2\sqrt{z}}e^{-z} - \sqrt{z}e^{-z}$.

28. $y' = \dfrac{\frac{2^z}{2\sqrt{z}} - (\sqrt{z})(\ln 2)(2^z)}{2^{2z}} = \dfrac{1-2z\ln 2}{2^{z+1}\sqrt{z}}$.

29. $f'(t) = 1\cdot e^{5-2t} + te^{5-2t}(-2) = e^{5-2t}(1-2t)$.

30. $y' = 2\left(\dfrac{x^2+2}{3}\right)\left(\dfrac{2x}{3}\right) = \dfrac{4}{9}x\left(x^2+2\right)$

31. We can write $h(x) = \left(\dfrac{x^2+9}{x+3}\right)^{1/2}$, so

$$h'(x) = \frac{1}{2}\left(\frac{x^2+9}{x+3}\right)^{-1/2}\left[\frac{2x(x+3)-(x^2+9)}{(x+3)^2}\right] = \frac{1}{2}\sqrt{\frac{x+3}{x^2+9}}\left[\frac{x^2+6x-9}{(x+3)^2}\right].$$

32. $\dfrac{dy}{dx} = \dfrac{2e^{2x}(x^2+1)-e^{2x}(2x)}{(x^2+1)^2} = \dfrac{2e^{2x}(x^2+1-x)}{(x^2+1)^2}$

33. $y' = \dfrac{-(3e^{3x}+2x)}{(e^{3x}+x^2)^2}$.

34. $h'(z) = \dfrac{-8b^4 z}{(a+z^2)^5}$

35. $h'(x) = (\ln 2)(3e^{3x})2^{e^{3x}} = 3e^{3x}2^{e^{3x}} \ln 2.$

36. $f'(z) = -2(e^z+1)^{-3} \cdot e^z = \dfrac{-2e^z}{(e^z+1)^3}.$

37. $f'(\theta) = -1(1+e^{-\theta})^{-2}(e^{-\theta})(-1) = \dfrac{e^{-\theta}}{(1+e^{-\theta})^2}.$

38. $f'(x) = 6(e^{5x})(5) + (e^{-x^2})(-2x) = 30e^{5x} - 2xe^{-x^2}.$

39.

$$\begin{aligned} f'(w) &= (e^{w^2})(10w) + (5w^2+3)(e^{w^2})(2w) \\ &= 2we^{w^2}(5+5w^2+3) \\ &= 2we^{w^2}(5w^2+8). \end{aligned}$$

40. $w' = (2t+3)(1-e^{-2t}) + (t^2+3t)(2e^{-2t}).$

41. $f(y) = \left[10^{(5-y)}\right]^{\frac{1}{2}} = 10^{\frac{5}{2}-\frac{1}{2}y}$

$f'(y) = (\ln 10)\left(10^{\frac{5}{2}-\frac{1}{2}y}\right)\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}(\ln 10)(10^{\frac{5}{2}-\frac{1}{2}y}).$

42. $f'(x) = e^{-(x-1)^2} \cdot (-2)(x-1).$

43. $f'(y) = e^{e^{(y^2)}}\left[(e^{y^2})(2y)\right] = 2ye^{[e^{(y^2)}+y^2]}.$

44. $f'(t) = 2(e^{-2e^{2t}})(-2e^{2t})2 = -8(e^{-2e^{2t}+2t}).$

45. Since a and b are constants, we have $f'(t) = ae^{bt}(b) = abe^{bt}.$

46. Since a and b are constants, we have $f'(x) = 3(ax^2+b)^2(2ax) = 6ax(ax^2+b)^2.$

47. We use the product rule. We have

$$f'(x) = (ax)(e^{-bx}(-b)) + (a)(e^{-bx}) = -abxe^{-bx} + ae^{-bx}.$$

48. $f'(x) = 6x(e^x - 4) + (3x^2+\pi)e^x = 6xe^x - 24x + 3x^2e^x + \pi e^x.$

Problems

49. We have $f(2) = (2-1)^3 = 1$, so $(2,1)$ is a point on the tangent line. Since $f'(x) = 3(x-1)^2$, the slope of the tangent line is

$$m = f'(2) = 3(2-1)^2 = 3.$$

The equation of the line is

$$y - 1 = 3(x-2) \quad \text{or} \quad y = 3x - 5.$$

50.

$$\begin{aligned} f(x) &= 6e^{5x} + e^{-x^2} \qquad & f'(x) &= 30e^{5x} - 2xe^{-x^2} \\ f(1) &= 6e^5 + e^{-1} & f'(1) &= 30e^5 - 2(1)e^{-1} \end{aligned}$$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (6e^5 + e^{-1}) &= (30e^5 - 2e^{-1})(x-1) \\ y - (6e^5 + e^{-1}) &= (30e^5 - 2e^{-1})x - (30e^5 - 2e^{-1}) \\ y &= (30e^5 - 2e^{-1})x - 30e^5 + 2e^{-1} + 6e^5 + e^{-1} \\ &\approx 4451.66x - 3560.81. \end{aligned}$$

51. The graph is concave down when $f''(x) < 0$.

$$\begin{aligned} f'(x) &= e^{-x^2}(-2x) \\ f''(x) &= \left[e^{-x^2}(-2x)\right](-2x) + e^{-x^2}(-2) \\ &= \frac{4x^2}{e^{x^2}} - \frac{2}{e^{x^2}} \\ &= \frac{4x^2-2}{e^{x^2}} < 0 \end{aligned}$$

The graph is concave down when $4x^2 < 2$. This occurs when $x^2 < \frac{1}{2}$, or $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

52.

$$\begin{aligned} f'(x) &= [10(2x+1)^9(2)][(3x-1)^7] + [(2x+1)^{10}][7(3x-1)^6(3)] \\ &= (2x+1)^9(3x-1)^6[20(3x-1)+21(2x+1)] \\ &= [(2x+1)^9(3x-1)^6](102x+1) \\ f''(x) &= [9(2x+1)^8(2)(3x-1)^6 + (2x+1)^9(6)(3x-1)^5(3)](102x+1) \\ &\quad +(2x+1)^9(3x-1)^6(102). \end{aligned}$$

53. **(a)** $H(x) = F(G(x))$
$H(4) = F(G(4)) = F(2) = 1$

(b) $H(x) = F(G(x))$
$H'(x) = F'(G(x)) \cdot G'(x)$
$H'(4) = F'(G(4)) \cdot G'(4) = F'(2) \cdot 6 = 5 \cdot 6 = 30$

(c) $H(x) = G(F(x))$
$H(4) = G(F(4)) = G(3) = 4$

(d) $H(x) = G(F(x))$
$H'(x) = G'(F(x)) \cdot F'(x)$
$H'(4) = G'(F(4)) \cdot F'(4) = G'(3) \cdot 7 = 8 \cdot 7 = 56$

(e) $H(x) = \frac{F(x)}{G(x)}$
$H'(x) = \frac{G(x)\cdot F'(x) - F(x)\cdot G'(x)}{[G(x)]^2}$
$H'(4) = \frac{G(4)\cdot F'(4) - F(4)\cdot G'(4)}{[G(4)]^2} = \frac{2\cdot 7 - 3\cdot 6}{2^2} = \frac{14-18}{4} = \frac{-4}{4} = -1$

54. **(a)** Differentiating $g(x) = \sqrt{f(x)} = (f(x))^{1/2}$, we have

$$\begin{aligned} g'(x) &= \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}} \\ g'(1) &= \frac{f'(1)}{2\sqrt{f(1)}} = \frac{3}{2\sqrt{4}} = \frac{3}{4}. \end{aligned}$$

(b) Differentiating $h(x) = f(\sqrt{x})$, we have

$$\begin{aligned} h'(x) &= f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ h'(1) &= f'(\sqrt{1}) \cdot \frac{1}{2\sqrt{1}} = \frac{f'(1)}{2} = \frac{3}{2}. \end{aligned}$$

55. **(a)** Since $h'(x) = f'(g(x)) \cdot g'(x)$, we have

$$h'(2) = f'(g(2)) \cdot g'(2) = f'(5) \cdot g'(2) = \pi\sqrt{2}.$$

(b) Since $h'(x) = g'(f(x)) \cdot f'(x)$, we have

$$h'(2) = g'(f(2)) \cdot f'(2) = g'(5) \cdot f'(2) = 7e.$$

(c) We have $h'(x) = f'(f(x)) \cdot f'(x)$, so

$$h'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2) = \pi e.$$

56. **(a)** If

$$p(x) = k(2x),$$

then

$$p'(x) = k'(2x) \cdot 2.$$

When $x = \frac{1}{2}$,

$$p'\left(\frac{1}{2}\right) = k'\left(2 \cdot \frac{1}{2}\right)(2) = 2 \cdot 2 = 4.$$

(b) If

$$q(x) = k(x+1),$$

then

$$q'(x) = k'(x+1) \cdot 1.$$

When $x = 0$,

$$q'(0) = k'(0+1)(1) = 2 \cdot 1 = 2.$$

(c) If

$$r(x) = k\left(\frac{1}{4}x\right),$$

then

$$r'(x) = k'\left(\frac{1}{4}x\right) \cdot \frac{1}{4}.$$

When $x = 4$,

$$r'(4) = k'\left(\frac{1}{4}4\right)\frac{1}{4} = 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

57. Yes. To see why, simply plug $x = \sqrt[3]{2t+5}$ into the expression $3x^2\dfrac{dx}{dt}$ and evaluate it. To do this, first we calculate $\dfrac{dx}{dt}$. By the chain rule,

$$\frac{dx}{dt} = \frac{d}{dt}(2t+5)^{\frac{1}{3}} = \frac{2}{3}(2t+5)^{-\frac{2}{3}} = \frac{2}{3}[(2t+5)^{\frac{1}{3}}]^{-2}.$$

But since $x = (2t+5)^{\frac{1}{3}}$, we have (by substitution)

$$\frac{dx}{dt} = \frac{2}{3}x^{-2}.$$

It follows that $3x^2\dfrac{dx}{dt} = 3x^2\left(\dfrac{2}{3}x^{-2}\right) = 2.$

58. We see that $m'(x)$ is nearly of the form $f'(g(x)) \cdot g'(x)$ where

$$f(g) = e^g \quad \text{and} \quad g(x) = x^6,$$

but $g'(x)$ is off by a multiple of 6. Therefore, using the chain rule, let

$$m(x) = \frac{f(g(x))}{6} = \frac{e^{(x^6)}}{6}.$$

59. We can find the rate the balance changes by differentiating B with respect to time: $B'(t) = 5000e^{0.08t} \cdot 0.08 = 400e^{0.08t}$. Calculating B' at time $t = 5$, we have $B'(5) = \$596.73$/yr. In 5 years, the account is generating \$597 per year of interest.

60. The concentration of the drug in the body after 4 hours is

$$f(4) = 27e^{-0.14(4)} = 15.4 \text{ ng/ml}.$$

The rate of change of the concentration is the derivative

$$f'(t) = 27e^{-0.14t}(-0.14) = -3.78e^{-0.14t}.$$

At $t = 4$, the concentration is changing at a rate of

$$f'(4) = -3.78e^{-0.14(4)} = -2.16 \text{ ng/ml per hour}.$$

61. We have $f(0) = 6$ and $f(10) = 6e^{0.013(10)} = 6.833$. The derivative of $f(t)$ is

$$f'(t) = 6e^{0.013t} \cdot 0.013 = 0.078e^{0.013t},$$

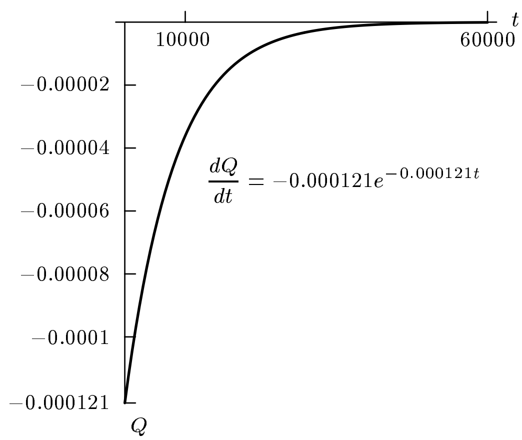
and so $f'(0) = 0.078$ and $f'(10) = 0.089$.

These values tell us that in 1999 (at $t = 0$), the population of the world was 6 billion people and the population was growing at a rate of 0.078 billion people per year. In the year 2009 (at $t = 10$), this model predicts that the population of the world will be 6.833 billion people and growing at a rate of 0.089 billion people per year.

62. (a)

$$\begin{aligned}\frac{dQ}{dt} &= \frac{d}{dt}e^{-0.000121t} \\ &= -0.000121e^{-0.000121t}\end{aligned}.$$

(b)



63. (a)

$$\frac{dH}{dt} = \frac{d}{dt}(40 + 30e^{-2t}) = 30(-2)e^{-2t} = -60e^{-2t}.$$

(b) Since e^{-2t} is always positive, $\dfrac{dH}{dt} < 0$; this makes sense because the temperature of the soda is decreasing.

(c) The magnitude of $\dfrac{dH}{dt}$ is

$$\left|\frac{dH}{dt}\right| = \left|-60e^{-2t}\right| = 60e^{-2t} \le 60 = \left|\frac{dH}{dt}_{t=0}\right|,$$

since $e^{-2t} \le 1$ for all $t \ge 0$ and $e^0 = 1$. This is just saying that at the moment that the can of soda is put in the refrigerator (at $t = 0$), the temperature difference between the soda and the inside of the refrigerator is the greatest, so the temperature of the soda is dropping the quickest.

64. (a) $\dfrac{dB}{dt} = P\left(1 + \dfrac{r}{100}\right)^t \ln\left(1 + \dfrac{r}{100}\right)$. The expression $\dfrac{dB}{dt}$ tells us how fast the amount of money in the bank is changing with respect to time for fixed initial investment P and interest rate r.

(b) $\dfrac{dB}{dr} = Pt\left(1 + \dfrac{r}{100}\right)^{t-1} \dfrac{1}{100}$. The expression $\dfrac{dB}{dr}$ indicates how fast the amount of money changes with respect to the interest rate r , assuming fixed initial investment P and time t.

65. The ripple's area and radius are related by $A(t) = \pi[r(t)]^2$. Taking derivatives and using the chain rule gives

$$\frac{dA}{dt} = \pi \cdot 2r\frac{dr}{dt}.$$

We know that $dr/dt = 10$ cm/sec, so when $r = 20$ cm we have

$$\frac{dA}{dt} = \pi \cdot 2 \cdot 20 \cdot 10\text{cm}^2/\text{sec} = 400\pi\text{cm}^2/\text{sec}.$$

66. **(a)**

$$\frac{dm}{dv} = \frac{d}{dv}\left[m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]$$
$$= m_0\left(-\frac{1}{2}\right)\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)$$
$$= \frac{m_0 v}{c^2}\frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)^3}}.$$

(b) $\dfrac{dm}{dv}$ represents the rate of change of mass with respect to the speed v.

67. **(a)** For $t<0, I=\dfrac{dQ}{dt}=0$.
For $t>0, I=\dfrac{dQ}{dt}=-\dfrac{Q_0}{RC}e^{-t/RC}$.

(b) For $t>0$, $t\to 0$ (that is, as $t\to 0^+$),

$$I = -\frac{Q_0}{RC}e^{-t/RC} \to -\frac{Q_0}{RC}.$$

Since $I=0$ just to the left of $t=0$ and $I=-Q_0/RC$ just to the right of $t=0$, it is not possible to define I at $t=0$.

(c) Q is not differentiable at $t=0$ because there is no tangent line at $t=0$.

68. The time constant for Q is the time, T_Q, such that $Q=Q_0/e$. Thus, T_Q satisfies

$$\frac{Q_0}{e} = Q_0 e^{-T_Q/RC}.$$

Canceling Q_0 and taking natural logs gives

$$e^{-T_Q/RC} = \frac{1}{e} = e^{-1}$$
$$\frac{-T_Q}{RC} = -1$$
$$T_Q = RC.$$

To find $I=dQ/dt$, differentiate Q:

$$I = \frac{dQ}{dt} = \frac{-Q_0}{RC}e^{-t/RC}.$$

Since the exponent of e is unchanged, so is the time constant. We know that the initial current is

$$I_0 = \frac{-Q_0}{RC}.$$

If T_I is the time constant for I, we know

$$\frac{1}{e}\left(\frac{-Q_0}{RC}\right) = \frac{-Q_0}{RC}e^{-T_I/RC}.$$

Canceling $-Q_0/RC$ gives

$$\frac{1}{e} = e^{-T_I/RC}.$$

This is the same equation as the one we solved for T_Q, so

$$T_I = RC.$$

69. Recall that $v = dx/dt$. We want to find the acceleration, dv/dt, when $x = 2$. Differentiating the expression for v with respect to t using the chain rule and substituting for v gives

$$\frac{dv}{dt} = \frac{d}{dx}(x^2 + 3x - 2) \cdot \frac{dx}{dt} = (2x + 3)v = (2x + 3)(x^2 + 3x - 2).$$

Substituting $x = 2$ gives

$$\text{Acceleration} = \left.\frac{dv}{dt}\right|_{x=2} = (2(2) + 3)(2^2 + 3 \cdot 2 - 2) = 56 \text{ cm/sec}^2.$$

70. **(a)** The population is increasing if $dP/dt > 0$, that is, if

$$kP(L - P) > 0.$$

Since $P \geq 0$ and $k, L > 0$, we must have $P > 0$ and $L - P > 0$ for this to be true. Thus, the population is increasing if $0 < P < L$.

The population is decreasing if $dP/dt < 0$, that is, if $P > L$.

The population remains constant if $dP/dt = 0$, so $P = 0$ or $P = L$.

(b) Differentiating with respect to t using the chain rule gives

$$\begin{aligned}\frac{d^2P}{dt^2} = \frac{d}{dt}\left(kP(L - P)\right) = \frac{d}{dP}(kLP - kP^2) \cdot \frac{dP}{dt} &= (kL - 2kP)(kP(L - P)) \\ &= k^2P(L - 2P)(L - P).\end{aligned}$$

71. Let f have a zero of multiplicity m at $x = a$ so that

$$f(x) = (x - a)^m h(x), \quad h(a) \neq 0.$$

Differentiating this expression gives

$$f'(x) = (x - a)^m h'(x) + m(x - a)^{(m-1)} h(x)$$

and both terms in the sum are zero when $x = a$ so $f'(a) = 0$. Taking another derivative gives

$$f''(x) = (x - a)^m h''(x) + 2m(x - a)^{(m-1)} h'(x) + m(m - 1)(x - a)^{(m-2)} h(x).$$

Again, each term in the sum contains a factor of $(x - a)$ to some positive power, so at $x = a$ this will evaluate to 0. Differentiating repeatedly, all derivatives will have positive integer powers of $(x - a)$ until the m^{th} and will therefore vanish. However,

$$f^{(m)}(a) = m!h(a) \neq 0.$$

Solutions for Section 3.5

Exercises

1.

Table 3.1

x	$\cos x$	Difference Quotient	$-\sin x$
0	1.0	−0.0005	0.0
0.1	0.995	−0.10033	−0.099833
0.2	0.98007	−0.19916	−0.19867
0.3	0.95534	−0.296	−0.29552
0.4	0.92106	−0.38988	−0.38942
0.5	0.87758	−0.47986	−0.47943
0.6	0.82534	−0.56506	−0.56464

2. $r'(\theta) = \cos\theta - \sin\theta$.

3. $s'(\theta) = -\sin\theta\sin\theta + \cos\theta\cos\theta = \cos^2\theta - \sin^2\theta = \cos 2\theta$.

4. $z' = -4\sin(4\theta)$.

5. $f'(x) = \cos(3x)\cdot 3 = 3\cos(3x)$.

6. $\dfrac{d}{dx}\sin(2-3x) = \cos(2-3x)\dfrac{d}{dx}(2-3x) = -3\cos(2-3x)$.

7. Using the chain rule gives $R'(x) = 3\pi\sin(\pi x)$.

8. $g'(\theta) = 2\sin(2\theta)\cos(2\theta)\cdot 2 - \pi = 4\sin(2\theta)\cos(2\theta) - \pi$

9. $f'(x) = (2x)(\cos x) + x^2(-\sin x) = 2x\cos x - x^2\sin x$.

10. $w' = e^t\cos(e^t)$.

11. $f'(x) = (e^{\cos x})(-\sin x) = -\sin x e^{\cos x}$.

12. $f'(y) = (\cos y)e^{\sin y}$.

13. $z' = e^{\cos\theta} - \theta(\sin\theta)e^{\cos\theta}$.

14. Using the chain rule gives $R'(\theta) = 3\cos(3\theta)e^{\sin(3\theta)}$.

15. $g'(\theta) = \dfrac{\cos(\tan\theta)}{\cos^2\theta}$

16. $w'(x) = \dfrac{2x}{\cos^2(x^2)}$

17.

$$\begin{aligned} f(x) &= (1-\cos x)^{\frac{1}{2}} \\ f'(x) &= \frac{1}{2}(1-\cos x)^{-\frac{1}{2}}(-(-\sin x)) \\ &= \frac{\sin x}{2\sqrt{1-\cos x}}. \end{aligned}$$

18. $f'(x) = [-\sin(\sin x)](\cos x)$.

19. $f'(x) = \dfrac{\cos x}{\cos^2(\sin x)}$.

20. $k'(x) = \frac{3}{2}\sqrt{\sin(2x)}(2\cos(2x)) = 3\cos(2x)\sqrt{\sin(2x)}$.

21. $f'(x) = 2\cdot[\sin(3x)] + 2x[\cos(3x)]\cdot 3 = 2\sin(3x) + 6x\cos(3x)$

22. $y' = e^\theta\sin(2\theta) + 2e^\theta\cos(2\theta)$.

23. $f'(x) = (e^{-2x})(-2)(\sin x) + (e^{-2x})(\cos x) = -2\sin x(e^{-2x}) + (e^{-2x})(\cos x) = e^{-2x}[\cos x - 2\sin x]$.

24. $z' = \dfrac{\cos t}{2\sqrt{\sin t}}$.

25. $y' = 5\sin^4\theta\cos\theta$.

26. $g'(z) = \dfrac{e^z}{\cos^2(e^z)}$.

27. $z' = \dfrac{-3e^{-3\theta}}{\cos^2(e^{-3\theta})}$.

28. $w' = (-\cos\theta)e^{-\sin\theta}$.

29. $h'(t) = 1\cdot(\cos t) + t(-\sin t) + \frac{1}{\cos^2 t} = \cos t - t\sin t + \frac{1}{\cos^2 t}$.

30. $f'(\alpha) = -\sin\alpha + 3\cos\alpha$

31. $k'(\alpha) = (5\sin^4\alpha\cos\alpha)\cos^3\alpha + \sin^5\alpha(3\cos^2\alpha(-\sin\alpha)) = 5\sin^4\alpha\cos^4\alpha - 3\sin^6\alpha\cos^2\alpha$

32. $t'(\theta) = \dfrac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta} = -\dfrac{(\sin^2\theta + \cos^2\theta)}{\sin^2\theta} = -\dfrac{1}{\sin^2\theta}$.

33. Using the power and quotient rules gives

$$f'(x) = \frac{1}{2}\left(\frac{1-\sin x}{1-\cos x}\right)^{-1/2}\left[\frac{-\cos x(1-\cos x) - (1-\sin x)\sin x}{(1-\cos x)^2}\right]$$

$$= \frac{1}{2}\sqrt{\frac{1-\cos x}{1-\sin x}}\left[\frac{-\cos x(1-\cos x)-(1-\sin x)\sin x}{(1-\cos x)^2}\right]$$
$$= \frac{1}{2}\sqrt{\frac{1-\cos x}{1-\sin x}}\left[\frac{1-\cos x-\sin x}{(1-\cos x)^2}\right].$$

34. The quotient rule gives $G'(x) = \dfrac{2\sin x\cos x(\cos^2 x+1)+2\sin x\cos x(\sin^2 x+1)}{(\cos^2 x+1)^2}$

or, using $\sin^2 x+\cos^2 x = 1$,

$G'(x) = \dfrac{6\sin x\cos x}{(\cos^2 x+1)^2}$.

35. $\dfrac{d}{dy}\left(\dfrac{y}{\cos y+a}\right) = \dfrac{\cos y+a-y(-\sin y)}{(\cos y+a)^2} = \dfrac{\cos y+a+y\sin y}{(\cos y+a)^2}$.

36. $h'(x) = (\ln 2)2^{\sin x}\cos x$.

37. $w' = (\ln 2)(2^{2\sin x+e^x})(2\cos x+e^x)$.

38. $f'(x) = 2\cos(2x)\sin(3x)+3\sin(2x)\cos(3x)$.

39. $f'(\theta) = 2\theta\sin\theta+\theta^2\cos\theta+2\cos\theta-2\theta\sin\theta-2\cos\theta = \theta^2\cos\theta$.

40. $f'(x) = \cos(\cos x+\sin x)(\cos x-\sin x)$

41. $f'(w) = -2\cos w\sin w-\sin(w^2)(2w) = -2(\cos w\sin w+w\sin(w^2))$

Problems

42. The pattern in the table below allows us to generalize and say that the $(4n)^{\text{th}}$ derivative of $\cos x$ is $\cos x$, i.e.,

$$\frac{d^4y}{dx^4} = \frac{d^8y}{dx^8} = \cdots = \frac{d^{4n}y}{dx^{4n}} = \cos x.$$

Thus we can say that $d^{48}y/dx^{48} = \cos x$. From there we differentiate twice more to obtain $d^{50}y/dx^{50} = -\cos x$.

n	1	2	3	4	$\cdots$	48	49	50
n^{th} derivative	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$		$\cos x$	$-\sin x$	$-\cos x$

43. We see that $q'(x)$ is of the form

$$\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{(g(x))^2},$$

with $f(x) = e^x$ and $g(x) = \sin x$. Therefore, using the quotient rule, let

$$q(x) = \frac{f(x)}{g(x)} = \frac{e^x}{\sin x}.$$

44. Since $F'(x)$ is of the form $\sin u$, we can make an initial guess that

$$F(x) = \cos(4x),$$

then

$$F'(x) = -4\sin(4x)$$

so we're off by a factor of -4. To fix this problem, we modify our guess by a factor of -4, so the next try is

$$F(x) = -(1/4)\cos(4x),$$

which has

$$F'(x) = \sin(4x).$$

45. We begin by taking the derivative of $y = \sin(x^4)$ and evaluating at $x = 10$:

$$\frac{dy}{dx} = \cos(x^4) \cdot 4x^3.$$

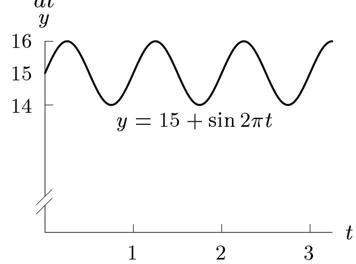
Evaluating $\cos(10{,}000)$ on a calculator (in radians) we see $\cos(10{,}000) < 0$, so we know that $dy/dx < 0$, and therefore the function is decreasing.
Next, we take the second derivative and evaluate it at $x = 10$;

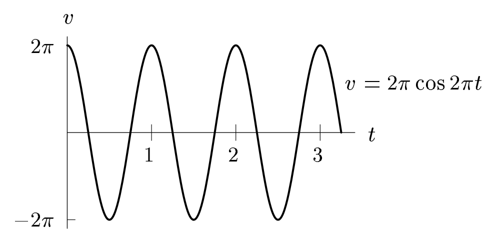
$$\frac{d^2y}{dx^2} = \underbrace{\cos(x^4) \cdot (12x^2)}_{\text{negative}} + \underbrace{4x^3 \cdot (-\sin(x^4))(4x^3)}_{\substack{\text{positive, but much}\\ \text{larger in magnitude}}}.$$

From this we can see that $d^2y/dx^2 > 0$, thus the graph is concave up.

46. **(a)** $v(t) = \dfrac{dy}{dt} = \dfrac{d}{dt}(15 + \sin(2\pi t)) = 2\pi\cos(2\pi t).$

(b)





47. **(a)** Differentiating gives

$$\frac{dy}{dt} = -\frac{4.9\pi}{6}\sin\left(\frac{\pi}{6}t\right).$$

The derivative represents the rate of change of the depth of the water in feet/hour.

(b) The derivative, dy/dt, is zero where the tangent line to the curve y is horizontal. This occurs when $dy/dt = \sin(\frac{\pi}{6}t) = 0$, or at $t = 6$, 12, 18 and 24 (6 am, noon, 6 pm, and midnight). When $dy/dt = 0$, the depth of the water is no longer changing. Therefore, it has either just finished rising or just finished falling, and we know that the harbor's level is at a maximum or a minimum.

48. **(a)** Differentiating, we find

$$\begin{aligned}\text{Rate of change of voltage with time} = \frac{dV}{dt} &= -120\pi \cdot 156\sin(120\pi t)\\ &= -18720\pi\sin(120\pi t)\text{ volts per second.}\end{aligned}$$

(b) The rate of change of voltage with time is zero when $\sin(120\pi t) = 0$. This occurs when $120\pi t$ equals any multiple of π. For example, $\sin(120\pi t) = 0$ when $120\pi t = \pi$, or at $t = 1/120$ seconds. Since there are an infinite number of multiples of π, there are many times when the rate of change dV/dt is zero.

(c) The maximum value of the rate of change is $18720\pi = 58810.6$ volts/sec.

49. **(a)** When $\sqrt{\frac{k}{m}}t = \frac{\pi}{2}$ the spring is farthest from the equilibrium position. This occurs at time $t = \frac{\pi}{2}\sqrt{\frac{m}{k}}$
$v = A\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)$, so the maximum velocity occurs when $t = 0$
$a = -A\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}t\right)$, so the maximum acceleration occurs when $\sqrt{\frac{k}{m}}t = \frac{3\pi}{2}$, which is at time $t = \frac{3\pi}{2}\sqrt{\frac{m}{k}}$

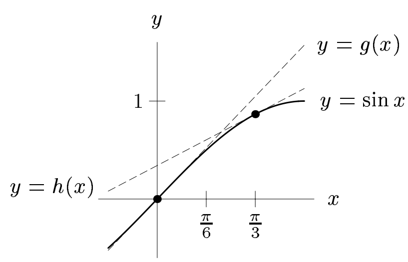
(b) $T = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}}$

(c) $\dfrac{dT}{dm} = \dfrac{2\pi}{\sqrt{k}} \cdot \dfrac{1}{2}m^{-\frac{1}{2}} = \dfrac{\pi}{\sqrt{km}}$
Since $\dfrac{dT}{dm} > 0$, an increase in the mass causes the period to increase.

50. The tangent lines to $f(x) = \sin x$ have slope $\frac{d}{dx}(\sin x) = \cos x$. The tangent line at $x = 0$ has slope $f'(0) = \cos 0 = 1$ and goes through the point $(0, 0)$. Consequently, its equation is $y = g(x) = x$. The approximate value of $\sin\frac{\pi}{6}$ given by this equation is then $g(\frac{\pi}{6}) = \frac{\pi}{6} \approx 0.524$.

Similarly, the tangent line at $x = \frac{\pi}{3}$ has slope $f'(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2}$ and goes through the point $(\frac{\pi}{3}, \frac{\sqrt{3}}{2})$. Consequently, its equation is $y = h(x) = \frac{1}{2}x + \frac{3\sqrt{3}-\pi}{6}$. The approximate value of $\sin\frac{\pi}{6}$ given by this equation is then $h(\frac{\pi}{6}) = \frac{6\sqrt{3}-\pi}{12} \approx 0.604$.

The actual value of $\sin\frac{\pi}{6}$ is $\frac{1}{2}$, so the approximation from 0 is better than that from $\frac{\pi}{3}$. This is because the slope of the function changes less between $x = 0$ and $x = \frac{\pi}{6}$ than it does between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$. This is illustrated below.



51. If the graphs of $y = \sin x$ and $y = ke^{-x}$ are tangent, then the y-values and the derivatives, $\frac{dy}{dx} = \cos x$ and $\frac{dy}{dx} = -ke^{-x}$, are equal at that point, so

$$\sin x = ke^{-x} \qquad \text{and} \qquad \cos x = -ke^{-x}.$$

Thus $\sin x = -\cos x$ so $\tan x = -1$. The smallest x-value is $x = 3\pi/4$, which leads to the smallest k value

$$k = \frac{\sin(3\pi/4)}{e^{-3\pi/4}} = 7.46.$$

When $x = \frac{3\pi}{4}$, we have $y = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$ so the point is $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

52. Differentiating with respect to t using the chain rule and substituting for dx/dt gives

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dx}(x\sin x)\cdot\frac{dx}{dt} = (\sin x + x\cos x)x\sin x.$$

53. **(a)** If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0}\frac{(\sin x\cos h + \sin h\cos x) - \sin x}{h} \\ &= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \sin h\cos x}{h} \\ &= \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}. \end{aligned}$$

(b) $\frac{\cos h-1}{h} \to 0$ and $\frac{\sin h}{h} \to 1$, as $h \to 0$. Thus, $f'(x) = \sin x\cdot 0 + \cos x\cdot 1 = \cos x$.

(c) Similarly,

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\ &= \lim_{h\to 0}\frac{(\cos x\cos h - \sin x\sin h) - \cos x}{h} \\ &= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x\sin h}{h} \\ &= \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} \\ &= -\sin x. \end{aligned}$$

54. **(a)** Sector OAQ is a sector of a circle with radius $\frac{1}{\cos\theta}$ and angle $\Delta\theta$. Thus its area is the left side of the inequality. Similarly, the area of Sector OBR is the right side of the equality. The area of the triangle OQR is $\frac{1}{2}\Delta\tan\theta$ since it is a triangle with base $\Delta\tan\theta$ (the segment QR) and height 1 (if you turn it sideways, it is easier to see this). Thus, using the given fact about areas (which is also clear from looking at the picture), we have

$$\frac{\Delta\theta}{2\pi}\cdot\pi\left(\frac{1}{\cos\theta}\right)^2 \le \frac{1}{2}\cdot\Delta(\tan\theta) \le \frac{\Delta\theta}{2\pi}\cdot\pi\left(\frac{1}{\cos(\theta+\Delta\theta)}\right)^2.$$

(b) Dividing the inequality through by $\frac{\Delta\theta}{2}$ and canceling the π's gives:

$$\left(\frac{1}{\cos\theta}\right)^2 \le \frac{\Delta\tan\theta}{\Delta\theta} \le \left(\frac{1}{\cos(\theta+\Delta\theta)}\right)^2$$

Then as $\Delta\theta \to 0$, the right and left sides both tend towards $\left(\frac{1}{\cos\theta}\right)^2$ while the middle (which is the difference quotient for tangent) tends to $(\tan\theta)'$. Thus, the derivative of tangent is "squeezed" between two values heading towards the same thing and must, itself, also tend to that value. Therefore, $(\tan\theta)' = \left(\frac{1}{\cos\theta}\right)^2$.

(c) Take the identity $\sin^2\theta + \cos^2\theta = 1$ and divide through by $\cos^2\theta$ to get $(\tan\theta)^2 + 1 = (\frac{1}{\cos\theta})^2$. Differentiating with respect to θ yields:

$$\begin{aligned}
2(\tan\theta)\cdot(\tan\theta)' &= 2\left(\frac{1}{\cos\theta}\right)\cdot\left(\frac{1}{\cos\theta}\right)' \\
2\left(\frac{\sin\theta}{\cos\theta}\right)\cdot\left(\frac{1}{\cos\theta}\right)^2 &= 2\left(\frac{1}{\cos\theta}\right)\cdot(-1)\left(\frac{1}{\cos\theta}\right)^2(\cos\theta)' \\
2\frac{\sin\theta}{\cos^3\theta} &= (-1)2\frac{1}{\cos^3\theta}(\cos\theta)' \\
-\sin\theta &= (\cos\theta)'.
\end{aligned}$$

(d)

$$\begin{aligned}
\frac{d}{d\theta}\left(\sin^2\theta + \cos^2\theta\right) &= \frac{d}{d\theta}(1) \\
2\sin\theta\cdot(\sin\theta)' + 2\cos\theta\cdot(\cos\theta)' &= 0 \\
2\sin\theta\cdot(\sin\theta)' + 2\cos\theta\cdot(-\sin\theta) &= 0 \\
(\sin\theta)' - \cos\theta &= 0 \\
(\sin\theta)' &= \cos\theta.
\end{aligned}$$

Solutions for Section 3.6

Exercises

1. $f'(t) = \frac{2t}{t^2+1}$.

2. $f'(x) = \frac{-1}{1-x} = \frac{1}{x-1}$.

3. Since $\ln(e^{2x}) = 2x$, the derivative $f'(x) = 2$.

4. Since $e^{\ln(e^{2x^2+3})} = e^{2x^2+3}$, the derivative $f'(x) = 4xe^{2x^2+3}$.

5. $f'(z) = -1(\ln z)^{-2}\cdot\frac{1}{z} = \frac{-1}{z(\ln z)^2}$.

6. $f'(\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$.

7. $f'(x) = \frac{1}{1-e^{-x}}\cdot -e^{-x}(-1) = \frac{e^{-x}}{1-e^{-x}}$.

8. $f'(\alpha) = \frac{1}{\sin\alpha}\cdot\cos\alpha = \frac{\cos\alpha}{\sin\alpha}$.

9. $f'(x) = \frac{1}{e^x+1}\cdot e^x$.

10. $\dfrac{dy}{dx} = \ln x + x\left(\dfrac{1}{x}\right) - 1 = \ln x$

11. $j'(x) = \dfrac{ae^{ax}}{(e^{ax}+b)}$

12. Using the product and chain rules gives $h'(w) = 3w^2\ln(10w) + w^3\dfrac{10}{10w} = 3w^2\ln(10w) + w^2$.

13. $f'(x) = \frac{1}{e^{7x}} \cdot (e^{7x})7 = 7.$
(Note also that $\ln(e^{7x}) = 7x$ implies $f'(x) = 7$.)

14. Note that $f(x) = e^{\ln x} \cdot e^1 = x \cdot e = ex$. So $f'(x) = e$. (Remember, e is just a constant.) You might also use the chain rule to get:
$f'(x) = e^{(\ln x)+1} \cdot \frac{1}{x}$.
[Are the two answers the same? Of course they are, since

$$e^{(\ln x)+1}\left(\frac{1}{x}\right) = e^{\ln x}\cdot e\left(\frac{1}{x}\right) = xe\left(\frac{1}{x}\right) = e.]$$

15. $f'(w) = \frac{1}{\cos(w-1)}[-\sin(w-1)] = -\tan(w-1)$.
[This could be done easily using the answer from Problem 6 and the chain rule.]

16. $f(t) = \ln t$ (because $\ln e^x = x$ *or* because $e^{\ln t} = t$), so $f'(t) = \frac{1}{t}$.

17. $f'(y) = \dfrac{2y}{\sqrt{1-y^4}}$.

18. $g'(t) = \dfrac{3}{(3t-4)^2+1}$.

19. $g(\alpha) = \alpha$, so $g'(\alpha) = 1$.

20. $g'(t) = e^{\arctan(3t^2)}\left(\dfrac{1}{1+(3t^2)^2}\right)(6t) = e^{\arctan(3t^2)}\left(\dfrac{6t}{1+9t^4}\right)$.

21. $g'(t) = \dfrac{-\sin(\ln t)}{t}$.

22. $h'(z) = (\ln 2)z^{(\ln 2-1)}$.

23. $h'(w) = \arcsin w + \dfrac{w}{\sqrt{1-w^2}}$.

24. Note that $f(x) = kx$ so, $f'(x) = k$.

25. Using the chain rule gives $r'(t) = \dfrac{2}{\sqrt{1-4t^2}}$.

26. $j'(x) = -\sin\left(\sin^{-1}x\right)\cdot\left[\dfrac{1}{\sqrt{1-x^2}}\right] = -\dfrac{x}{\sqrt{1-x^2}}$

27. $f'(x) = -\sin(\arctan 3x)\left(\dfrac{1}{1+(3x)^2}\right)(3) = \dfrac{-3\sin(\arctan 3x)}{1+9x^2}$.

28. Note that $g(x) = \arcsin(\sin \pi x) = \pi x$.
Thus, $g'(x) = \pi$.

29. Using the quotient rule gives

$$\begin{aligned} f'(x) &= \frac{1+\ln x - x(\frac{1}{x})}{(1+\ln x)^2} \\ &= \frac{\ln x}{(1+\ln x)^2}. \end{aligned}$$

30. $\dfrac{dy}{dx} = 2(\ln x + \ln 2) + 2x\left(\dfrac{1}{x}\right) - 2 = 2(\ln x + \ln 2) = 2\ln(2x)$

31. Using the chain rule gives $f'(x) = \dfrac{\cos x - \sin x}{\sin x + \cos x}$.

32. $f'(t) = \dfrac{1}{\ln t}\cdot\dfrac{1}{t} = \dfrac{1}{t\ln t}$

33. Using the chain rule gives

$$\begin{aligned}T'(u) &= \left[\frac{1}{1+\left(\frac{u}{1+u}\right)^2}\right]\left[\frac{(1+u)-u}{(1+u)^2}\right]\\ &= \frac{(1+u)^2}{(1+u)^2+u^2}\left[\frac{1}{(1+u)^2}\right]\\ &= \frac{1}{1+2u+2u^2}.\end{aligned}$$

34. Since $\ln\left[\left(\frac{1-\cos t}{1+\cos t}\right)^4\right] = 4\ln\left[\left(\frac{1-\cos t}{1+\cos t}\right)\right]$ we have

$$\begin{aligned}a'(t) &= 4\left(\frac{1+\cos t}{1-\cos t}\right)\left[\frac{\sin t(1+\cos t)+\sin t(1-\cos t)}{(1+\cos t)^2}\right]\\ &= \left[\frac{1+\cos t}{1-\cos t}\right]\left[\frac{8\sin t}{(1+\cos t)^2}\right]\\ &= \frac{8\sin t}{1-\cos^2 t}\\ &= \frac{8}{\sin t}.\end{aligned}$$

35. $f'(x) = -\sin(\arcsin(x+1))(\frac{1}{\sqrt{1-(x+1)^2}}) = \frac{-(x+1)}{\sqrt{1-(x+1)^2}}.$

Problems

36. Differentiating

$$\begin{aligned}f'(x) &= \frac{1}{x^2+1}\cdot 2x = 2x(x^2+1)^{-1}\\ f''(x) &= 2(x^2+1)^{-1} - 2x(x^2+1)^{-2}\cdot 2x\\ &= \frac{2}{(x^2+1)} - \frac{4x^2}{(x^2+1)^2} = \frac{2x^2+2}{(x^2+1)^2} - \frac{4x^2}{(x^2+1)^2}\\ &= \frac{2(1-x^2)}{(x^2+1)^2}.\end{aligned}$$

Since $(x^2+1)^2 > 0$ for all x, we see that $f''(0) > 0$ for $1-x^2 > 0$ or $x^2 < 1$. That is, $\ln(x^2+1)$ is concave up on the interval $-1 < x < 1$.

37. Let

$$g(x) = \arcsin x$$

so

$$\sin[g(x)] = x.$$

Differentiating,

$$\begin{aligned}\cos[g(x)]\cdot g'(x) &= 1\\ g'(x) &= \frac{1}{\cos[g(x)]}\end{aligned}$$

Using the fact that $\sin^2\theta + \cos^2\theta = 1$, and $\cos[g(x)] \geq 0$, since $-\frac{\pi}{2} \leq g(x) \leq \frac{\pi}{2}$, we get

$$\cos[g(x)] = \sqrt{1-(\sin[g(x)])^2}.$$

Therefore,

$$g'(x) = \frac{1}{\sqrt{1-(\sin[g(x)])^2}}$$

Since $\sin[g(x)] = x$, we have

$$g'(x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1.$$

38. Let

$$g(x) = \log x.$$

Then

$$10^{g(x)} = x.$$

Differentiating,

$$(\ln 10)[10^{g(x)}]g'(x) = 1$$
$$g'(x) = \frac{1}{(\ln 10)[10^{g(x)}]}$$
$$g'(x) = \frac{1}{(\ln 10)x}.$$

39. **(a)** From the second figure in the problem, we see that $\theta \approx 3.3$ when $t = 2$. The coordinates of P are given by $x = \cos\theta$, $y = \sin\theta$. When $t = 2$, the coordinates of P are

$$(x, y) \approx (\cos 3.3, \sin 3.3) = (-0.99, -0.16).$$

(b) Using the chain rule, the velocity in the x-direction is given by

$$v_x = \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = -\sin\theta\cdot\frac{d\theta}{dt}.$$

From Figure 3.5, we estimate that when $t = 2$,

$$\left.\frac{d\theta}{dt}\right|_{t=2} \approx 2.$$

So

$$v_x = \frac{dx}{dt} \approx -(-0.16)\cdot(2) = 0.32.$$

Similarly, the velocity in the y-direction is given by

$$v_y = \frac{dy}{dt} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dt} = \cos\theta\cdot\frac{d\theta}{dt}.$$

When $t = 2$

$$v_y = \frac{dy}{dt} \approx (-0.99)\cdot(2) = -1.98.$$

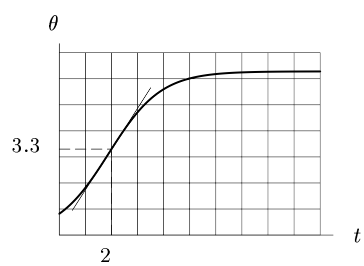


Figure 3.5

40. **(a)** The definition of the derivative of $\ln(1+x)$ at $x = 0$ is

$$\lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \left.\frac{1}{1+x}\right|_{x=0} = 1.$$

(b) The rules of logarithms give

$$\lim_{h\to 0}\frac{\ln(1+h)}{h} = \lim_{h\to 0}\frac{1}{h}\ln(1+h) = \lim_{h\to 0}\ln(1+h)^{1/h} = 1.$$

Thus, taking e to both sides and using the fact that $e^{\ln A} = A$, we have

$$e^{\lim_{h\to 0} \ln(1+h)^{1/h}} = \lim_{h\to 0} e^{\ln(1+h)^{1/h}} = e^1$$
$$\lim_{h\to 0}(1+h)^{1/h} = e.$$

This limit is sometimes used as the definition of e.

(c) Let $n = 1/h$. Then as $h \to 0^+$, we have $n \to \infty$. Since

$$\lim_{h\to 0^+}(1+h)^{1/h} = \lim_{h\to 0}(1+h)^{1/h} = e,$$

we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

This limit is also sometimes used as the definition of e.

41. pH $= 2 = -\log x$ means $\log x = -2$ so $x = 10^{-2}$. Rate of change of pH with hydrogen ion concentration is

$$\frac{d}{dx}\text{pH} = -\frac{d}{dx}(\log x) = \frac{-1}{x(\ln 10)} = -\frac{1}{(10^{-2})\ln 10} = -43.4$$

42. The closer you look at the function, the more it begins to look like a line with slope equal to the derivative of the function at $x = 0$. Hence, functions whose derivatives at $x = 0$ are equal will look the same there.

The following functions look like the line $y = x$ since, in all cases, $y' = 1$ at $x = 0$.

$y = x$ $\quad y' = 1$
$y = \sin x$ $\quad y' = \cos x$
$y = \tan x$ $\quad y' = \frac{1}{\cos^2 x}$
$y = \ln(x+1)$ $\quad y' = \frac{1}{x+1}$

The following functions look like the line $y = 0$ since, in all cases, $y' = 0$ at $x = 0$.

$y = x^2$ $\quad y' = 2x$
$y = x\sin x$ $\quad y' = x\cos x + \sin x$
$y = x^3$ $\quad y' = 3x^2$
$y = \frac{1}{2}\ln(x^2+1)$ $\quad y' = 2x\cdot\frac{1}{2}\cdot\frac{1}{x^2+1} = \frac{x}{x^2+1}$
$y = 1 - \cos x$ $\quad y' = \sin x$

The following functions look like the line $x = 0$ since, in all cases, as $x \to 0^+$, the slope $y' \to \infty$.

$y = \sqrt{x}$ $\quad y' = \frac{1}{2\sqrt{x}}$
$y = \sqrt{\frac{x}{x+1}}$ $\quad y' = \frac{(x+1)-x}{(x+1)^2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{\frac{x}{x+1}}} = \frac{1}{2(x+1)^2}\cdot\sqrt{\frac{x+1}{x}}$
$y = \sqrt{2x - x^2}$ $\quad y' = (2-2x)\frac{1}{2}\cdot\frac{1}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{2x-x^2}}$

43. **(a)**

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\cdot\left(-\frac{1}{x^2}\right) \\ &= \frac{1}{1+x^2} + \left(-\frac{1}{x^2+1}\right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0 \end{aligned}$$

(b) f is a constant function. Checking at a few values of x,

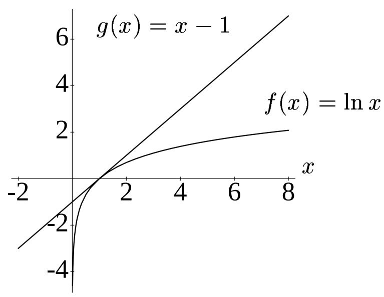
Table 3.2

x	$\arctan x$	$\arctan x^{-1}$	$f(x) = \arctan x + \arctan x^{-1}$
1	0.785392	0.7853982	1.5707963
2	1.1071487	0.4636476	1.5707963
3	1.2490458	0.3217506	1.5707963

44. **(a)** $y = \ln x$, $y' = \frac{1}{x}$; $f'(1) = \frac{1}{1} = 1$.
$y - y_1 = m(x - x_1)$, $y - 0 = 1(x - 1)$; $y = g(x) = x - 1$.

(b) $g(1.1) = 1.1 - 1 = 0.1$; $g(2) = 2 - 1 = 1$.

(c) $f(1.1)$ and $f(2)$ are below $g(x) = x - 1$. $f(0.9)$ and $f(0.5)$ are also below $g(x)$. This would be true for any approximation of this function by a tangent line since f is concave down ($f''(x) = -\frac{1}{x^2} < 0$ for all $x \neq 0$). See figure below. Thus, for a given x-value, the y-value given by the function is always below the value given by the tangent line.



45. **(a)** Let $g(x) = ax^2 + bx + c$ be our quadratic and $f(x) = \ln x$. For the best approximation, we want to find a quadratic with the same value as $\ln x$ at $x = 1$ and the same first and second derivatives as $\ln x$ at $x = 1$. $g'(x) = 2ax + b$, $g''(x) = 2a$, $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$.

$$\begin{aligned} g(1) &= a(1)^2 + b(1) + c & f(1) &= 0 \\ g'(1) &= 2a(1) + b & f'(1) &= 1 \\ g''(1) &= 2a & f''(1) &= -1 \end{aligned}$$

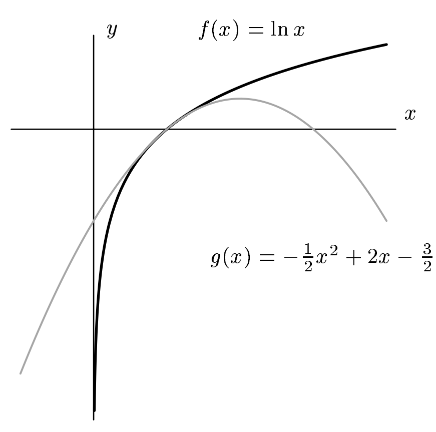
Thus, we obtain the equations

$$\begin{aligned} a + b + c &= 0 \\ 2a + b &= 1 \\ 2a &= -1 \end{aligned}$$

We find $a = -\frac{1}{2}$, $b = 2$ and $c = -\frac{3}{2}$. Thus our approximation is:

$$g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$$

(b) From the graph below, we notice that around $x = 1$, the value of $f(x) = \ln x$ and the value of $g(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}$ are very close.



(c) $g(1.1) = 0.095 \quad g(2) = 0.5$
Compare with $f(1.1) = 0.0953$, $f(2) = 0.693$.

46. We differentiate $F = k/r^2$ with respect to t using the chain rule to give

$$\frac{dF}{dt} = -\frac{2k}{r^3} \cdot \frac{dr}{dt}.$$

We know that $k = 10^{13}$ newton $\cdot$ km^2 and that the rocket is moving at 0.2 km/sec when $r = 10^4$ km. In other words, $dr/dt = 0.2$ km/sec when $r = 10^4$. Substituting gives

$$\frac{dF}{dt} = -\frac{2 \cdot 10^{13}}{(10^4)^3} \cdot 0.2 = -4 \text{ newtons/sec.}$$

47. **(a)** Assuming that $T(1) = 98.6 - 2 = 96.6$, we get

$$\begin{aligned} 96.6 &= 68 + 30.6e^{-k \cdot 1} \\ 28.6 &= 30.6e^{-k} \\ 0.935 &= e^{-k}. \end{aligned}$$

So

$$k = -\ln(0.935) \approx 0.067.$$

(b) We're looking for a value of t which gives $T'(t) = -1$. First we find $T'(t)$:

$$\begin{aligned} T(t) &= 68 + 30.6e^{-0.067t} \\ T'(t) &= (30.6)(-0.067)e^{-0.067t} \approx -2e^{-0.067t}. \end{aligned}$$

Setting this equal to -1°F per hour gives

$$\begin{aligned} -1 &= -2e^{-0.067t} \\ \ln(0.5) &= -0.067t \\ t &= -\frac{\ln(0.5)}{0.067} \approx 10.3. \end{aligned}$$

Thus, when $t \approx 10.3$ hours, we have $T'(t) \approx -1^\circ$F per hour.

(c) The coroner's rule of thumb predicts that in 24 hours the body temperature will decrease 25°F, to about 73.6°F. The formula predicts a temperature of

$$T(24) = 68 + 30.6e^{-0.067 \cdot 24} \approx 74.1^\circ\text{F}.$$

48. **(a)** Since $P = 1$ when $V = 20$, we have

$$k = 1 \cdot (20^{1.4}) = 66.29.$$

Thus, we have

$$P = 66.29V^{-1.4}.$$

Differentiating gives

$$\frac{dP}{dV} = 66.29(-1.4V^{-2.4}) = -92.8V^{-2.4} \text{ atmospheres/cm}^3.$$

(b) We are given that $dV/dt = 2\,\text{cm}^3/\text{min}$ when $V = 30\,\text{cm}^3$. Using the chain rule, we have

$$\begin{aligned} \frac{dP}{dt} = \frac{dP}{dV} \cdot \frac{dV}{dt} &= \left(-92.8V^{-2.4} \frac{\text{atm}}{\text{cm}^3}\right) \left(2\frac{\text{cm}^3}{\text{min}}\right) \\ &= -92.8\left(30^{-2.4}\right) 2\frac{\text{atm}}{\text{min}} \\ &= -0.0529 \text{ atmospheres/min} \end{aligned}$$

Thus, the pressure is decreasing at 0.0529 atmospheres per minute.

49. If V is the volume of the balloon and r is its radius, then

$$V = \frac{4}{3}\pi r^3.$$

We want to know the rate at which air is being blown into the balloon, which is the rate at which the volume is increasing, dV/dt. We are told that

$$\frac{dr}{dt} = 2 \text{ cm/sec} \quad \text{when} \quad r = 10 \text{ cm}.$$

Using the chain rule, we have

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Substituting gives

$$\frac{dV}{dt} = 4\pi(10)^2 2 = 800\pi = 2513.3 \text{ cm}^3\text{/sec.}$$

50. We are given that the volume is increasing at a constant rate $\frac{dV}{dt} = 400$. The radius r is related to the volume by the formula $V = \frac{4}{3}\pi r^3$. By implicit differentiation, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Plugging in $\frac{dV}{dt} = 400$ and $r = 10$, we have

$$400 = 400\pi \frac{dr}{dt}$$

so $\frac{dr}{dt} = \frac{1}{\pi} \approx 0.32\mu$m/day.

51. Let r be the radius of the raindrop. Then its volume $V = \frac{4}{3}\pi r^3$ cm^3 and its surface area is $S = 4\pi r^2$ cm^2. It is given that

$$\frac{dV}{dt} = 2S = 8\pi r^2.$$

Furthermore,

$$\frac{dV}{dr} = 4\pi r^2,$$

so from the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \quad \text{and thus} \quad \frac{dr}{dt} = \frac{dV/dt}{dV/dr} = 2.$$

Since dr/dt is a constant, $dr/dt = 2$, the radius is increasing at a constant rate of 2 cm/sec.

52. The volume, V, of a cone of height h and radius r is

$$V = \frac{1}{3}\pi r^2 h.$$

Since the angle of the cone is $\pi/6$, so $r = h\tan(\pi/6) = h/\sqrt{3}$

$$V = \frac{1}{3}\pi \left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{1}{9}\pi h^3.$$

Differentiating gives

$$\frac{dV}{dh} = \frac{1}{3}\pi h^2.$$

To find dh/dt, use the chain rule to obtain

$$\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}.$$

So,

$$\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{0.1\text{meters/hour}}{\pi h^2/3} = \frac{0.3}{\pi h^2} \text{ meters/hour.}$$

Since $r = h\tan(\pi/6) = h/\sqrt{3}$, we have

$$\frac{dr}{dt} = \frac{dh}{dt}\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\frac{0.3}{\pi h^2} \text{ meters/hour.}$$

53. **(a)** Using Pythagoras' theorem, we see

$$z^2 = 0.5^2 + x^2$$

so

$$z = \sqrt{0.25 + x^2}.$$

(b) We want to calculate dz/dt. Using the chain rule, we have

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} = \frac{2x}{2\sqrt{0.25 + x^2}} \frac{dx}{dt}.$$

Because the train is moving at 0.8 km/hr, we know that

$$\frac{dx}{dt} = 0.8 \text{ km/hr}.$$

At the moment we are interested in $z = 1$ km so

$$1^2 = 0.25 + x^2$$

giving

$$x = \sqrt{0.75} = 0.866 \text{ km}.$$

Therefore

$$\frac{dz}{dt} = \frac{2(0.866)}{2\sqrt{0.25 + 0.75}} \cdot 0.8 = 0.866 \cdot 0.8 = 0.693 \text{ km/min}.$$

(c) We want to know $d\theta/dt$, where θ is as shown in Figure 3.6. Since

$$\frac{x}{0.5} = \tan\theta$$

we know

$$\theta = \arctan\left(\frac{x}{0.5}\right),$$

so

$$\frac{d\theta}{dt} = \frac{1}{1 + (x/0.5)^2} \cdot \frac{1}{0.5} \frac{dx}{dt}.$$

We know that $dx/dt = 0.8$ km/min and, at the moment we are interested in, $x = \sqrt{0.75}$. Substituting gives

$$\frac{d\theta}{dt} = \frac{1}{1 + 0.75/0.25} \cdot \frac{1}{0.5} \cdot 0.8 = 0.4 \text{ radians/min}.$$

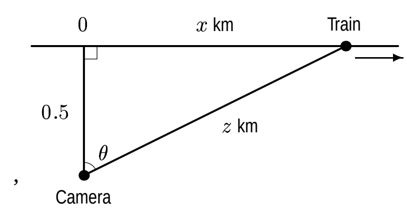


Figure 3.6

54. Using the triangle OSL in Figure 3.7, we label the distance x.

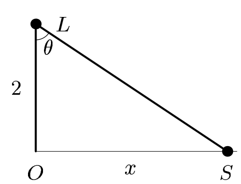


Figure 3.7

We want to calculate $dx/d\theta$. First we must find x as a function of θ. From the triangle, we see

$$\frac{x}{2} = \tan\theta \quad \text{so} \quad x = 2\tan\theta.$$

Thus,

$$\frac{dx}{d\theta} = \frac{2}{\cos^2\theta}.$$

55. (a) Since the elevator is descending at 30 ft/sec, its height from the ground is given by $h(t) = 300 - 30t$, for $0 \leq t \leq 10$.

(b) From the triangle in the figure,

$$\tan\theta = \frac{h(t)-100}{150} = \frac{300-30t-100}{150} = \frac{200-30t}{150}.$$

Therefore

$$\theta = \arctan\left(\frac{200-30t}{150}\right)$$

and

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{200-30t}{150}\right)^2}\cdot\left(\frac{-30}{150}\right) = -\frac{1}{5}\left(\frac{150^2}{150^2+(200-30t)^2}\right).$$

Notice that $\frac{d\theta}{dt}$ is always negative, which is reasonable since θ decreases as the elevator descends.

(c) If we want to know when θ changes (decreases) the fastest, we want to find out when $d\theta/dt$ has the largest magnitude. This will occur when the denominator, $150^2 + (200-30t)^2$, in the expression for $d\theta/dt$ is the smallest, or when $200 - 30t = 0$. This occurs when $t = \frac{200}{30}$ seconds, and so $h(\frac{200}{30}) = 100$ feet, i.e., when the elevator is at the level of the observer.

Solutions for Section 3.7

Exercises

1. We differentiate implicitly both sides of the equation with respect to x.

$$2x + 2y\frac{dy}{dx} = 0\,,$$
$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

2. We differentiate implicitly both sides of the equation with respect to x.

$$2x + \left(y + x\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = y^2 + x(2y)\frac{dy}{dx}\,,$$
$$x\frac{dy}{dx} - 3y^2\frac{dy}{dx} - 2xy\frac{dy}{dx} = y^2 - y - 2x\,,$$
$$\frac{dy}{dx} = \frac{y^2-y-2x}{x-3y^2-2xy}.$$

3. We differentiate implicitly both sides of the equation with respect to x.

$$x^{1/2} = 5y^{1/2}$$
$$\frac{1}{2}x^{-1/2} = \frac{5}{2}y^{-1/2}\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-1/2}}{\frac{5}{2}y^{-1/2}} = \frac{1}{5}\sqrt{\frac{y}{x}} = \frac{1}{25}.$$

We can also obtain this answer by realizing that the original equation represents part of the line $x = 25y$ which has slope $1/25$.

4. We differentiate implicitly both sides of the equation with respect to x.

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 25\,,$$
$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0\,,$$
$$\frac{dy}{dx} = -\frac{\frac{1}{2}x^{-\frac{1}{2}}}{\frac{1}{2}y^{-\frac{1}{2}}} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{\sqrt{y}}{\sqrt{x}} = -\sqrt{\frac{y}{x}}.$$

5. We differentiate implicity with respect to x.

$$\begin{aligned} y + x\frac{dy}{dx} - 1 - \frac{3dy}{dx} &= 0 \\ (x-3)\frac{dy}{dx} &= 1 - y \\ \frac{dy}{dx} &= \frac{1-y}{x-3} \end{aligned}$$

6.

$$\begin{aligned} 12x + 8y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-12x}{8y} = \frac{-3x}{2y} \end{aligned}$$

7.

$$\begin{aligned} 2ax - 2by\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-2ax}{-2by} = \frac{ax}{by} \end{aligned}$$

8. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} \ln x + \ln(y^2) &= 3 \\ \frac{1}{x} + \frac{1}{y^2}(2y)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-1/x}{2y/y^2} = -\frac{y}{2x}. \end{aligned}$$

9. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} e^{x^2} + \ln y &= 0 \\ 2xe^{x^2} + \frac{1}{y}\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -2xye^{x^2}. \end{aligned}$$

10. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned} \arctan(x^2y) &= xy^2 \\ \frac{1}{1+x^4y^2}(2xy + x^2\frac{dy}{dx}) &= y^2 + 2xy\frac{dy}{dx} \\ 2xy + x^2\frac{dy}{dx} &= [1 + x^4y^2][y^2 + 2xy\frac{dy}{dx}] \\ \frac{dy}{dx}[x^2 - (1 + x^4y^2)(2xy)] &= (1 + x^4y^2)y^2 - 2xy \\ \frac{dy}{dx} &= \frac{y^2 + x^4y^4 - 2xy}{x^2 - 2xy - 2x^5y^3}. \end{aligned}$$

11. We differentiate implicitly both sides of the equation with respect to x.

$$\ln y + x\frac{1}{y}\frac{dy}{dx} + 3y^2\frac{dy}{dx} = \frac{1}{x}$$

$$\begin{aligned}\frac{x}{y}\frac{dy}{dx}+3y^2\frac{dy}{dx}&=\frac{1}{x}-\ln y\\ \frac{dy}{dx}\left(\frac{x}{y}+3y^2\right)&=\frac{1-x\ln y}{x}\\ \frac{dy}{dx}\left(\frac{x+3y^3}{y}\right)&=\frac{1-x\ln y}{x}\\ \frac{dy}{dx}&=\frac{(1-x\ln y)}{x}\cdot\frac{y}{(x+3y^3)}\end{aligned}$$

12. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned}\cos(xy)\left(y+x\frac{dy}{dx}\right)&=2\\ y\cos(xy)+x\cos(xy)\frac{dy}{dx}&=2\\ \frac{dy}{dx}&=\frac{2-y\cos(xy)}{x\cos(xy)}.\end{aligned}$$

13. $\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx}=0$, $\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$.

14. We differentiate implicitly both sides of the equation with respect to x.

$$\begin{aligned}e^{\cos y}(-\sin y)\frac{dy}{dx}&=3x^2\arctan y+x^3\frac{1}{1+y^2}\frac{dy}{dx}\\ \frac{dy}{dx}\left(-e^{\cos y}\sin y-\frac{x^3}{1+y^2}\right)&=3x^2\arctan y\\ \frac{dy}{dx}&=\frac{3x^2\arctan y}{-e^{\cos y}\sin y-x^3(1+y^2)^{-1}}.\end{aligned}$$

15. Using the relation $\cos^2 y+\sin^2 y=1$, the equation becomes:
$1=y+2$ or $y=-1$. Hence, $\frac{dy}{dx}=0$.

16. Differentiating $x^2+y^2=1$ with respect to x gives

$$2x+2yy'=0$$

so that

$$y'=-\frac{x}{y}$$

At the point $(0,1)$ the slope is 0.

17. Differentiating $\sin(xy)=x$ with respect to x gives

$$(y+xy')\cos(xy)=1$$

or

$$xy'\cos(xy)=1-y\cos(xy)$$

so that

$$y'=\frac{1-y\cos(xy)}{x\cos(xy)}.$$

As we move along the curve to the point $(1,\frac{\pi}{2})$, the value of $dy/dx\to\infty$, which tells us the tangent to the curve at $(1,\frac{\pi}{2})$ has infinite slope; the tangent is the vertical line $x=1$.

18. The slope is given by dy/dx, which we find using implicit differentiation. Notice that the product rule is needed for the second term. We differentiate to obtain:

$$\begin{aligned} 3x^2 + 5x^2\frac{dy}{dx} + 10xy + 4y\frac{dy}{dx} &= 4\frac{dy}{dx} \\ (5x^2 + 4y - 4)\frac{dy}{dx} &= -3x^2 - 10xy \\ \frac{dy}{dx} &= \frac{-3x^2 - 10xy}{5x^2 + 4y - 4}. \end{aligned}$$

At the point $(1, 2)$, we have $dy/dx = (-3 - 20)/(5 + 8 - 4) = -23/9$. The slope of this curve at the point $(1, 2)$ is $-23/9$.

19. Differentiating with respect to x gives

$$3x^2 + 2xy' + 2y + 2yy' = 0$$

so that

$$y' = -\frac{3x^2 + 2y}{2x + 2y}$$

At the point $(1, 1)$ the slope is $-\frac{5}{4}$.

20. First, we must find the slope of the tangent, i.e. $\left.\frac{dy}{dx}\right|_{(1,-1)}$. Differentiating implicitly, we have:

$$\begin{aligned} y^2 + x(2y)\frac{dy}{dx} &= 0, \\ \frac{dy}{dx} &= -\frac{y^2}{2xy} = -\frac{y}{2x}. \end{aligned}$$

Substitution yields $\left.\frac{dy}{dx}\right|_{(1,-1)} = -\frac{-1}{2} = \frac{1}{2}$. Using the point-slope formula for a line, we have that the equation for the tangent line is $y + 1 = \frac{1}{2}(x - 1)$ or $y = \frac{1}{2}x - \frac{3}{2}$.

21. First we must find the slope of the tangent, $\frac{dy}{dx}$, at $(1, e^2)$. Differentiating implicitly, we have:

$$\begin{aligned} \frac{1}{xy}\left(x\frac{dy}{dx} + y\right) &= 2 \\ \frac{dy}{dx} &= \frac{2xy - y}{x}. \end{aligned}$$

Evaluating dy/dx at $(1, e^2)$ yields $(2(1)e^2 - e^2)/1 = e^2$. Using the point-slope formula for the equation of the line, we have:

$$y - e^2 = e^2(x - 1),$$

or

$$y = e^2x.$$

22. First, we must find the slope of the tangent, $\left.\frac{dy}{dx}\right|_{(4,2)}$. Implicit differentiation yields:

$$2y\frac{dy}{dx} = \frac{2x(xy - 4) - x^2\left(x\frac{dy}{dx} + y\right)}{(xy - 4)^2}.$$

Given the complexity of the above equation, we first want to substitute 4 for x and 2 for y (the coordinates of the point where we are constructing our tangent line), then solve for $\frac{dy}{dx}$. Substitution yields:

$$2 \cdot 2\frac{dy}{dx} = \frac{(2 \cdot 4)(4 \cdot 2 - 4) - 4^2\left(4\frac{dy}{dx} + 2\right)}{(4 \cdot 2 - 4)^2} = \frac{8(4) - 16(4\frac{dy}{dx} + 2)}{16} = -4\frac{dy}{dx}.$$

$$4\frac{dy}{dx} = -4\frac{dy}{dx},$$

Solving for $\frac{dy}{dx}$, we have:

$$\frac{dy}{dx} = 0.$$

The tangent is a horizontal line through $(4, 2)$, hence its equation is $y = 2$.

23. First, we must find the slope of the tangent, $\left.\frac{dy}{dx}\right|_{(a,0)}$. We differentiate implicitly, obtaining:

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0\ ,$$

$$\frac{dy}{dx} = -\frac{\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}\ .$$

Substitution yields, $\left.\frac{dy}{dx}\right|_{(a,0)} = \frac{\sqrt[3]{0}}{\sqrt[3]{a}} = 0$. The tangent is a horizontal line through $(a, 0)$, hence its equation is $y = 0$.

Problems

24. **(a)** By implicit differentiation, we have:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} - 4 + 7\frac{dy}{dx} &= 0 \\ (2y+7)\frac{dy}{dx} &= 4 - 2x \\ \frac{dy}{dx} &= \frac{4-2x}{2y+7}. \end{aligned}$$

(b) The curve has a horizontal tangent line when $dy/dx = 0$, which occurs when $4 - 2x = 0$ or $x = 2$. The curve has a horizontal tangent line at all points where $x = 2$.
The curve has a vertical tangent line when dy/dx is undefined, which occurs when $2y + 7 = 0$ or when $y = -7/2$. The curve has a vertical tangent line at all points where $y = -7/2$.

25. **(a)** Taking derivatives implicitly, we get

$$\begin{aligned} \frac{2}{25}x + \frac{2}{9}y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-9x}{25y} \end{aligned}$$

(b) The slope is not defined anywhere along the line $y = 0$. This ellipse intersects that line in two places, $(-5, 0)$ and $(5, 0)$. (These are, of course, the "ends" of the ellipse where the tangent is vertical.)

26. **(a)** If $x = 4$ then $16 + y^2 = 25$, so $y = \pm 3$. We find $\frac{dy}{dx}$ implicitly:

$$\begin{aligned} 2x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

So the slope at $(4, 3)$ is $-\frac{4}{3}$ and at $(4, -3)$ is $\frac{4}{3}$. The tangent lines are:

$$(y-3) = -\frac{4}{3}(x-4) \quad \text{and} \quad (y+3) = \frac{4}{3}(x-4)$$

(b) The normal lines have slopes that are the negative of the reciprocal of the slopes of the tangent lines. Thus,

$$(y-3) = \frac{3}{4}(x-4) \quad \text{so} \quad y = \frac{3}{4}x$$

and

$$(y+3) = -\frac{3}{4}(x-4) \quad \text{so} \quad y = -\frac{3}{4}x$$

are the normal lines.

(c) These lines meet at the origin, which is the center of the circle.

27. (a) Solving for $\frac{dy}{dx}$ by implicit differentiation yields

$$3x^2 + 3y^2\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{y^2 - 3x^2}{3y^2 - 2xy}.$$

(b) We can approximate the curve near $x = 1$, $y = 2$ by its tangent line. The tangent line will have slope $\frac{(2)^2-3(1)^2}{3(2)^2-2(1)(2)} = \frac{1}{8} = 0.125$. Thus its equation is

$$y = 0.125x + 1.875$$

Using the y-values of the tangent line to approximate the y-values of the curve, we get:

x	0.96	0.98	1	1.02	1.04
approximate y	1.995	1.9975	2.000	2.0025	2.005

(c) When $x = 0.96$, we get the equation $0.96^3 + y^3 - 0.96y^2 = 5$, whose solution by numerical methods is 1.9945, which is close to the one above.

(d) The tangent line is horizontal when $\frac{dy}{dx}$ is zero and vertical when $\frac{dy}{dx}$ is undefined. These will occur when the numerator is zero and when the denominator is zero, respectively.

Thus, we know that the tangent is horizontal when $y^2 - 3x^2 = 0 \Rightarrow y = \pm\sqrt{3}x$. To find the points that satisfy this condition, we substitute back into the original equation for the curve:

$$x^3 + y^3 - xy^2 = 5$$
$$x^3 \pm 3\sqrt{3}x^3 - 3x^3 = 5$$
$$x^3 = \frac{5}{\pm 3\sqrt{3} - 2}$$
$$\text{So } x \approx 1.1609 \text{ or } x \approx -0.8857.$$

Substituting,

$$y = \pm\sqrt{3}x \quad \text{so} \quad y \approx 2.0107 \quad \text{or} \quad y \approx 1.5341.$$

Thus, the tangent line is horizontal at $(1.1609, 2.0107)$ and $(-0.8857, 1.5341)$.

Also, we know that the tangent is vertical whenever $3y^2 - 2xy = 0$, that is, when $y = \frac{2}{3}x$ or $y = 0$. Substituting into the original equation for the curve gives us $x^3 + (\frac{2}{3}x)^3 - (\frac{2}{3})^2x^3 = 5$. This means $x^3 \approx 5.8696$, so $x \approx 1.8039$, $y \approx 1.2026$. The other vertical tangent is at $y = 0$, $x = \sqrt[3]{5}$.

28. The slope of the tangent to the curve $y = x^2$ at $x = 1$ is 2 so the equation of such a tangent will be of the form $y = 2x + c$. As the tangent must pass through $(1, 1)$, $c = -1$ and so the required tangent is $y = 2x - 1$.

Any circle centered at $(8, 0)$ will be of the form

$$(x - 8)^2 + y^2 = R^2.$$

The slope of this curve at (x, y) is given by implicit differentiation:

$$2(x - 8) + 2yy' = 0$$

or

$$y' = \frac{8 - x}{y}$$

For the tangent to the parabola to be tangential to the circle we need

$$\frac{8 - x}{y} = 2$$

so that at the point of contact of the circle and the line the coordinates are given by (x, y) when $y = 4 - x/2$. Substituting into the equation of the tangent line gives $x = 2$ and $y = 3$. From this we conclude that $R^2 = 45$ so that the equation of the circle is

$$(x - 8)^2 + y^2 = 45.$$

29. Let the point of intersection of the tangent line with the smaller circle be (x_1, y_1) and the point of intersection with the larger be (x_2, y_2). Let the tangent line be $y = mx + c$. Then at (x_1, y_1) and (x_2, y_2) the slopes of $x^2 + y^2 = 1$ and $y^2 + (x-3)^2 = 4$ are also m. The slope of $x^2+y^2 = 1$ is found by implicit differentiation: $2x+2yy' = 0$ so $y' = -x/y$. Similarly, the slope of $y^2 + (x-3)^2 = 4$ is $y' = -(x-3)/y$. Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1}{y_1} = -\frac{(x_2 - 3)}{y_2},$$

where $y_1 = \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4 - (x_2 - 3)^2}$. The positive values for y_1 and y_2 follow from Figure 3.8 and from our choice of $m > 0$. We obtain

$$\begin{aligned}
\frac{x_1}{\sqrt{1 - x_1^2}} &= \frac{x_2 - 3}{\sqrt{4 - (x_2 - 3)^2}} \\
\frac{x_1^2}{1 - x_1^2} &= \frac{(x_2 - 3)^2}{4 - (x_2 - 3)^2} \\
x_1^2[4 - (x_2 - 3)^2] &= (1 - x_1^2)(x_2 - 3)^2 \\
4x_1^2 - (x_1^2)(x_2 - 3)^2 &= (x_2 - 3)^2 - x_1^2(x_2 - 3)^2 \\
4x_1^2 &= (x_2 - 3)^2 \\
2|x_1| &= |x_2 - 3|.
\end{aligned}$$

From the picture $x_1 < 0$ and $x_2 < 3$. This gives $x_2 = 2x_1 + 3$ and $y_2 = 2y_1$. From

$$\frac{y_2 - y_1}{x_2 - x_1} = -\frac{x_1}{y_1},$$

substituting $y_1 = \sqrt{1 - x_1^2}$, $y_2 = 2y_1$ and $x_2 = 2x_1 + 3$ gives

$$x_1 = -\frac{1}{3}.$$

From $x_2 = 2x_1 + 3$ we get $x_2 = 7/3$. In addition, $y_1 = \sqrt{1 - x_1^2}$ gives $y_1 = 2\sqrt{2}/3$, and finally $y_2 = 2y_1$ gives $y_2 = 4\sqrt{2}/3$.

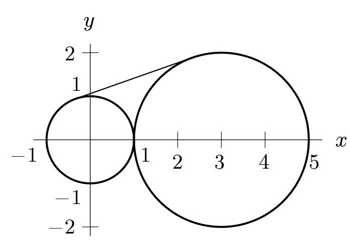


Figure 3.8

30. $y = x^{\frac{m}{n}}$. Taking n^{th} powers of both sides of this expression yields $(y)^n = (x^{\frac{m}{n}})^n$, or $y^n = x^m$.

$$\begin{aligned}
\frac{d}{dx}(y^n) &= \frac{d}{dx}(x^m) \\
ny^{n-1}\frac{dy}{dx} &= mx^{m-1} \\
\frac{dy}{dx} &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \\
&= \frac{m}{n}\frac{x^{m-1}}{x^{m-\frac{m}{n}}} \\
&= \frac{m}{n}x^{(m-1)-(m-\frac{m}{n})} = \frac{m}{n}x^{\frac{m}{n}-1}.
\end{aligned}$$

Solutions for Section 3.8

Exercises

1. Between times $t = 0$ and $t = 1$, x goes at a constant rate from 0 to 1 and y goes at a constant rate from 1 to 0. So the particle moves in a straight line from $(0, 1)$ to $(1, 0)$. Similarly, between times $t = 1$ and $t = 2$, it goes in a straight line to $(0, -1)$, then to $(-1, 0)$, then back to $(0, 1)$. So it traces out the diamond shown in Figure 3.9.

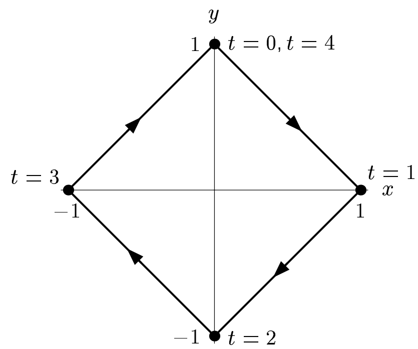


Figure 3.9

2. This is like Example 2, except that the x-coordinate goes all the way to 2 and back. So the particle traces out the rectangle shown in Figure 3.10.

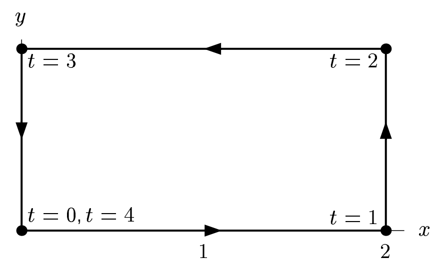


Figure 3.10

3. Between times $t = 0$ and $t = 1$, x goes from -1 to 1, while y stays fixed at 1. So the particle goes in a straight line from $(-1, 1)$ to $(1, 1)$. Then both the x- and y-coordinates decrease at a constant rate from 1 to -1. So the particle goes in a straight line from $(1, 1)$ to $(-1, -1)$. Then it moves across to $(1, -1)$, then back diagonally to $(-1, 1)$. See Figure 3.11.

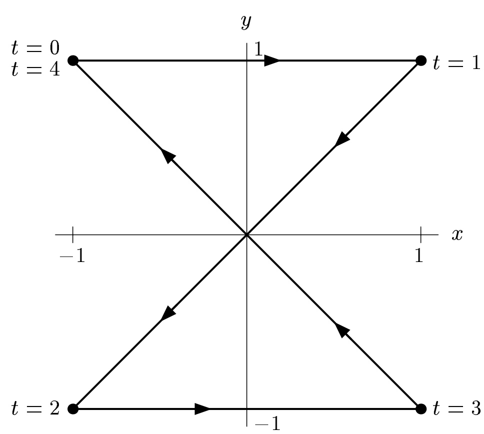


Figure 3.11

4. As the x-coordinate goes at a constant rate from 2 to 0, the y-coordinate goes from 0 to 1, then down to -1, then back to 0. So the particle zigs and zags from $(2,0)$ to $(1.5,1)$ to $(1,0)$ to $(.5,-1)$ to $(0,0)$. Then it zigs and zags back again, forming the shape in Figure 3.12.

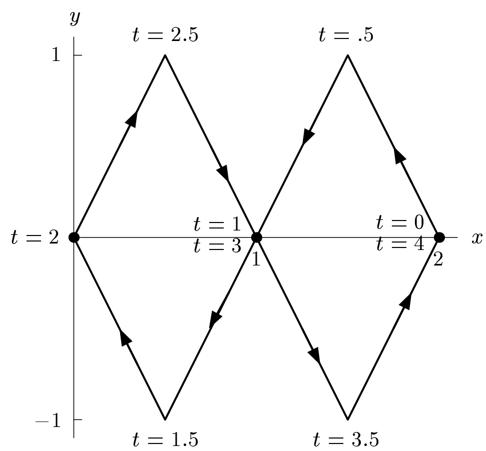


Figure 3.12

5. The particle moves clockwise: For $0 \le t \le \frac{\pi}{2}$, we have $x = \cos t$ decreasing and $y = -\sin t$ decreasing. Similarly, for the time intervals $\frac{\pi}{2} \le t \le \pi, \pi \le t \le \frac{3\pi}{2}$, and $\frac{3\pi}{2} \le t \le 2\pi$, we see that the particle moves clockwise.

6. For $0 \le t \le \frac{\pi}{2}$, we have $x = \sin t$ increasing and $y = \cos t$ decreasing, so the motion is clockwise for $0 \le t \le \frac{\pi}{2}$. Similarly, we see that the motion is clockwise for the time intervals $\frac{\pi}{2} \le t \le \pi, \pi \le t \le \frac{3\pi}{2}$, and $\frac{3\pi}{2} \le t \le 2\pi$.

7. Let $f(t) = t^2$. The particle is moving clockwise when $f(t)$ is decreasing, that is, when $f'(t) = 2t < 0$, so when $t < 0$. The particle is moving counterclockwise when $f'(t) = 2t > 0$, so when $t > 0$.

8. Let $f(t) = t^3 - t$. The particle is moving clockwise when $f(t)$ is decreasing, that is, when $f'(t) = 3t^2 - 1 < 0$, and counterclockwise when $f'(t) = 3t^2 - 1 > 0$. That is, it moves clockwise when $-\sqrt{\frac{1}{3}} < t < \sqrt{\frac{1}{3}}$, between $(\cos((-\sqrt{\frac{1}{3}})^3 + \sqrt{\frac{1}{3}}), \sin((-\sqrt{\frac{1}{3}})^3 + \sqrt{\frac{1}{3}}))$ and $(\cos((\sqrt{\frac{1}{3}})^3 - \sqrt{\frac{1}{3}}), \sin((\sqrt{\frac{1}{3}})^3 - \sqrt{\frac{1}{3}})$, and counterclockwise when $t < -\sqrt{\frac{1}{3}}$ or $t > \sqrt{\frac{1}{3}}$.

9. Let $f(t) = \ln t$. Then $f'(t) = \frac{1}{t}$. The particle is moving counterclockwise when $f'(t) > 0$, that is, when $t > 0$. Any other time, when $t \le 0$, the position is not defined.

10. Let $f(t) = \cos t$. Then $f'(t) = -\sin t$. The particle is moving clockwise when $f'(t) < 0$, or $-\sin t < 0$, that is, when

$$2k\pi < t < (2k+1)\pi,$$

where k is an integer. The particle is otherwise moving counterclockwise, that is, when

$$(2k-1)\pi < t < 2k\pi,$$

where k is an integer. Actually, the particle does not fully trace out a circle. The range of $f(t)$ is $[-1,1]$ so the particle oscillates between the points $(\cos(-1), \sin(-1))$ and $(\cos 1, \sin 1)$.

11. One possible answer is $x = 3\cos t, y = -3\sin t, 0 \le t \le 2\pi$.

12. One possible answer is $x = -2, y = t$.

13. One possible answer is $x = 2 + 5\cos t, y = 1 + 5\sin t, 0 \le t \le 2\pi$.

14. The parameterization $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, is a circle of radius 2 traced out counterclockwise starting at the point $(2,0)$. To start at $(-2,0)$, put a negative in front of the first coordinate

$$x = -2\cos t \quad y = 2\sin t, \qquad 0 \le t \le 2\pi.$$

Now we must check whether this parameterization traces out the circle clockwise or counterclockwise. Since when t increases from 0, $\sin t$ is positive, the point (x,y) moves from $(-2,0)$ into the second quadrant. Thus, the circle is traced out clockwise and so this is one possible parameterization.

15. The slope of the line is

$$m = \frac{3-(-1)}{1-2} = -4.$$

The equation of the line with slope -4 through the point $(2,-1)$ is $y-(-1) = (-4)(x-2)$, so one possible parameterization is $x = t$ and $y = -4t+8-1 = -4t+7$.

16. The ellipse $x^2/25 + y^2/49 = 1$ can be parameterized by $x = 5\cos t, y = 7\sin t, 0 \le t \le 2\pi$.

17. The parameterization $x = -3\cos t$, $y = 7\sin t$, $0 \le t \le 2\pi$, starts at the right point but sweeps out the ellipse in the wrong direction (the y-coordinate becomes positive as t increases). Thus, a possible parameterization is $x = -3\cos(-t) = -3\cos t, y = 7\sin(-t) = -7\sin t, 0 \le t \le 2\pi$.

18. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-1}.$$

Thus when $t = 2$, the slope of the tangent line is $4/11$. Also when $t = 2$, we have

$$x = 2^3 - 2 = 6, \quad y = 2^2 = 4.$$

Therefore the equation of the tangent line is

$$(y-4) = \frac{4}{11}(x-6).$$

19. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\cos(4t)}{3\cos(3t)}.$$

Thus when $t = \pi$, the slope of the tangent line is $-4/3$. Since $x = 0$ and $y = 0$ when $t = \pi$, the equation of the tangent line is $y = -(4/3)x$.

20. We have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+2}{2t-2}.$$

When $t = 1$, the denominator is zero and the numerator is nonzero, so the tangent line is vertical. Since $x = -1$ when $t = 1$, the equation of the tangent line is $x = -1$.

21. We have $dx/dt = 2t$ and $dy/dt = 3t^2$. Therefore, the speed of the particle is

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{((2t)^2 + (3t^2)^2)} = |t| \cdot \sqrt{(4+9t^2)}.$$

The particle comes to a complete stop when its speed is 0, that is, if $t\sqrt{4+9t^2} = 0$, and so when $t = 0$.

22. We have $dx/dt = -2t\sin(t^2)$ and $dy/dt = 2t\cos(t^2)$. Therefore, the speed of the particle is given by

$$\begin{aligned} v &= \sqrt{(-2t\sin(t^2))^2 + (2t\cos(t^2))^2} \\ &= \sqrt{4t^2(\sin(t^2))^2 + 4t^2(\cos(t^2))^2} \\ &= 2|t|\sqrt{\sin^2(t^2) + \cos^2(t^2)} \\ &= 2|t|. \end{aligned}$$

The particle comes to a complete stop when speed is 0, that is, if $2|t| = 0$, and so when $t = 0$.

23. We have

$$\frac{dx}{dt} = -2\sin 2t, \frac{dy}{dt} = \cos t.$$

The speed is

$$v = \sqrt{4\sin^2(2t) + \cos^2 t}.$$

Thus, $v = 0$ when $\sin(2t) = \cos t = 0$, and so the particle stops when $t = \pm\pi/2, \pm 3\pi/2, \ldots$ or $t = (2n+1)\frac{\pi}{2}$, for any integer n.

24. We have

$$\frac{dx}{dt} = (2t-2), \frac{dy}{dt} = (3t^2-3).$$

The speed is given by:

$$v = \sqrt{(2t-2)^2 + (3t^2-3)^2}.$$

The particle stops when $2t - 2 = 0$ and $3t^2 - 3 = 0$. Since these are both satisfied only by $t = 1$, this is the only time that the particle stops.

25. At $t = 2$, the position is $(2^2, 2^3) = (4, 8)$, the velocity in the x-direction is $2 \cdot 2 = 4$, and the velocity in the y-direction is $3 \cdot 2^2 = 12$. So we want the line going through the point $(4, 8)$ at the time $t = 2$, with the given x- and y-velocities:

$$x = 4 + 4(t - 2), \quad y = 8 + 12(t - 2).$$

Problems

26. **(a)** Eliminating t between

$$x = 2 + t, \quad y = 4 + 3t$$

gives

$$\begin{aligned} y - 4 &= 3(x - 2), \\ y &= 3x - 2. \end{aligned}$$

Eliminating t between

$$x = 1 - 2t, \quad y = 1 - 6t$$

gives

$$\begin{aligned} y - 1 &= 3(x - 1), \\ y &= 3x - 2. \end{aligned}$$

Since both parametric equations give rise to the same equation in x and y, they both parameterize the same line.

(b) Slope $= 3$, y-intercept $= -2$.

27. **(a)** We get the part of the line with $x < 10$ and $y < 0$.

(b) We get the part of the line between the points $(10, 0)$ and $(11, 2)$.

28. **(a)** If $t \geq 0$, we have $x \geq 2, y \geq 4$, so we get the part of the line to the right of and above the point $(2, 4)$.

(b) When $t = 0, (x, y) = (2, 4)$. When $t = -1, (x, y) = (-1, -3)$. Restricting t to the interval $-1 \leq t \leq 0$ gives the part of the line between these two points.

(c) If $x < 0$, giving $2 + 3t < 0$ or $t < -2/3$. Thus $t < -2/3$ gives the points on the line to the left of the y-axis.

29. **(a)** The curve is a spiral as shown in Figure 3.13.

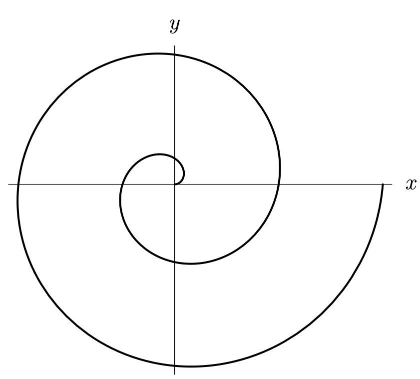


Figure 3.13: The spiral
$x = t\cos t, y = t\sin t$ for $0 \leq t \leq 4\pi$

(b) At $t = 2$, the position is $(2\cos 2, 2\sin 2) = (-0.8323, 1.8186)$, and at $t = 2.01$ the position is $(2.01\cos 2.01, 2.01\sin 2.01) = (-0.8546, 1.8192)$. The distance between these points is

$$\sqrt{(-0.8546 - (-0.8323))^2 + (1.8192 - 1.8186)^2} \approx 0.022.$$

Thus the speed is approximately $0.022/0.01 \approx 2.2$.

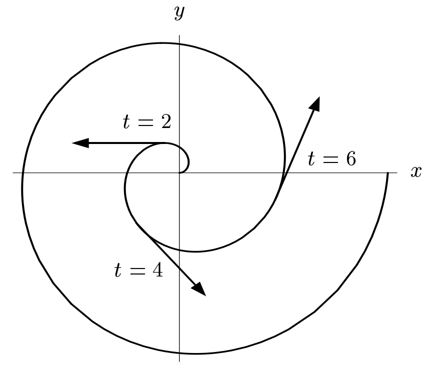


Figure 3.14: The spiral $x = t\cos t, y = t\sin t$ and three velocity vectors

(c) Evaluating the exact formula

$$v = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$$

gives :

$$v(2) = \sqrt{(-2.235)^2 + (0.077)^2} = 2.2363.$$

30. (a) In order for the particle to stop, its velocity both dx/dt and dy/dt must be zero,

$$\frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1) = 0,$$
$$\frac{dy}{dt} = 2t - 2 = 2(t-1) = 0.$$

The value $t = 1$ is the only solution. Therefore, the particle stops when $t = 1$ at the point $(t^3 - 3t,\ t^2 - 2t)|_{t=1} = (-2, -1)$.

(b) In order for the particle to be traveling straight up or down, the velocity in the x-direction must be 0. Thus, we solve $dx/dt = 3t^2 - 3 = 0$ and obtain $t = \pm 1$. However, at $t = 1$ the particle has no vertical motion, as we saw in part (a). Thus, the particle is moving straight up or down only when $t = -1$. The position at that time is $(t^3 - 3t,\ t^2 - 2t)|_{t=-1} = (2, 3)$.

(c) For horizontal motion we need $dy/dt = 0$. That happens when $dy/dt = 2t - 2 = 0$, and so $t = 1$. But from part (a) we also have $dx/dt = 0$ also at $t = 1$, so the particle is not moving at all when $t = 1$. Thus, there is no time when the motion is horizontal.

31. In all three cases, $y = x^2$, so that the motion takes place on the parabola $y = x^2$.

In case (a), the x-coordinate always increases at a constant rate of one unit distance per unit time, so the equations describe a particle moving to the right on the parabola at constant horizontal speed.

In case (b), the x-coordinate is never negative, so the particle is confined to the right half of the parabola. As t moves from $-\infty$ to $+\infty$, $x = t^2$ goes from ∞ to 0 to ∞. Thus the particle first comes down the right half of the parabola, reaching the origin $(0, 0)$ at time $t = 0$, where it reverses direction and goes back up the right half of the parabola.

In case (c), as in case (a), the particle traces out the entire parabola $y = x^2$ from left to right. The difference is that the horizontal speed is not constant. This is because a unit change in t causes larger and larger changes in $x = t^3$ as t approaches $-\infty$ or ∞. The horizontal motion of the particle is faster when it is farther from the origin.

32. (I) has a positive slope and so must be l_1 or l_2. Since its y-intercept is negative, these equations must describe l_2. (II) has a negative slope and positive x-intercept, so these equations must describe l_3.

33. **(a)** C_1 has center at the origin and radius 5, so $a = b = 0, k = 5$ or -5.

(b) C_2 has center at $(0, 5)$ and radius 5, so $a = 0, b = 5, k = 5$ or -5.

(c) C_3 has center at $(10, -10)$, so $a = 10, b = -10$. The radius of C_3 is $\sqrt{10^2 + (-10)^2} = \sqrt{200}$, so $k = \sqrt{200}$ or $k = -\sqrt{200}$.

34. It is a straight line through the point $(3, 5)$ with slope -1. A linear parameterization of the same line is $x = 3 + t$, $y = 5 - t$.

35. **(a)** To find the equations of the moon's motion relative to the star, you must first calculate the equation of the planet's motion relative to the star, and then the moon's motion relative to the planet, and then add the two together.

The distance from the planet to the star is R, and the time to make one revolution is one unit, so the parametric equations for the planet relative to the star are $x = R\cos t$, $y = R\sin t$.

The distance from the moon to the planet is 1, and the time to make one revolution is twelve units, therefore, the parametric equations for the moon relative to the planet are $x = \cos 12t$, $y = \sin 12t$.

Adding these together, we get:

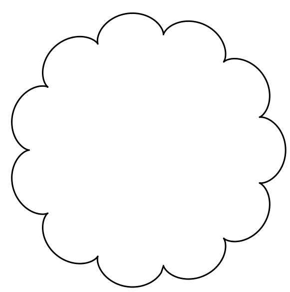
$$x = R\cos t + \cos 12t,$$
$$y = R\sin t + \sin 12t.$$

(b) For the moon to stop completely at time t, the velocity of the moon must be equal to zero. Therefore,

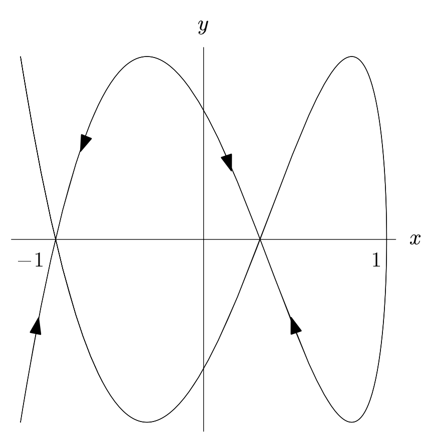
$$\frac{dx}{dt} = -R\sin t - 12\sin 12t = 0,$$
$$\frac{dy}{dt} = R\cos t + 12\cos 12t = 0.$$

There are many possible values to choose for R and t that make both of these equations equal to zero. We choose $t = \pi$, and $R = 12$.

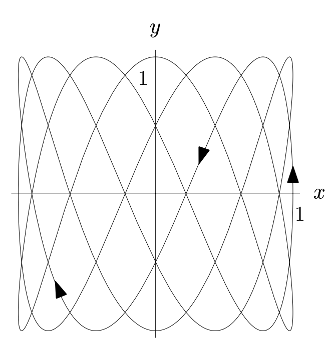
(c) The graph with $R = 12$ is shown below.



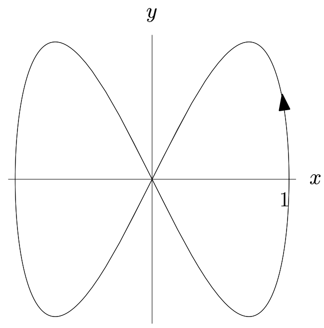
36. For $0 \le t \le 2\pi$



37. For $0 \le t \le 2\pi$



38. For $0 \le t \le 2\pi$



39. This curve never closes on itself. The plot for $0 \le t \le 8\pi$ is in Figure 3.15.

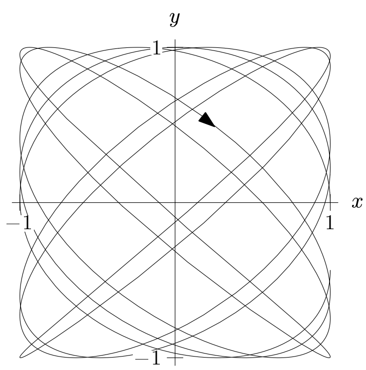


Figure 3.15

40. **(a)** Since $x = t^3 + t$ and $y = t^2$, we have

$$w = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2+1}.$$

Differentiating w with respect to t, we get

$$\frac{dw}{dt} = \frac{(3t^2+1)2 - (2t)(6t)}{(3t^2+1)^2} = \frac{-6t^2+2}{(3t^2+1)^2},$$

so

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw/dt}{dx/dt} = \frac{-6t^2+2}{(3t^2+1)^3}.$$

When $t = 1$, we have $d^2y/dx^2 = -1/16 < 0$, so the curve is concave down.

(b) We have

$$w = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

To find dw/dt, we use the quotient rule:

$$\frac{dw}{dt} = \frac{(dx/dt)(d^2y/dt^2) - (dy/dt)(d^2x/dt^2)}{(dx/dt)^2}.$$

We then divide this by dx/dt again to get the required formula, since

$$\frac{d^2y}{dx^2} = \frac{dw}{dx} = \frac{dw/dt}{dx/dt}.$$

Solutions for Section 3.9

Exercises

1. With $f(x)=\sqrt{1+x}$, the chain rule gives $f'(x)=1/(2\sqrt{1+x})$, so $f(0)=1$ and $f'(0)=1/2$. Therefore the tangent line approximation of f near $x=0$,

$$f(x)\approx f(0)+f'(0)(x-0),$$

becomes

$$\sqrt{1+x}\approx 1+\frac{x}{2}.$$

This means that, near $x=0$, the function $\sqrt{1+x}$ can be approximated by its tangent line $y=1+x/2$. (See Figure 3.16.)

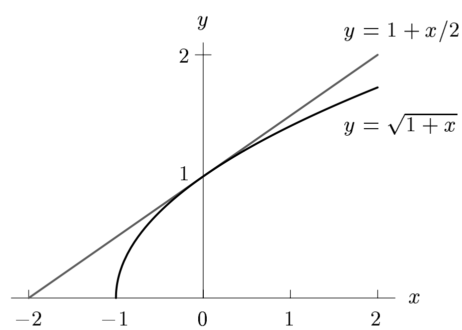


Figure 3.16

2. With $f(x)=e^x$, the tangent line approximation to f near $x=0$ is $f(x)\approx f(0)+f'(0)(x-0)$ which becomes $e^x\approx e^0+e^0x=1+1x=1+x$. Thus, our local linearization of e^x near $x=0$ is $e^x\approx 1+x$.

3. With $f(x)=1/x$, we see that the tangent line approximation to f near $x=1$ is

$$f(x)\approx f(1)+f'(1)(x-1),$$

which becomes

$$\frac{1}{x}\approx 1+f'(1)(x-1).$$

Since $f'(x)=-1/x^2$, $f'(1)=-1$. Thus our formula reduces to

$$\frac{1}{x}\approx 1-(x-1)=2-x.$$

This is the local linearization of $1/x$ near $x=1$.

4. With $f(x)=1/(\sqrt{1+x})$, we see that the tangent line approximation to f near $x=0$ is

$$f(x)\approx f(0)+f'(0)(x-0),$$

which becomes

$$\frac{1}{\sqrt{1+x}}\approx 1+f'(0)x.$$

Since $f'(x)=(-1/2)(1+x)^{-3/2}$, $f'(0)=-1/2$. Thus our formula reduces to

$$\frac{1}{\sqrt{1+x}}\approx 1-x/2.$$

This is the local linearization of $\dfrac{1}{\sqrt{1+x}}$ near $x=0$.

5. Let $f(x)=e^{-x}$. Then $f'(x)=-e^{-x}$. So $f(0)=1$, $f'(0)=-e^0=-1$. Therefore, $e^{-x}\approx f(0)+f'(0)x=1-x$.

6. With $f(x)=e^{x^2}$, we get a tangent line approximation of $f(x)\approx f(1)+f'(1)(x-1)$ which becomes $e^{x^2}\approx e+\left(2xe^{x^2}\right)\Big|_{x=1}(x-1)=e+2e(x-1)=2ex-e$. Thus, our local linearization of e^{x^2} near $x=1$ is $e^{x^2}\approx 2ex-e$.

Problems

7. **(a)** Let $f(x) = (1+x)^k$. Then $f'(x) = k(1+x)^{k-1}$. Since

$$f(x) \approx f(0) + f'(0)(x-0)$$

is the tangent line approximation, and $f(0) = 1$, $f'(0) = k$, for small x we get

$$f(x) \approx 1 + kx.$$

(b) Since $\sqrt{1.1} = (1+0.1)^{1/2} \approx 1 + (1/2)0.1 = 1.05$ by the above method, this estimate is about right.

(c) The real answer is less than 1.05. Since $(1.05)^2 = (1+0.05)^2 = 1 + 2(1)(0.05) + (0.05)^2 = 1.1 + (0.05)^2 > 1.1$, we have $(1.05)^2 > 1.1$ Therefore

$$\sqrt{1.1} < 1.05.$$

Graphically, this because the graph of $\sqrt{1+x}$ is concave down, so it bends below its tangent line. Therefore the true value $(\sqrt{1.1})$ which is on the curve is below the approximate value (1.05) which is on the tangent line.

8. The local linearization of e^x near $x = 0$ is $1 + 1x$ so

$$e^x \approx 1 + x.$$

Squaring this yields, for small x,

$$e^{2x} = (e^x)^2 \approx (1+x)^2 = 1 + 2x + x^2.$$

Local linearization of e^{2x} directly yields

$$e^{2x} \approx 1 + 2x$$

for small x. The two approximations are consistent because they agree: the tangent line approximation to $1 + 2x + x^2$ is just $1 + 2x$.

The first approximation is more accurate. One can see this numerically or by noting that the approximation for e^{2x} given by $1 + 2x$ is really the same as approximating e^y at $y = 2x$. Since the other approximation approximates e^y at $y = x$, which is twice as close to 0 and therefore a better general estimate, it's more likely to be correct.

9. **(a)** Let $f(x) = 1/(1+x)$. Then $f'(x) = -1/(1+x)^2$ by the chain rule. So $f(0) = 1$, and $f'(0) = -1$. Therefore, for x near 0, $1/(1+x) \approx f(0) + f'(0)x = 1 - x$.

(b) We know that for small y, $1/(1+y) \approx 1 - y$. Let $y = x^2$; when x is small, so is $y = x^2$. Hence, for small x, $1/(1+x^2) \approx 1 - x^2$.

(c) Since the linearization of $1/(1+x^2)$ is the line $y = 1$, and this line has a slope of 0, the derivative of $1/(1+x^2)$ is zero at $x = 0$.

10. The local linearizations of $f(x) = e^x$ and $g(x) = \sin x$ near $x = 0$ are

$$f(x) = e^x \approx 1 + x$$

and

$$g(x) = \sin x \approx x.$$

Thus, the local linearization of $e^x \sin x$ is the local linearization of the product:

$$e^x \sin x \approx (1+x)x = x + x^2 \approx x.$$

We therefore know that the derivative of $e^x \sin x$ at $x = 0$ must be 1. Similarly, using the local linearization of $1/(1+x)$ near $x = 0$, $1/(1+x) \approx 1 - x$, we have

$$\frac{e^x \sin x}{1+x} = (e^x)(\sin x)\left(\frac{1}{1+x}\right) \approx (1+x)(x)(1-x) = x - x^3$$

so the local linearization of the triple product $\frac{e^x \sin x}{1+x}$ at $x = 0$ is simply x. And therefore the derivative of $\frac{e^x \sin x}{1+x}$ at $x = 0$ is 1.

11. **(a)** Suppose

$$g = f(r) = \frac{GM}{r^2}.$$

Then

$$f'(r) = \frac{-2GM}{r^3}.$$

So

$$f(r+\Delta r) \approx f(r) - \frac{2GM}{r^3}(\Delta r).$$

Since $f(r+\Delta r) - f(r) = \Delta g$, and $g = GM/r^2$, we have

$$\Delta g \approx -2\frac{GM}{r^3}(\Delta r) = -2g\frac{\Delta r}{r}.$$

(b) The negative sign tells us that the acceleration due to gravity decreases as the distance from the center of the earth increases.

(c) The fractional change in g is given by

$$\frac{\Delta g}{g} \approx -2\frac{\Delta r}{r}.$$

So, since $\Delta r = 4.315$ km and $r = 6400$ km, we have

$$\frac{\Delta g}{g} \approx -2\left(\frac{4.315}{6400}\right) = -0.00135 = -0.135\%.$$

12. (a) Suppose g is a constant and

$$T = f(l) = 2\pi\sqrt{\frac{l}{g}}.$$

Then

$$f'(l) = \frac{2\pi}{\sqrt{g}}\frac{1}{2}l^{-1/2} = \frac{\pi}{\sqrt{gl}}.$$

Thus, local linearity tells us that

$$f(l+\Delta l) \approx f(l) + \frac{\pi}{\sqrt{gl}}\Delta l.$$

Now $T = f(l)$ and $\Delta T = f(l+\Delta l) - f(l)$, so

$$\Delta T \approx \frac{\pi}{\sqrt{gl}}\Delta l = 2\pi\sqrt{\frac{l}{g}}\cdot\frac{1}{2}\frac{\Delta l}{l} = \frac{T}{2}\frac{\Delta l}{l}.$$

(b) Knowing that the length of the pendulum increases by 2% tells us that

$$\frac{\Delta l}{l} = 0.02.$$

Thus,

$$\Delta T \approx \frac{T}{2}(0.02) = 0.01T.$$

So

$$\frac{\Delta T}{T} \approx 0.01.$$

Thus, T increases by 1%.

13. (a) Considering l as a constant, we have

$$T = f(g) = 2\pi\sqrt{\frac{l}{g}}.$$

Then,

$$f'(g) = 2\pi\sqrt{l}\left(-\frac{1}{2}g^{-3/2}\right) = -\pi\sqrt{\frac{l}{g^3}}.$$

Thus, local linearity gives

$$f(g+\Delta g) \approx f(g) - \pi\sqrt{\frac{l}{g^3}}(\Delta g).$$

Since $T = f(g)$ and $\Delta T = f(g+\Delta g) - f(g)$, we have

$$\Delta T \approx -\pi\sqrt{\frac{l}{g^3}}\Delta g = -2\pi\sqrt{\frac{l}{g}}\frac{\Delta g}{2g} = \frac{-T}{2}\frac{\Delta g}{g}.$$

$$\Delta T \approx \frac{-T}{2}\frac{\Delta g}{g}.$$

(b) If g increases by 1%, we know

$$\frac{\Delta g}{g} = 0.01.$$

Thus,

$$\frac{\Delta T}{T} \approx -\frac{1}{2}\frac{\Delta g}{g} = -\frac{1}{2}(0.01) = -0.005,$$

So, T decreases by 0.5%.

14. Since f has a positive second derivative, its graph is concave up, as in Figure 3.17 or 3.18. This means that the graph of $f(x)$ is above its tangent line. We see that in both cases

$$f(1+\Delta x) \geq f(1) + f'(1)\Delta x.$$

(The diagrams show Δx positive, but the result is also true if Δx is negative.)

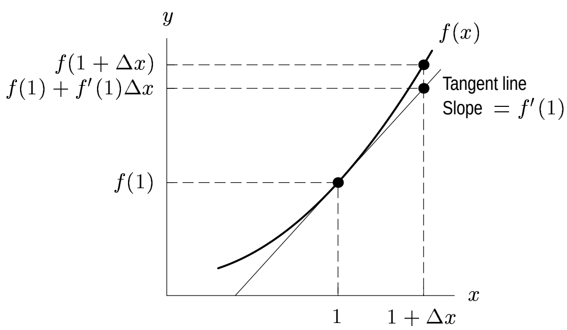


Figure 3.17

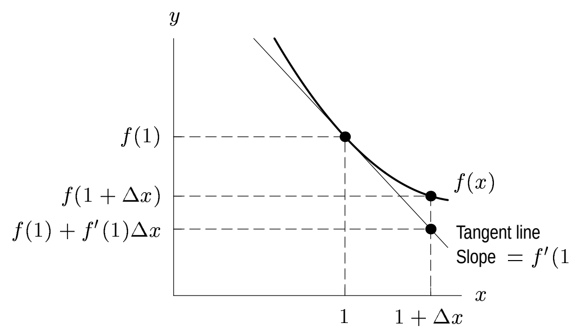


Figure 3.18

15. (a) Since f' is decreasing, $f'(5)$ is larger.
(b) Since f' is decreasing, its derivative, f'', is negative. Thus, $f''(5)$ is negative, so 0 is larger.
(c) Since $f''(x)$ is negative for all x, the graph of f is concave down. Thus the graph of $f(x)$ is below its tangent line. From Figure 3.19, we see that $f(5+\Delta x)$ is below $f(5)+f'(5)\Delta x$. Thus, $f(5)+f'(5)\Delta x$ is larger.

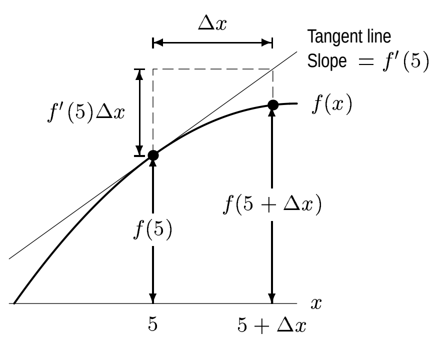


Figure 3.19

16. Note that

$$[f(x)g(x)]' = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

We use the hint: For small h, $f(x+h) \approx f(x) + f'(x)h$, and $g(x+h) \approx g(x) + g'(x)h$. Therefore

$$\begin{aligned} f(x+h)g(x+h) - f(x)g(x) &\approx [f(x) + hf'(x)][g(x) + hg'(x)] - f(x)g(x) \\ &= f(x)g(x) + hf'(x)g(x) + hf(x)g'(x) \\ &\quad + h^2 f'(x)g'(x) - f(x)g(x) \\ &= hf'(x)g(x) + hf(x)g'(x) + h^2 f'(x)g'(x). \end{aligned}$$

Therefore

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} &= \lim_{h\to 0}\frac{hf'(x)g(x)+hf(x)g'(x)+h^2f'(x)g'(x)}{h}\\ &= \lim_{h\to 0}\frac{h\left(f'(x)g(x)+f(x)g'(x)+hf'(x)g'(x)\right)}{h}\\ &= \lim_{h\to 0}\left(f'(x)g(x)+f(x)g'(x)+hf'(x)g'(x)\right)\\ &= f'(x)g(x)+f(x)g'(x).\end{aligned}$$

A more complete derivation can be given using the error term discussed in the section on Differentiability and Linear Approximation in Chapter 2. Adapting the notation of that section to this problem, we write

$$f(x+h)=f(x)+f'(x)h+E_f(h) \quad \text{and} \quad g(x+h)=g(x)+g'(x)h+E_g(h),$$

where $\lim_{h\to 0}\frac{E_f(h)}{h}=\lim_{h\to 0}\frac{E_g(h)}{h}=0$. (This implies that $\lim_{h\to 0}E_f(h)=\lim_{h\to 0}E_g(h)=0$.)

We have

$$\begin{aligned}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} &= \frac{f(x)g(x)}{h}+f(x)g'(x)+f'(x)g(x)+f(x)\frac{E_g(h)}{h}+g(x)\frac{E_f(h)}{h}\\ &\quad +f'(x)g'(x)h+f'(x)E_g(h)+g'(x)E_f(h)+\frac{E_f(h)E_g(h)}{h}-\frac{f(x)g(x)}{h}\end{aligned}$$

The terms $f(x)g(x)/h$ and $-f(x)g(x)/h$ cancel out. All the remaining terms on the right, with the exception of the second and third terms, go to zero as $h\to 0$. Thus, we have

$$[f(x)g(x)]'=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}=f(x)g'(x)+f'(x)g(x).$$

17. Note that

$$[f(g(x))]'=\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}.$$

Using the local linearizations of f and g, we get that

$$\begin{aligned}f(g(x+h))-f(g(x)) &\approx f\left(g(x)+g'(x)h\right)-f(g(x))\\ &\approx f\left(g(x)\right)+f'(g(x))g'(x)h-f(g(x))\\ &= f'(g(x))g'(x)h.\end{aligned}$$

Therefore,

$$\begin{aligned}[f(g(x))]' &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\ &= \lim_{h\to 0}\frac{f'(g(x))g'(x)h}{h}\\ &= \lim_{h\to 0}f'(g(x))g'(x)=f'(g(x))g'(x).\end{aligned}$$

A more complete derivation can be given using the error term discussed in the section on Differentiability and Linear Approximation in Chapter 2. Adapting the notation of that section to this problem, we write

$$f(z+k)=f(z)+f'(z)k+E_f(k) \quad \text{and} \quad g(x+h)=g(x)+g'(x)h+E_g(h),$$

where $\lim_{h\to 0}\frac{E_g(h)}{h}=\lim_{k\to 0}\frac{E_f(k)}{k}=0$.

Now we let $z=g(x)$ and $k=g(x+h)-g(x)$. Then we have $k=g'(x)h+E_g(h)$. Thus,

$$\begin{aligned}\frac{f(g(x+h))-f(g(x))}{h} &= \frac{f(z+k)-f(z)}{h}\\ &= \frac{f(z)+f'(z)k+E_f(k)-f(z)}{h}=\frac{f'(z)k+E_f(k)}{h}\end{aligned}$$

$$= \frac{f'(z)g'(x)h + f'(z)E_g(h)}{h} + \frac{E_f(k)}{k} \cdot \left(\frac{k}{h}\right)$$
$$= f'(z)g'(x) + \frac{f'(z)E_g(h)}{h} + \frac{E_f(k)}{k}\left[\frac{g'(x)h + E_g(h)}{h}\right]$$
$$= f'(z)g'(x) + \frac{f'(z)E_g(h)}{h} + \frac{g'(x)E_f(k)}{k} + \frac{E_g(h) \cdot E_f(k)}{h \cdot k}$$

Now, if $h \to 0$ then $k \to 0$ as well, and all the terms on the right except the first go to zero, leaving us with the term $f'(z)g'(x)$. Substituting $g(x)$ for z, we obtain

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x))g'(x).$$

18. We want to show that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = L.$$

Substituting for $f(x)$ we have

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{f(a) + L(x-a) + E_L(x) - f(a)}{x - a}$$
$$= \lim_{x\to a}\left(L + \frac{E_L(x)}{x-a}\right) = L + \lim_{x\to 0}\frac{E_L(x)}{x-a} = L.$$

Thus, we have shown that f is differentiable at $x = a$ and that its derivative is L, that is, $f'(a) = L$.

Solutions for Section 3.10

Exercises

1. Since $f'(a) > 0$ and $g'(a) < 0$, l'Hopital's rule tells us that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} < 0.$$

2. Since $f'(a) < 0$ and $g'(a) < 0$, l'Hopital's rule tells us that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} > 0.$$

3. Here $f(a) = g(a) = f'(a) = g'(a) = 0$, and $f''(a) > 0$ and $g''(a) < 0$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f''(a)}{g''(a)} < 0$$

4. Note that $f(0) = g(0) = 0$ and $f'(0) = g'(0)$. Since $x = 0$ looks like a point of inflection for each curve, $f''(0) = g''(0) = 0$. Therefore, applying l'Hopital's rule successively gives us

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{f'''(x)}{g'''(x)}.$$

Now notice how the concavity of f changes: for $x < 0$, it is concave up, so $f''(x) > 0$, and for $x > 0$ it is concave down, so $f''(x) < 0$. Thus $f''(x)$ is a decreasing function at 0 and so $f'''(0)$ is negative. Similarly, for $x < 0$, we see g is concave down and for $x > 0$ it is concave up, so $g''(x)$ is increasing at 0 and so $g'''(0)$ is positive. Consequently,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'''(x)}{g'''(0)} < 0.$$

5. The denominator approaches zero as x goes to zero and the numerator goes to zero even faster, so you should expect that the limit to be 0. You can check this by substituting several values of x close to zero. Alternatively, using l'Hopital's rule, we have

$$\lim_{x\to 0}\frac{x^2}{\sin x}=\lim_{x\to 0}\frac{2x}{\cos x}=0.$$

6. The numerator goes to zero faster than the denominator, so you should expect the limit to be zero. Using l'Hopital's rule, we have

$$\lim_{x\to 0}\frac{\sin^2 x}{x}=\lim_{x\to 0}\frac{2\sin x\cos x}{1}=0.$$

7. The denominator goes to zero more slowly than x does, so the numerator goes to zero faster than the denominator, so you should expect the limit to be zero. With l'Hopital's rule,

$$\lim_{x\to 0}\frac{\sin x}{x^{1/3}}=\lim_{x\to 0}\frac{\cos x}{\frac{1}{3}x^{-2/3}}=\lim_{x\to 0}3x^{2/3}\cos x=0.$$

8. The denominator goes to zero more slowly than x. Therefore, you should expect that the limit to be 0. Using l'Hopital's rule,

$$\lim_{x\to 0}\frac{x}{(\sin x)^{1/3}}=\lim_{x\to 0}\frac{1}{\frac{1}{3}(\sin x)^{-2/3}\cos x}=\lim_{x\to 0}\frac{3(\sin x)^{2/3}}{\cos x}=0,$$

since $\sin 0=0$ and $\cos 0=1$.

9. The larger power dominates. Using l'Hopital's rule

$$\lim_{x\to\infty}\frac{x^5}{0.1x^7}=\lim_{x\to\infty}\frac{5x^4}{0.7x^6}=\lim_{x\to\infty}\frac{20x^3}{4.2x^5}$$
$$=\lim_{x\to\infty}\frac{60x^2}{21x^4}=\lim_{x\to\infty}\frac{120x}{84x^3}=\lim_{x\to\infty}\frac{120}{252x^2}=0$$

so $0.1x^7$ dominates.

10. We apply l'Hopital's rule twice to the ratio $50x^2/0.01x^3$:

$$\lim_{x\to\infty}\frac{50x^2}{0.01x^3}=\lim_{x\to\infty}\frac{100x}{0.03x^2}=\lim_{x\to\infty}\frac{100}{0.06x}=0.$$

Since the limit is 0, we see that $0.01x^3$ is much larger than $50x^2$ as $x\to\infty$.

11. The power function dominates. Using l'Hopital's rule

$$\lim_{x\to\infty}\frac{\ln(x+3)}{x^{0.2}}=\lim_{x\to\infty}\frac{\frac{1}{(x+3)}}{0.2x^{-0.8}}=\lim_{x\to\infty}\frac{x^{0.8}}{0.2(x+3)}.$$

Using l'Hopital's rule again gives

$$\lim_{x\to\infty}\frac{x^{0.8}}{0.2(x+3)}=\lim_{x\to\infty}\frac{0.8x^{-0.2}}{0.2}=0,$$

so $x^{0.2}$ dominates.

12. The exponential dominates. After 10 applications of l'Hopital's rule

$$\lim_{x\to\infty}\frac{x^{10}}{e^0.1x}=\lim_{x\to\infty}\frac{10x^9}{0.1e^{0.1x}}=\cdots=\lim_{x\to\infty}\frac{10!}{(0.1)^{10}e^{0.1x}}=0.$$

so $e^{0.1x}$ dominates.

Problems

13. Let $f(x)=\ln x$ and $g(x)=1/x$ so $f'(x)=1/x$ and $g'(x)=-1/x^2$ and

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}\frac{x}{-1}=0.$$

14. **(a)** Since $f'(x) = 3\cos(3x)$, we have $f'(0) = 3$.
(b) Since $g'(x) = 5$, we have $g'(0) = 5$.
(c) Since $f(x) = \sin 3x$ and $g(x) = 5x$ are both 0 at $x = 0$, we apply l'Hopital's rule to obtain

$$\lim_{x\to 0} \frac{\sin(3x)}{5x} = \frac{f'(0)}{g'(0)} = \frac{3}{5}.$$

15. Let $f(x) = \sin(2x)$ and $g(x) = x$. Observe that $f(1) = \sin 2 \neq 0$ and $g(1) = 1 \neq 0$. Therefore l'Hopital's rule does not apply. However,

$$\lim_{x\to 1} \frac{\sin 2x}{x} = \frac{\sin 2}{1} = 0.909297.$$

16. Let $f(x) = \cos x$ and $g(x) = x$. Observe that since $f(0) = 1$, l'Hopital's rule does not apply. But since $g(0) = 0$,

$$\lim_{x\to 0} \frac{\cos x}{x} \quad \text{does not exist.}$$

17. Let $f(x) = e^{-x}$ and $g(x) = \sin x$. Observe that as x increases, $f(x)$ approaches 0 but $g(x)$ oscillates between -1 and 1. Since $g(x)$ does not approach 0 in the limit, l'Hopital's rule does not apply. Because $g(x)$ is in the denominator and oscillates through 0 forever, the limit does not exist.

18. We want to find $\lim_{x\to\infty} f(x)$, which we do by three applications of l'Hopital's rule:

$$\lim_{x\to\infty} \frac{2x^3+5x^2}{3x^3-1} = \lim_{x\to\infty} \frac{6x^2+10x}{9x^2} = \lim_{x\to\infty} \frac{12x+10}{18x} = \lim_{x\to\infty} \frac{12}{18} = \frac{2}{3}.$$

So the line $y = 2/3$ is the horizontal asymptote.

19. Observe that both $f(4)$ and $g(4)$ are zero. Also, $f'(4) = 1.4$ and $g'(4) = -0.7$, so by l'Hopital's rule,

$$\lim_{x\to 4} \frac{f(x)}{g(x)} = \frac{f'(4)}{g'(4)} = \frac{1.4}{-0.7} = -2.$$

Solutions for Chapter 3 Review

Exercises

1. $f'(t) = \dfrac{d}{dt}\left(2te^t - \dfrac{1}{\sqrt{t}}\right) = 2e^t + 2te^t + \dfrac{1}{2t^{3/2}}$.

2.

$$\begin{aligned}\frac{dw}{dz} &= \frac{(-3)(5+3z)-(5-3z)(3)}{(5+3z)^2}\\ &= \frac{-15-9z-15+9z}{(5+3z)^2} = \frac{-30}{(5+3z)^2}\end{aligned}$$

3. $\dfrac{d}{dy}\ln\ln(2y^3) = \dfrac{1}{\ln(2y^3)}\dfrac{1}{2y^3}6y^2 = \dfrac{3}{y\ln(2y^3)}$.

4. $f'(x) = \dfrac{3x^2}{9}(3\ln x - 1) + \dfrac{x^3}{9}\left(\dfrac{3}{x}\right) = x^2\ln x - \dfrac{x^2}{3} + \dfrac{x^2}{3} = x^2\ln x$

5. $g'(x) = \dfrac{d}{dx}\left(x^k + k^x\right) = kx^{k-1} + k^x\ln k$.

6. $\dfrac{dz}{d\theta} = 3\sin^2\theta\cos\theta$

7.

$$\begin{aligned}f'(t) &= 2\cos(3t+5)\cdot(-\sin(3t+5))3\\ &= -6\cos(3t+5)\cdot\sin(3t+5)\end{aligned}$$

8.

$$\begin{aligned} M'(\alpha) &= 2\tan(2+3\alpha)\cdot\frac{1}{\cos^2(2+3\alpha)}\cdot 3 \\ &= 6\cdot\frac{\tan(2+3\alpha)}{\cos^2(2+3\alpha)} \end{aligned}$$

9. $s'(\theta) = \dfrac{d}{d\theta}\sin^2(3\theta-\pi) = 6\cos(3\theta-\pi)\sin(3\theta-\pi)$.

10. $h'(t) = \dfrac{1}{e^{-t}-t}\left(-e^{-t}-1\right)$.

11.

$$\begin{aligned} \frac{d}{d\theta}\left(\frac{\sin(5-\theta)}{\theta^2}\right) &= \frac{\cos(5-\theta)(-1)\theta^2 - \sin(5-\theta)(2\theta)}{\theta^4} \\ &= -\frac{\theta\cos(5-\theta)+2\sin(5-\theta)}{\theta^3}. \end{aligned}$$

12. $w'(\theta) = \dfrac{1}{\sin^2\theta} - \dfrac{2\theta\cos\theta}{\sin^3\theta}$

13. $g'(x) = \dfrac{d}{dx}\left(x^{\frac{1}{2}} + x^{-1} + x^{-\frac{3}{2}}\right) = \dfrac{1}{2}x^{-\frac{1}{2}} - x^{-2} - \dfrac{3}{2}x^{-\frac{5}{2}}$.

14. $s'(x) = \dfrac{d}{dx}\left(\arctan(2-x)\right) = \dfrac{-1}{1+(2-x)^2}$.

15. $r'(\theta) = \dfrac{d}{d\theta}\left(e^{\left(e^{\theta}+e^{-\theta}\right)}\right) = e^{\left(e^{\theta}+e^{-\theta}\right)}\left(e^{\theta}-e^{-\theta}\right)$.

16. Using the chain rule, we get:

$$m'(n) = \cos(e^n)\cdot(e^n)$$

17. Using the chain rule we get:

$$k'(\alpha) = e^{\tan(\sin\alpha)}(\tan(\sin\alpha))' = e^{\tan(\sin\alpha)}\cdot\frac{1}{\cos^2(\sin\alpha)}\cdot\cos\alpha.$$

18. Here we use the product rule, and then the chain rule, and then the product rule.

$$\begin{aligned} g'(t) = \cos(\sqrt{t}e^t) + t(\cos\sqrt{t}e^t)' &= \cos(\sqrt{t}e^t) + t(-\sin(\sqrt{t}e^t)\cdot(\sqrt{t}e^t)') \\ &= \cos(\sqrt{t}e^t) - t\sin(\sqrt{t}e^t)\cdot\left(\sqrt{t}e^t + \frac{1}{2\sqrt{t}}e^t\right) \end{aligned}$$

19. $f'(r) = e(\tan 2 + \tan r)^{e-1}(\tan 2+\tan r)' = e(\tan 2+\tan r)^{e-1}\left(\dfrac{1}{\cos^2 r}\right)$

20. $y' = 0$

21. $\dfrac{d}{dx}xe^{\tan x} = e^{\tan x} + xe^{\tan x}\dfrac{1}{\cos^2 x}$.

22. $\dfrac{dy}{dx} = 2e^{2x}\sin^2(3x) + e^{2x}(2\sin(3x)\cos(3x)3) = 2e^{2x}\sin(3x)(\sin(3x)+3\cos(3x))$

23. $g'(x) = \dfrac{6x}{1+(3x^2+1)^2} = \dfrac{6x}{9x^4+6x^2+2}$

24. $g'(w) = \dfrac{d}{dw}\left(\dfrac{1}{2^w+e^w}\right) = -\dfrac{2^w\ln 2 + e^w}{(2^w+e^w)^2}$.

25. $\dfrac{dy}{dx} = (\ln 2)2^{\sin x}\cos x\cdot\cos x + 2^{\sin x}(-\sin x) = 2^{\sin x}\left((\ln 2)\cos^2 x - \sin x\right)$

26. $h(x) = ax\cdot\ln e = ax$, so $h'(x) = a$.

27. $k'(x) = a$

28. $f'(\theta) = ke^{k\theta}$

29. Using the product rule and factoring gives $f'(t) = e^{-4kt}(\cos t - 4k\sin t)$.

30. Using the chain rule gives $f'(x) = 5\ln(a)a^{5x}$.

31. Using the quotient rule gives

$$\begin{aligned} f'(x) &= \frac{(-2x)(a^2+x^2)-(2x)(a^2-x^2)}{(a^2+x^2)^2} \\ &= \frac{-4a^2x}{(a^2+x^2)^2}. \end{aligned}$$

32. Using the quotient rule gives

$$\begin{aligned} w'(r) &= \frac{2ar(b+r^3)-3r^2(ar^2)}{(b+r^3)^2} \\ &= \frac{2abr-ar^4}{(b+r^3)^2}. \end{aligned}$$

33. Using the quotient rule gives

$$\begin{aligned} f'(s) &= \frac{-2s\sqrt{a^2+s^2}-\frac{s}{\sqrt{a^2+s^2}}(a^2-s^2)}{(a^2+s^2)} \\ &= \frac{-2s(a^2+s^2)-s(a^2-s^2)}{(a^2+s^2)^{3/2}} \\ &= \frac{-2a^2s-2s^3-a^2s+s^3}{(a^2+s^2)^{3/2}} \\ &= \frac{-3a^2s-s^3}{(a^2+s^2)^{3/2}}. \end{aligned}$$

34. Using the product rule gives

$$\begin{aligned} H'(t) &= 2ate^{-ct}-c(at^2+b)e^{-ct} \\ &= (-cat^2+2at-bc)e^{-ct}. \end{aligned}$$

35. $\dfrac{d}{d\theta}\sqrt{a^2-\sin^2\theta} = \dfrac{1}{2\sqrt{a^2-\sin^2\theta}}(-2\sin\theta\cos\theta) = -\dfrac{\sin\theta\cos\theta}{\sqrt{a^2-\sin^2\theta}}$.

36. $\dfrac{dy}{dx} = \dfrac{1}{1+\left(\frac{2}{x}\right)^2}\left(\dfrac{-2}{x^2}\right) = \dfrac{-2}{x^2+4}$

37. Using the chain rule gives $r'(t) = \dfrac{\cos(\frac{t}{k})}{\sin(\frac{t}{k})}\left(\dfrac{1}{k}\right)$.

38. $g'(u) = \dfrac{ae^{au}}{a^2+b^2}$

39. Since $g(w) = 5(a^2-w^2)^{-2},\ g'(w) = -10(a^2-w^2)^{-3}(-2w) = \dfrac{20w}{(a^2-w^2)^3}$

40.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2} \\ &= \frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2} = \frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2} \\ &= \frac{4}{(e^x+e^{-x})^2} \end{aligned}$$

41. Using the quotient and chain rules, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{(ae^{ax}+ae^{-ax})(e^{ax}+e^{-ax})-(e^{ax}-e^{-ax})(ae^{ax}-ae^{-ax})}{(e^{ax}+e^{-ax})^2}\\ &= \frac{a(e^{ax}+e^{-ax})^2-a(e^{ax}-e^{-ax})^2}{(e^{ax}+e^{-ax})^2}\\ &= \frac{a[(e^{2ax}+2+e^{-2ax})-(e^{2ax}-2+e^{-2ax})]}{(e^{ax}+e^{-ax})^2}\\ &= \frac{4a}{(e^{ax}+e^{-ax})^2}\end{aligned}$$

42.

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{1}{2}(\cos(5\theta))^{-\frac{1}{2}}(-\sin(5\theta)\cdot 5)+2\sin(6\theta)\cos(6\theta)\cdot 6\\ &= -\frac{5}{2}\frac{\sin(5\theta)}{\sqrt{\cos(5\theta)}}+12\sin(6\theta)\cos(6\theta)\end{aligned}$$

43. $r'(\theta) = \frac{d}{d\theta}\sin[(3\theta-\pi)^2] = \cos[(3\theta-\pi)^2]\cdot 2(3\theta-\pi)\cdot 3 = 6(3\theta-\pi)\cos[(3\theta-\pi)^2]$.

44.

$$\begin{aligned}\frac{dy}{dz} &= 3(x^2+5)^2(2x)(3x^3-2)^2+(x^2+5)^3[2(3x^3-2)(9x^2)]\\ &= 3(2x)(x^2+5)^2(3x^3-2)[(3x^3-2)+(x^2+5)(3x)]\\ &= 6x(x^2+5)^2(3x^3-2)[6x^3+15x-2]\end{aligned}$$

45. Since $\tan(\arctan(k\theta)) = k\theta$, because tan and arctan are inverse functions, we have $N'(\theta) = k$.

46. Using the product rule gives $h'(t) = ke^{kt}(\sin at+\cos bt)+e^{kt}(a\cos at-b\sin bt)$.

47. $f'(x) = \frac{d}{dx}(2-4x-3x^2)(6x^e-3\pi) = (-4-6x)(6x^e-3\pi)+(2-4x-3x^2)(6ex^{e-1})$.

48. $f'(t) = 4(\sin(2t)-\cos(3t))^3[2\cos(2t)+3\sin(3t)]$

49. Since $\cos^2 y+\sin^2 y = 1$, we have $s(y) = \sqrt[3]{1+3} = \sqrt[3]{4}$. Thus $s'(y) = 0$.

50.

$$\begin{aligned}f'(x) &= (-2x+6x^2)(6-4x+x^7)+(4-x^2+2x^3)(-4+7x^6)\\ &= (-12x+44x^2-24x^3-2x^8+6x^9)+(-16+4x^2-8x^3+28x^6-7x^8+14x^9)\\ &= -16-12x+48x^2-32x^3+28x^6-9x^8+20x^9\end{aligned}$$

51.

$$\begin{aligned}h'(x) &= \left(-\frac{1}{x^2}+\frac{2}{x^3}\right)\left(2x^3+4\right)+\left(\frac{1}{x}-\frac{1}{x^2}\right)\left(6x^2\right)\\ &= -2x+4-\frac{4}{x^2}+\frac{8}{x^3}+6x-6\\ &= 4x-2-4x^{-2}+8x^{-3}\end{aligned}$$

52. Note: $f(z) = (5z)^{1/2}+5z^{1/2}+5z^{-1/2}-\sqrt{5}z^{-1/2}+\sqrt{5}$, so $f'(z) = \frac{5}{2}(5z)^{-1/2}+\frac{5}{2}z^{-1/2}-\frac{5}{2}z^{-3/2}+\frac{\sqrt{5}}{2}z^{-3/2}$.

53. We wish to find the slope $m = dy/dx$. To do this, we can implicitly differentiate the given formula in terms of x:

$$\begin{aligned}x^2+3y^2 &= 7\\ 2x+6y\frac{dy}{dx} &= \frac{d}{dx}(7) = 0\\ \frac{dy}{dx} &= \frac{-2x}{6y} = \frac{-x}{3y}.\end{aligned}$$

Thus, at $(2,-1)$, $m = -(2)/3(-1) = 2/3$.

54. Taking derivatives implicitly, we find

$$\begin{aligned}\frac{dy}{dx} + \cos y\frac{dy}{dx} + 2x &= 0\\ \frac{dy}{dx} &= \frac{-2x}{1+\cos y}\end{aligned}$$

So, at the point $x = 3, y = 0$,

$$\frac{dy}{dx} = \frac{(-2)(3)}{1+\cos 0} = \frac{-6}{2} = -3.$$

55.

$$\begin{aligned}2xy + x^2\frac{dy}{dx} - 2\frac{dy}{dx} &= 0\\ (x^2-2)\frac{dy}{dx} &= -2xy\\ \frac{dy}{dx} &= \frac{-2xy}{(x^2-2)}\end{aligned}$$

56.

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} - 8xy - 4x^2\frac{dy}{dx} &= 0\\ (3y^2-4x^2)\frac{dy}{dx} &= 8xy - 3x^2\\ \frac{dy}{dx} &= \frac{8xy-3x^2}{3y^2-4x^2}\end{aligned}$$

57. Differentiating implicitly on both sides with respect to x,

$$\begin{aligned}a\cos(ay)\frac{dy}{dx} - b\sin(bx) &= y + x\frac{dy}{dx}\\ (a\cos(ay) - x)\frac{dy}{dx} &= y + b\sin(bx)\\ \frac{dy}{dx} &= \frac{y+b\sin(bx)}{a\cos(ay)-x}.\end{aligned}$$

58. First, we differentiate with respect to x:

$$\begin{aligned}x\cdot\frac{dy}{dx} + y\cdot 1 + 2y\frac{dy}{dx} &= 0\\ \frac{dy}{dx}(x+2y) &= -y\\ \frac{dy}{dx} &= \frac{-y}{x+2y}.\end{aligned}$$

At $x = 3$, we have

$$\begin{aligned}3y + y^2 &= 4\\ y^2 + 3y - 4 &= 0\\ (y-1)(y+4) &= 0.\end{aligned}$$

Our two points, then, are $(3, 1)$ and $(3, -4)$.

$$\text{At } (3,1), \quad \frac{dy}{dx} = \frac{-1}{3+2(1)} = -\frac{1}{5}; \quad \text{Tangent line: } (y-1) = -\frac{1}{5}(x-3).$$

$$\text{At } (3,-4), \quad \frac{dy}{dx} = \frac{-(-4)}{3+2(-4)} = -\frac{4}{5}; \quad \text{Tangent line: } (y+4) = -\frac{4}{5}(x-3).$$

Problems

59. Since W is proportional to r^3, we have $W = kr^3$ for some constant k. Thus, $dW/dr = k(3r^2) = 3kr^2$. Thus, dW/dr is proportional to r^2.

60. Taking the values of f, f', g, and g' from the table we get:

(a) $h(4) = f(g(4)) = f(3) = 1$.
(b) $h'(4) = f'(g(4))g'(4) = f'(3) \cdot 1 = 2$.
(c) $h(4) = g(f(4)) = g(4) = 3$.
(d) $h'(4) = g'(f(4))f'(4) = g'(4) \cdot 3 = 3$.
(e) $h'(4) = (f(4)g'(4) - g(4)f'(4))/f^2(4) = -5/16$.
(f) $h'(4) = f(4)g'(4) + g(4)f'(4) = 13$.

61. **(a)** $H'(2) = r'(2)s(2) + r(2)s'(2) = -1 \cdot 1 + 4 \cdot 3 = 11$.
(b) $H'(2) = \dfrac{r'(2)}{2\sqrt{r(2)}} = \dfrac{-1}{2\sqrt{4}} = -\dfrac{1}{4}$.
(c) $H'(2) = r'(s(2))s'(2) = r'(1) \cdot 3$, but we don't know $r'(1)$.
(d) $H'(2) = s'(r(2))r'(2) = s'(4)r'(2) = -3$.

62. **(a)** $f(x) = x^2 - 4g(x)$
$f'(x) = 2x - 4g'(x)$
$f'(2) = 2(2) - 4(-4) = 4 + 16 = 20$
(b) $f(x) = \frac{x}{g(x)}$
$f'(x) = \frac{g(x) - xg'(x)}{(g(x))^2}$
$f'(2) = \frac{g(2) - 2g'(2)}{(g(2))^2} = \frac{3 - 2(-4)}{(3)^2} = \frac{11}{9}$
(c) $f(x) = x^2 g(x)$
$f'(x) = 2xg(x) + x^2 g'(x)$
$f'(2) = 2(2)(3) + (2)^2(-4) = 12 - 16 = -4$
(d) $f(x) = (g(x))^2$
$f'(x) = 2g(x) \cdot g'(x)$
$f'(2) = 2(3)(-4) = -24$
(e) $f(x) = x \sin(g(x))$
$f'(x) = \sin(g(x)) + x \cos(g(x)) \cdot g'(x)$

$$\begin{aligned} f'(2) &= \sin(g(2)) + 2\cos(g(2)) \cdot g'(2) \\ &= \sin 3 + 2\cos(3) \cdot (-4) \\ &= \sin 3 - 8\cos 3 \end{aligned}$$

(f) $f(x) = x^2 \ln(g(x))$
$f'(x) = 2x \ln(g(x)) + x^2(\frac{g'(x)}{g(x)})$

$$\begin{aligned} f'(2) &= 2(2)\ln 3 + (2)^2(\tfrac{-4}{3}) \\ &= 4\ln 3 - \tfrac{16}{3} \end{aligned}$$

63. **(a)** $f(x) = x^2 - 4g(x)$
$f(2) = 4 - 4(3) = -8$
$f'(2) = 20$
Thus, we have a point $(2, -8)$ and slope $m = 20$. This gives

$$\begin{aligned} -8 &= 2(20) + b \\ b &= -48, \quad \text{so} \\ y &= 20x - 48. \end{aligned}$$

(b) $f(x) = \dfrac{x}{g(x)}$
$f(2) = \dfrac{2}{3}$
$f'(2) = \dfrac{11}{9}$
Thus, we have point $(2, \frac{2}{3})$ and slope $m = \frac{11}{9}$. This gives

$$\frac{2}{3} = (\frac{11}{9})(2) + b$$

$$b = \frac{2}{3} - \frac{22}{9} = \frac{-16}{9}, \quad \text{so}$$
$$y = \frac{11}{9}x - \frac{16}{9}.$$

(c) $f(x) = x^2 g(x)$
$f(2) = 4 \cdot g(2) = 4(3) = 12$
$f'(2) = -4$
Thus, we have point $(2, 12)$ and slope $m = -4$. This gives

$$12 = 2(-4) + b$$
$$b = 20, \quad \text{so}$$
$$y = -4x + 20.$$

(d) $f(x) = (g(x))^2$
$f(2) = (g(2))^2 = (3)^2 = 9$
$f'(2) = -24$
Thus, we have point $(2, 9)$ and slope $m = -24$. This gives

$$9 = 2(-24) + b$$
$$b = 57, \quad \text{so}$$
$$y = -24x + 57.$$

(e) $f(x) = x\sin(g(x))$
$f(2) = 2\sin(g(2)) = 2\sin 3$
$f'(2) = \sin 3 - 8\cos 3$
We will use a decimal approximation for $f(2)$ and $f'(2)$, so the point $(2, 2\sin 3) \approx (2, 0.28)$ and $m \approx 8.06$. Thus,

$$0.28 = 2(8.06) + b$$
$$b = -15.84, \quad \text{so}$$
$$y = 8.06x - 15.84.$$

(f) $f(x) = x^2 \ln g(x)$
$f(2) = 4\ln g(2) = 4\ln 3 \approx 4.39$
$f'(2) = 4\ln 3 - \dfrac{16}{3} \approx -0.94.$
Thus, we have point $(2, 4.39)$ and slope $m = -0.94$. This gives

$$4.39 = 2(-0.94) + b$$
$$b = 6.27, \quad \text{so}$$
$$y = -0.94x + 6.27.$$

64. When we zoom in on the origin, we find that two functions are not defined there. The other functions all look like straight lines through the origin. The only way we can tell them apart is their slope.

The following functions all have slope 0 and are therefore indistinguishable:
$\sin x - \tan x$, $\frac{x^2}{x^2+1}$, $x - \sin x$, and $\frac{1-\cos x}{\cos x}$.

These functions all have slope 1 at the origin, and are thus indistinguishable:
$\arcsin x$, $\frac{\sin x}{1+\sin x}$, $\arctan x$, $e^x - 1$, $\frac{x}{x+1}$, and $\frac{x}{x^2+1}$.

Now, $\frac{\sin x}{x} - 1$ and $-x\ln x$ both are undefined at the origin, so they are distinguishable from the other functions. In addition, while $\frac{\sin x}{x} - 1$ has a slope that approaches zero near the origin, $-x\ln x$ becomes vertical near the origin, so they are distinguishable from each other.

Finally, $x^{10} + \sqrt[10]{x}$ is the only function defined at the origin and with a vertical tangent there, so it is distinguishable from the others.

65. It makes sense to define the angle between two curves to be the angle between their tangent lines. (The tangent lines are the best linear approximations to the curves). See Figure 3.20. The functions $\sin x$ and $\cos x$ are equal at $x = \frac{\pi}{4}$.

$$\text{For } f_1(x) = \sin x, \quad f_1'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$
$$\text{For } f_2(x) = \cos x, \quad f_2'(\frac{\pi}{4}) = -\sin(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}.$$

Using the point $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$ for each tangent line we get $y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4})$ and $y = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 + \frac{\pi}{4})$, respectively.

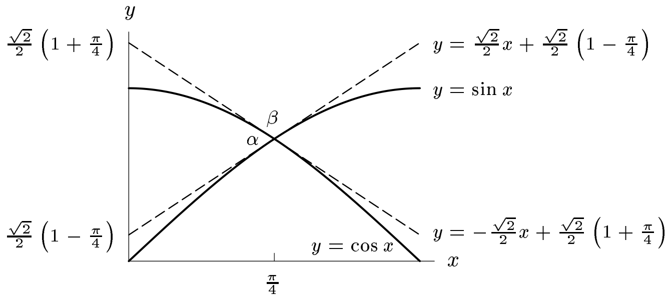


Figure 3.20

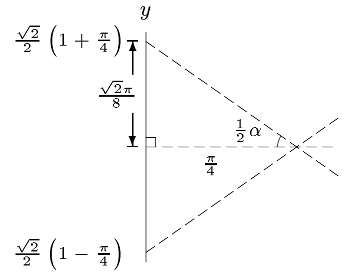


Figure 3.21

There are two possibilities of how to define the angle between the tangent lines, indicated by α and β above. The choice is arbitrary, so we will solve for both. To find the angle, α, we consider the triangle formed by these two lines and the y-axis. See Figure 3.21.

$$\tan\left(\frac{1}{2}\alpha\right) = \frac{\sqrt{2}\pi/8}{\pi/4} = \frac{\sqrt{2}}{2}$$
$$\frac{1}{2}\alpha = 0.61548 \text{ radians}$$
$$\alpha = 1.231 \text{ radians, or } 70.5^\circ.$$

Now let us solve for β, the other possible measure of the angle between the two tangent lines. Since α and β are supplementary, $\beta = \pi - 1.231 = 1.909$ radians, or 109.4°.

66. The curves meet when $1 + x - x^2 = 1 - x + x^2$, that is when $2x(1-x) = 0$ so that $x = 1$ or $x = 0$. Let

$$y_1(x) = 1 + x - x^2 \quad \text{and} \quad y_2(x) = 1 - x + x^2.$$

Then

$$y_1{}' = 1 - 2x \quad \text{and} \quad y_2{}' = -1 + 2x.$$

At $x = 0$, $y_1{}' = 1$, $y_2{}' = -1$ so that $y_1{}' \cdot y_2{}' = -1$ and the curves are perpendicular. At $x = 1$, $y_1{}' = -1$, $y_2{}' = 1$ so that $y_1{}' \cdot y_2{}' = -1$ and the curves are perpendicular.

67. The curves meet when $1 - x^3/3 = x - 1$, that is when $x^3 + 3x - 6 = 0$. So the roots of this equation give us the x-coordinates of the intersection point. By numerical methods, we see there is one solution near $x = 1.3$. See Figure 3.22. Let

$$y_1(x) = 1 - \frac{x^3}{3} \quad \text{and} \quad y_2(x) = x - 1.$$

So we have

$$y_1{}' = -x^2 \quad \text{and} \quad y_2{}' = 1.$$

However, $y_2{}'(x) = +1$, so if the curves are to be perpendicular when they cross, then $y_1{}'$ must be -1. Since $y_1{}' = -x^2$, $y_1{}' = -1$ only at $x = \pm 1$ which is not the point of intersection. The curves are therefore not perpendicular when they cross.

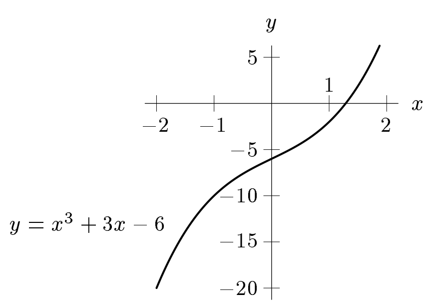


Figure 3.22

68. Differentiating gives $\dfrac{dy}{dx} = \ln x + 1 - b$.

To find the point at which the graph crosses the x-axis, set $y = 0$ and solve for x:

$$\begin{aligned} 0 &= x \ln x - bx \\ 0 &= x(\ln x - b). \end{aligned}$$

Since $x > 0$, we have

$$\begin{aligned} \ln x - b &= 0 \\ x &= e^b. \end{aligned}$$

At the point $(e^b, 0)$, the slope is

$$\frac{dy}{dx} = \ln\left(e^b\right) + 1 - b = b + 1 - b = 1.$$

Thus the equation of the tangent line is

$$\begin{aligned} y - 0 &= 1(x - e^b) \\ y &= x - e^b. \end{aligned}$$

69. **(a)** $\dfrac{dg}{dr} = GM\dfrac{d}{dr}\left(\dfrac{1}{r^2}\right) = GM\dfrac{d}{dr}\left(r^{-2}\right) = GM(-2)r^{-3} = -\dfrac{2GM}{r^3}$.

(b) $\dfrac{dg}{dr}$ is the rate of change of acceleration due to the pull of gravity. The further away from the center of the earth, the weaker the pull of gravity is. So g is decreasing and therefore its derivative, $\dfrac{dg}{dr}$, is negative.

(c) By part (a),

$$\left.\frac{dg}{dr}\right|_{r=6400} = -\left.\frac{2GM}{r^3}\right|_{r=6400} = -\frac{2(6.67 \times 10^{-20})(6 \times 10^{24})}{(6400)^3} \approx -3.05 \times 10^{-6}.$$

(d) It is reasonable to assume that g is a constant near the surface of the earth.

70. The population of Mexico is given by the formula

$$M = 84(1 + 0.026)^t = 84(1.026)^t \text{ million}$$

and that of the US by

$$U = 250(1 + 0.007)^t = 250(1.007)^t \text{ million},$$

where t is measured in years ($t = 0$ corresponds to the year 1990). So,

$$\left.\frac{dM}{dt}\right|_{t=0} = 84\frac{d}{dt}(1.026)^t\Big|_{t=0} = 84(1.026)^t\ln(1.026)\Big|_{t=0} \approx 2.156$$

and

$$\left.\frac{dU}{dt}\right|_{t=0} = 250\frac{d}{dt}(1.007)^t\Big|_{t=0} = 250(1.007)^t\ln(1.007)\Big|_{t=0} \approx 1.744$$

Since $\left.\dfrac{dM}{dt}\right|_{t=0} > \left.\dfrac{dU}{dt}\right|_{t=0}$, the population of Mexico was growing faster in 1990.

71. **(a)** If the distance $s(t) = 20e^{\frac{t}{2}}$, then the velocity, $v(t)$, is given by

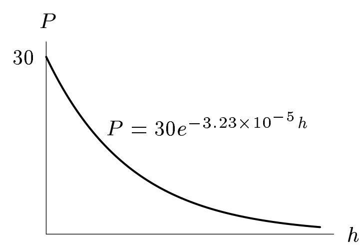
$$v(t) = s'(t) = \left(20e^{\frac{t}{2}}\right)' = \left(\frac{1}{2}\right)\left(20e^{\frac{t}{2}}\right) = 10e^{\frac{t}{2}}.$$

(b) Observing the differentiation in (a), we note that

$$s'(t) = v(t) = \frac{1}{2}\left(20e^{\frac{t}{2}}\right) = \frac{1}{2}s(t).$$

Substituting $s(t)$ for $20e^{\frac{t}{2}}$, we obtain $s'(t) = \frac{1}{2}s(t)$.

72. **(a)**



(b)

$$\frac{dP}{dh} = 30e^{-3.23\times10^{-5}h}(-3.23\times10^{-5})$$

so

$$\left.\frac{dP}{dh}\right|_{h=0} = -30(3.23\times10^{-5}) = -9.69\times10^{-4}$$

Hence, at $h = 0$, the slope of the tangent line is -9.69×10^{-4}, so the equation of the tangent line is

$$y - 30 = (-9.69\times10^{-4})(h - 0)$$
$$y = (-9.69\times10^{-4})h + 30.$$

(c) The rule of thumb says

$$\left(\begin{array}{c}\text{Drop in pressure from}\\ \text{sea level to height } h\end{array}\right) = \frac{h}{1000}$$

But since the pressure at sea level is 30 inches of mercury, this drop in pressure is also $(30 - P)$, so

$$30 - P = \frac{h}{1000}$$

giving

$$P = 30 - 0.001h.$$

(d) The equations in (b) and (c) are almost the same: both have P intercepts of 30, and the slopes are almost the same $(9.69\times10^{-4} \approx 0.001)$. The rule of thumb calculates values of P which are very close to the tangent lines, and therefore yields values very close to the curve.

(e) The tangent line is slightly below the curve, and the rule of thumb line, having a slightly more negative slope, is slightly below the tangent line (for $h > 0$). Thus, the rule of thumb values are slightly smaller.

73.

$$\frac{dy}{dt} = -7.5(0.507)\sin(0.507t) = -3.80\sin(0.507t)$$

(a) When $t = 6$, $\frac{dy}{dt} = -3.80\sin(0.507\cdot6) = -0.38$ meters/hour. So the tide is falling at 0.38 meters/hour.

(b) When $t = 9$, $\frac{dy}{dt} = -3.80\sin(0.507\cdot9) = 3.76$ meters/hour. So the tide is rising at 3.76 meters/hour.

(c) When $t = 12$, $\frac{dy}{dt} = -3.80\sin(0.507\cdot12) = 0.75$ meters/hour. So the tide is rising at 0.75 meters/hour.

(d) When $t = 18$, $\frac{dy}{dt} = -3.80\sin(0.507\cdot18) = -1.12$ meters/hour. So the tide is falling at 1.12 meters/hour.

74. Since we're given that the instantaneous rate of change of T at $t = 30$ is 2, we want to choose a and b so that the derivative of T agrees with this value. Differentiating, $T'(t) = ab \cdot e^{-bt}$. Then we have

$$2 = T'(30) = abe^{-30b} \text{ or } e^{-30b} = \frac{2}{ab}$$

We also know that at $t = 30$, $T = 120$, so

$$120 = T(30) = 200 - ae^{-30b} \text{ or } e^{-30b} = \frac{80}{a}$$

Thus $\frac{80}{a} = e^{-30b} = \frac{2}{ab}$, so $b = \frac{1}{40} = 0.025$ and $a = 169.36$.

75. (a) Differentiating, we see

$$v = \frac{dy}{dt} = -2\pi\omega y_0 \sin(2\pi\omega t)$$
$$a = \frac{dv}{dt} = -4\pi^2\omega^2 y_0 \cos(2\pi\omega t).$$

(b) We have

$$y = y_0 \cos(2\pi\omega t)$$
$$v = -2\pi\omega y_0 \sin(2\pi\omega t)$$
$$a = -4\pi^2\omega^2 y_0 \cos(2\pi\omega t).$$

So

Amplitude of y is $|y_0|$,
Amplitude of v is $|2\pi\omega y_0| = 2\pi\omega|y_0|$,
Amplitude of a is $|4\pi^2\omega^2 y_0| = 4\pi^2\omega^2|y_0|$.

The amplitudes are different (provided $2\pi\omega \neq 1$). The periods of the three functions are all the same, namely $1/\omega$.

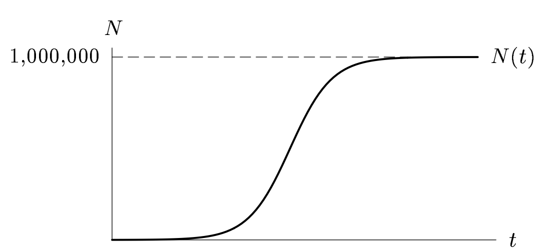
(c) Looking at the answer to part (a), we see

$$\frac{d^2y}{dt^2} = a = -4\pi^2\omega^2 \left(y_0 \cos(2\pi\omega t)\right) = -4\pi^2\omega^2 y.$$

So we see that

$$\frac{d^2y}{dt^2} + 4\pi^2\omega^2 y = 0.$$

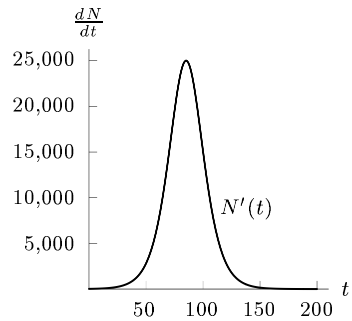
76. (a) Since $\lim_{t\to\infty} e^{-0.1t} = 0$, we see that $\lim_{t\to\infty} \frac{1000000}{1 + 5000e^{-0.1t}} = 1000000$. Thus, in the long run, close to 1,000,000 people will have had the disease. This can be seen in the figure below.



(b) The rate at which people fall sick is given by the first derivative $N'(t)$.
$N'(t) \approx \frac{\Delta N}{\Delta t}$, where $\Delta t = 1$ day.

$$N'(t) = \frac{500{,}000{,}000}{e^{0.1t}(1 + 5000e^{-0.1t})^2} = \frac{500{,}000{,}000}{e^{0.1t} + 25{,}000{,}000e^{-0.1t} + 10^4}$$

Graphing this we see that the maximum value of $N'(t)$ is approximately $25{,}000$. Therefore the maximum number of people to fall sick on any given day is $25{,}000$.



77. Let r be the radius of the balloon. Then its volume, V, is

$$V = \frac{4}{3}\pi r^3.$$

We need to find the rate of change of V with respect to time, that is dV/dt. Since $V = V(r)$,

$$\frac{dV}{dr} = 4\pi r^2$$

so that by the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \cdot 1.$$

When $r = 5$, $dV/dt = 100\pi$ cm^3/sec.

78. The radius r is related to the volume by the formula $V = \frac{4}{3}\pi r^3$. By implicit differentiation, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

The surface area of a sphere is $4\pi r^2$, so we have

$$\frac{dV}{dt} = s \cdot \frac{dr}{dt},$$

but since $\dfrac{dV}{dt} = \dfrac{1}{3}s$ was given, we have

$$\frac{dr}{dt} = \frac{1}{3}.$$

79. **(a)** Since $d\theta/dt$ represents the rate of change of θ with time, $d\theta/dt$ represents the angular velocity of the disk.
(b) Suppose P is the point on the rim shown in Figure 3.23.

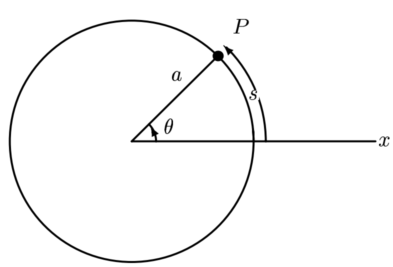


Figure 3.23

Any other point on the rim is moving at the same speed, though in a different direction. We know that since θ is in radians,

$$s = a\theta.$$

Since a is a constant, we know

$$\frac{ds}{dt} = a\frac{d\theta}{dt}.$$

But $ds/dt = v$, the speed of the point on the rim, so

$$v = a\frac{d\theta}{dt}.$$

80. Using Pythagoras' theorem, we see that the distance x between the aircraft's current position and the point 2 miles directly above the ground station are related to s by the formula $x = (s^2 - 2^2)^{1/2}$. See Figure 3.24. The speed along the aircraft's constant altitude flight path is

$$\frac{dx}{dt} = \left(\frac{1}{2}\right)(s^2 - 4)^{-1/2}(2s)\left(\frac{ds}{dt}\right) = \frac{s}{x}\frac{ds}{dt}.$$

When $s = 4.6$ and $ds/dt = 210$,

$$\begin{aligned}\frac{dx}{dt} &= \frac{4.6}{\sqrt{(4.6)^2 - 4}}210 \\ &= \frac{966}{\sqrt{21.16 - 4}} \\ &= \frac{966}{4.14} \approx 233.2 \text{ miles/hour.}\end{aligned}$$

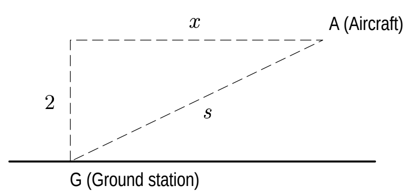


Figure 3.24

81. We want to find dP/dV. Solving $PV = k$ for P gives

$$P = k/V$$

so,

$$\frac{dP}{dV} = -\frac{k}{V^2}.$$

82. **(a)** Since $V = k/P$, the volume decreases.

(b) Since $PV = k$ and $P = 2$ when $V = 10$, we have $k = 20$, so

$$V = \frac{20}{P}.$$

We think of both P and V as functions of time, so by the chain rule

$$\begin{aligned}\frac{dV}{dt} &= \frac{dV}{dP}\frac{dP}{dt}, \\ \frac{dV}{dt} &= -\frac{20}{P^2}\frac{dP}{dt}.\end{aligned}$$

We know that $dP/dt = 0.05$ atm/min when $P = 2$ atm, so

$$\frac{dV}{dt} = -\frac{20}{2^2} \cdot (0.05) = -0.25 \text{ cm}^3\text{/min.}$$

83. **(a)** If $y = \ln x$, then

$$\begin{aligned} y' &= \frac{1}{x} \\ y'' &= -\frac{1}{x^2} \\ y''' &= \frac{2}{x^3} \\ y'''' &= -\frac{3 \cdot 2}{x^4} \end{aligned}$$

and so

$$y^{(n)} = (-1)^{n+1}(n-1)!x^{-n}.$$

(b) If $y = xe^x$, then

$$\begin{aligned} y' &= xe^x + e^x \\ y'' &= xe^x + 2e^x \\ y''' &= xe^x + 3e^x \end{aligned}$$

so that

$$y^{(n)} = xe^x + ne^x.$$

(c) If $y = e^x \cos x$, then

$$\begin{aligned} y' &= e^x(\cos x - \sin x) \\ y'' &= -2e^x \sin x \\ y''' &= e^x(-2\cos x - 2\sin x) \\ y^{(4)} &= -4e^x \cos x \\ y^{(5)} &= e^x(-4\cos x + 4\sin x) \\ y^{(6)} &= 8e^x \sin x. \end{aligned}$$

Combining these results we get

$$\begin{aligned} y^{(n)} &= (-4)^{(n-1)/4} e^x(\cos x - \sin x), && n = 4m+1, \quad m = 0,1,2,3,\ldots \\ y^{(n)} &= -2(-4)^{(n-2)/4} e^x \sin x, && n = 4m+2, \quad m = 0,1,2,3,\ldots \\ y^{(n)} &= -2(-4)^{(n-3)/4} e^x(\cos x + \sin x), && n = 4m+3, \quad m = 0,1,2,3,\ldots \\ y^{(n)} &= (-4)^{(n/4)} e^x \cos x, && n = 4m, \quad m = 1,2,3,\ldots. \end{aligned}$$

84. **(a)** We multiply through by $h = f \cdot g$ and cancel as follows:

$$\begin{aligned} \frac{f'}{f} + \frac{g'}{g} &= \frac{h'}{h} \\ \left(\frac{f'}{f} + \frac{g'}{g}\right) \cdot fg &= \frac{h'}{h} \cdot fg \\ \frac{f'}{f} \cdot fg + \frac{g'}{g} \cdot fg &= \frac{h'}{h} \cdot h \\ f' \cdot g + g' \cdot f &= h', \end{aligned}$$

which is the product rule.

(b) We start with the product rule, multiply through by $1/(fg)$ and cancel as follows:

$$\begin{aligned} f' \cdot g + g' \cdot f &= h' \\ (f' \cdot g + g' \cdot f) \cdot \frac{1}{fg} &= h' \cdot \frac{1}{fg} \\ (f' \cdot g) \cdot \frac{1}{fg} + (g' \cdot f) \cdot \frac{1}{fg} &= h' \cdot \frac{1}{fg} \\ \frac{f'}{f} + \frac{g'}{g} &= \frac{h'}{h}, \end{aligned}$$

which is the additive rule shown in part (a).

85. This problem can be solved by using either the quotient rule or the fact that

$$\frac{f'}{f} = \frac{d}{dx}(\ln f) \quad \text{and} \quad \frac{g'}{g} = \frac{d}{dx}(\ln g).$$

We use the second method. The relative rate of change of f/g is $(f/g)'/(f/g)$, so

$$\frac{(f/g)'}{f/g} = \frac{d}{dx}\ln\left(\frac{f}{g}\right) = \frac{d}{dx}(\ln f - \ln g) = \frac{d}{dx}(\ln f) - \frac{d}{dx}(\ln g) = \frac{f'}{f} - \frac{g'}{g}.$$

Thus, the relative rate of change of f/g is the difference between the relative rates of change of f and of g.

CAS Challenge Problems

86. **(a)** Answers from different computer algebra systems may be in different forms. One form is:

$$\frac{d}{dx}(x+1)^x = x(x+1)^{x-1} + (x+1)^x \ln(x+1)$$
$$\frac{d}{dx}(\sin x)^x = x\cos x(\sin x)^{x-1} + (\sin x)^x \ln(\sin x)$$

(b) Both the answers in part (a) follow the general rule:

$$\frac{d}{dx}f(x)^x = xf'(x)\,(f(x))^{x-1} + (f(x))^x \ln(f(x)).$$

(c) Applying this rule to $g(x)$, we get

$$\frac{d}{dx}(\ln x)^x = x(1/x)(\ln x)^{x-1} + (\ln x)^x \ln(\ln x) = (\ln x)^{x-1} + (\ln x)^x \ln(\ln x).$$

This agrees with the answer given by the computer algebra system.

(d) We can write $f(x) = e^{\ln(f(x))}$. So

$$(f(x))^x = (e^{\ln(f(x))})^x = e^{x\ln(f(x))}.$$

Therefore, using the chain rule and the product rule,

$$\begin{aligned}\frac{d}{dx}\,(f(x))^x &= \frac{d}{dx}\,(x\ln(f(x)))\cdot e^{x\ln(f(x))} = \left(\ln(f(x)) + x\frac{d}{dx}\ln(f(x))\right)e^{x\ln(f(x))}\\ &= \left(\ln(f(x)) + x\frac{f'(x)}{f(x)}\right)(f(x))^x = \ln(f(x))\,(f(x))^x + xf'(x)\,(f(x))^{x-1}\\ &= xf'(x)\,(f(x))^{x-1} + (f(x))^x\ln(f(x)).\end{aligned}$$

87. **(a)** A CAS gives $f'(x) = 1$.

(b) By the chain rule,

$$f'(x) = \cos(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}.$$

Now $\cos t = \pm\sqrt{1-\sin^2 t}$. Furthermore, if $-\pi/2 \le t \le \pi/2$ then $\cos t \ge 0$, so we take the positive square root and get $\cos t = \sqrt{1-\sin^2 t}$. Since $-\pi/2 \le \arcsin x \le \pi/2$ for all x in the domain of arcsin, we have

$$\cos(\arcsin x) = \sqrt{1-(\sin(\arcsin x))^2} = \sqrt{1-x^2},$$

so

$$\frac{d}{dx}\sin(\arcsin(x)) = \sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}} = 1.$$

(c) Since $\sin(\arcsin(x)) = x$, its derivative is 1.

88. **(a)** A CAS gives $g'(r) = 0$.

(b) Using the product rule,

$$\begin{aligned}g'(r) &= \frac{d}{dr}(2^{-2r})\cdot 4^r + 2^{-2r}\frac{d}{dr}(4^r) = -2\ln 2\cdot 2^{-2r}4^r + 2^{-2r}\ln 4\cdot 4^r\\ &= -\ln 4\cdot 2^{-2r}4^r + \ln 4\cdot 2^{-2r}4^r = (-\ln 4 + \ln 4)2^{-2r}4^r = 0\cdot 2^{-2r}4^r = 0.\end{aligned}$$

(c) By the laws of exponents, $4^r = (2^2)^r = 2^{2r}$, so $2^{-2r}4^r = 2^{-2r}2^{2r} = 2^0 = 1$. Therefore, its derivative is zero.

89. **(a)** A CAS gives $h'(t) = 0$

(b) By the chain rule

$$h'(t) = \frac{\frac{d}{dt}\left(1 - \frac{1}{t}\right)}{1 - \frac{1}{t}} + \frac{\frac{d}{dt}\left(\frac{t}{t-1}\right)}{\frac{t}{t-1}} = \frac{\frac{1}{t^2}}{\frac{t-1}{t}} + \frac{\frac{1}{t-1} - \frac{t}{(t-1)^2}}{\frac{t}{t-1}}$$

$$= \frac{1}{t^2 - t} + \frac{(t-1) - t}{t^2 - t} = \frac{1}{t^2 - t} + \frac{-1}{t^2 - t} = 0.$$

(c) The expression inside the first logarithm is $1 - (1/t) = (t-1)/t$. Using the property $\log A + \log B = \log(AB)$, we get

$$\ln\left(1 - \frac{1}{t}\right) + \ln\left(\frac{t}{t-1}\right) = \ln\left(\frac{t-1}{t}\right) + \ln\left(\frac{t}{t-1}\right)$$

$$= \ln\left(\frac{t-1}{t} \cdot \frac{t}{1-t}\right) = \ln 1 = 0.$$

Thus $h(t) = 0$, so $h'(t) = 0$ also.

CHECK YOUR UNDERSTANDING

1. True. Since $d(x^n)/dx = nx^{n-1}$, the derivative of a power function is a power function, so the derivative of a polynomial is a polynomial.

2. False, since

$$\frac{d}{dx}\left(\frac{\pi}{x^2}\right) = \frac{d}{dx}\left(\pi x^{-2}\right) = -2\pi x^{-3} = \frac{-2\pi}{x^3}.$$

3. True, since $\cos\theta$ and therefore $\cos^2\theta$ are periodic, and

$$\frac{d}{d\theta}(\tan\theta) = \frac{1}{\cos^2\theta}.$$

4. False. Since

$$\frac{d}{dx}\ln(x^2) = \frac{1}{x^2}\cdot 2x = \frac{2}{x} \quad \text{and} \quad \frac{d^2}{dx^2}\ln(x^2) = \frac{d}{dx}\left(\frac{2}{x}\right) = -\frac{2}{x^2},$$

we see that the second derivative of $\ln(x^2)$ is negative for $x > 0$. Thus, the graph is concave down.

5. True. Since $f'(x)$ is the limit

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h},$$

the function f must be defined for all x.

6. True. The slope of $f(x) + g(x)$ at $x = 2$ is the sum of the derivatives, $f'(2) + g'(2) = 3.1 + 7.3 = 10.4$.

7. False. The product rule gives

$$(fg)' = fg' + f'g.$$

Differentiating this and using the product rule again, we get

$$(fg)'' = f'g' + fg'' + f'g' + f''g = fg'' + 2f'g' + f''g.$$

Thus, the right hand side is not equal to $fg'' + f''g$ in general.

8. True. If $f(x)$ is periodic with period c, then $f(x+c) = f(x)$ for all x. By the definition of the derivative, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

and

$$f'(x+c) = \lim_{h\to 0}\frac{f(x+c+h) - f(x+c)}{h}.$$

Since f is periodic, for any $h \neq 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x+c+h) - f(x+c)}{h}.$$

Taking the limit as $h \to 0$, we get that $f'(x) = f'(x+c)$, so f' is periodic with the same period as $f(x)$.

9. True; differentiating the equation with respect to x, we get

$$2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0.$$

Solving for dy/dx, we get that

$$\frac{dy}{dx} = \frac{-y}{2y+x}.$$

Thus dy/dx exists where $2y + x \neq 0$. Now if $2y + x = 0$, then $x = -2y$. Substituting for x in the original equation, $y^2 + xy - 1 = 0$, we get

$$y^2 - 2y^2 - 1 = 0.$$

This simplifies to $y^2 + 1 = 0$, which has no solutions. Thus dy/dx exists everywhere.

10. False. The slope is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t\cos(t^2)}{-2t\sin(t^2)} = -\frac{\cos(t^2)}{\sin(t^2)}.$$

11. False. If $f(x) = |x|$, then $f(x)$ is not differentiable at $x = 0$ and $f'(x)$ does not exist at $x = 0$.

12. False. If $f(x) = \ln x$, then $f'(x) = 1/x$, which is decreasing for $x > 0$.

13. False; the fourth derivative of $\cos t + C$, where C is any constant, is indeed $\cos t$. But any function of the form $\cos t + p(t)$, where $p(t)$ is a polynomial of degree less than or equal to 3, also has its fourth derivative equal to $\cos t$. So $\cos t + t^2$ will work.

14. False; For example, the inverse function of $f(x) = x^3$ is $x^{1/3}$, and the derivative of $x^{1/3}$ is $(1/3)x^{-2/3}$, which is not $1/f'(x) = 1/(3x^2)$.

15. False; for example, if both $f(x)$ and $g(x)$ are constant functions, such as $f(x) = 6$, $g(x) = 10$, then $(fg)'(x) = 0$, and $f'(x) = 0$ and $g'(x) = 0$.

16. True; looking at the statement from the other direction, if both $f(x)$ and $g(x)$ are differentiable at $x = 1$, then so is their quotient, $f(x)/g(x)$, as long as it is defined there, which requires that $g(1) \neq 0$. So the only way in which $f(x)/g(x)$ can be defined but not differentiable at $x = 1$ is if either $f(x)$ or $g(x)$, or both, is not differentiable there.

17. False; for example, if both f and g are constant functions, then the derivative of $f(g(x))$ is zero, as is the derivative of $f(x)$. Another example is $f(x) = 5x + 7$ and $g(x) = x + 2$.

18. True. Since $f''(x) > 0$ and $g''(x) > 0$ for all x, we have $f''(x) + g''(x) > 0$ for all x, which means that $f(x) + g(x)$ is concave up.

19. False. Let $f(x) = x^2$ and $g(x) = x^2 - 1$. Let $h(x) = f(x)g(x)$. Then $h''(x) = 12x^2 - 2$. Since $h''(0) < 0$, clearly h is not concave up for all x.

20. False. Let $f(x) = 2x^2$ and $g(x) = x^2$. Then $f(x) - g(x) = x^2$, which is concave up for all x.

21. False. Let $f(x) = e^{-x}$ and $g(x) = x^2$. Let $h(x) = f(g(x)) = e^{-x^2}$. Then $h'(x) = -2xe^{-x^2}$ and $h''(x) = (-2 + 4x^2)e^{-x^2}$. Since $h''(0) < 0$, clearly h is not concave up for all x.

22. **(a)** False. Only if $k = f'(a)$ is L the local linearization of f.

(b) False. Since $f(a) = L(a)$ for any k, we have $\lim_{x\to a}(f(x) - L(x)) = f(a) - L(a) = 0$, but only if $k = f'(a)$ is L the local linearization of f.

23. **(a)** This is not a counterexample. Although the product rule says that $(fg)' = f'g + fg'$, that does not rule out the possibility that also $(fg)' = f'g'$. In fact, if f and g are both constant functions, then both $f'g + fg'$ and $f'g'$ are zero, so they are equal to each other.

(b) This is not a counterexample. In fact, it agrees with the product rule:

$$\frac{d}{dx}(xf(x)) = \left(\frac{d}{dx}(x)\right) f(x) + x\frac{d}{dx}f(x) = f(x) + xf'(x) = xf'(x) + f(x).$$

(c) This is not a counterexample. Although the product rule says that

$$\frac{d}{dx}(f(x)^2) = \frac{d}{dx}f(x)\cdot f(x) = f'(x)f(x) + f(x)f'(x) = 2f(x)f'(x),$$

it could be true that $f'(x) = 1$, so that the derivative is also just $2f(x)$. In fact, $f(x) = x$ is an example where this happens.

(d) This would be a counterexample. If $f'(a) = g'(a) = 0$, then

$$\frac{d}{dx}(f(x)g(x))\Big|_{x=a} = f'(a)g(a) + f(a)g'(a) = 0.$$

So fg cannot have positive slope at $x = a$. Of course such a counterexample could not exist, since the product rule is true.

PROJECTS FOR CHAPTER THREE

1. Let $r = i/100$. (For example if $i = 5\%$, $r = 0.05$.) Then the balance, $\$B$, after t years is given by

$$B = P(1 + r)^t,$$

where $\$P$ is the original deposit. If we are doubling our money, then $B = 2P$, so we wish to solve for t in the equation $2P = P(1 + r)^t$. This is equivalent to

$$2 = (1 + r)^t.$$

Taking natural logarithms of both sides and solving for t yields

$$\ln 2 = t \ln(1 + r),$$
$$t = \frac{\ln 2}{\ln(1 + r)}.$$

We now approximate $\ln(1 + r)$ near $r = 0$. Let $f(r) = \ln(1 + r)$. Then $f'(r) = 1/(1 + r)$. Thus, $f(0) = 0$ and $f'(0) = 1$, so

$$f(r) \approx f(0) + f'(0)r$$

becomes

$$\ln(1 + r) \approx r.$$

Therefore,

$$t = \frac{\ln 2}{\ln(1 + r)} \approx \frac{\ln 2}{r} = \frac{100 \ln 2}{i} \approx \frac{70}{i},$$

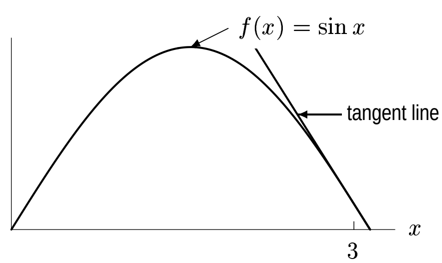
as claimed. We expect this approximation to hold for small values of i; it turns out that values of i up to 10 give good enough answers for most everyday purposes.

2. (a) (i) Set $f(x) = \sin x$, so $f'(x) = \cos x$. Guess $x_0 = 3$. Then

$$x_1 = 3 - \frac{\sin 3}{\cos 3} \approx 3.1425$$
$$x_2 = x_1 - \frac{\sin x_1}{\cos x_1} \approx 3.1415926533,$$

which is correct to one billionth!

(ii) Newton's method uses the tangent line at $x = 3$, i.e. $y - \sin 3 = \cos(3)(x - 3)$. Around $x = 3$, however, $\sin x$ is almost linear, since the second derivative $\sin''(\pi) = 0$. Thus using the tangent line to get an approximate value for the root gives us a very good approximation.



(iii) For $f(x) = \sin x$, we have

$$f(3) = 0.14112$$
$$f(4) = -0.7568,$$

so there is a root in $[3, 4]$. We now continue bisecting:

$$[3, 3.5] : f(3.5) = -0.35078 \text{ (bisection 1)}$$
$$[3, 3.25] : f(3.25) = -0.10819 \text{ (bisection 2)}$$
$$[3.125, 3.25] : f(3.125) = \quad 0.01659 \text{ (bisection 3)}$$
$$[3.125, 3.1875] : f(3.1875) = -0.04584 \text{ (bisection 4)}$$

We continue this process; after 11 bisections, we know the root lies between 3.1411 and 3.1416, which still is not as good an approximation as what we get from Newton's method in just two steps.

(b) (i) We have $f(x) = \sin x - \frac{2}{3}x$ and $f'(x) = \cos x - \frac{2}{3}$.
Using $x_0 = 0.904$,

$$x_1 = 0.904 - \frac{\sin(0.904) - \frac{2}{3}(0.904)}{\cos(0.904) - \frac{2}{3}} \approx 4.704,$$

$$x_2 = 4.704 - \frac{\sin(4.704) - \frac{2}{3}(4.704)}{\cos(4.704) - \frac{2}{3}} \approx -1.423,$$

$$x_3 = -1.433 - \frac{\sin(-1.423) - \frac{2}{3}(-1.423)}{\cos(-1.423) - \frac{2}{3}} \approx -1.501,$$

$$x_4 = -1.499 - \frac{\sin(-1.501) - \frac{2}{3}(-1.501)}{\cos(-1.501) - \frac{2}{3}} \approx -1.496,$$

$$x_5 = -1.496 - \frac{\sin(-1.496) - \frac{2}{3}(-1.496)}{\cos(-1.496) - \frac{2}{3}} \approx -1.496.$$

Using $x_0 = 0.905$,

$$x_1 = 0.905 - \frac{\sin(0.905) - \frac{2}{3}(0.905)}{\cos(0.905) - \frac{2}{3}} \approx 4.643,$$

$$x_2 = 4.643 - \frac{\sin(4.643) - \frac{2}{3}(4.643)}{\cos(4.643) - \frac{2}{3}} \approx -0.918,$$

$$x_3 = -0.918 - \frac{\sin(-0.918) - \frac{2}{3}(-0.918)}{\cos(-0.918) - \frac{2}{3}} \approx -3.996,$$

$$x_4 = -3.996 - \frac{\sin(-3.996) - \frac{2}{3}(-3.996)}{\cos(-3.996) - \frac{2}{3}} \approx -1.413,$$

$$x_5 = -1.413 - \frac{\sin(-1.413) - \frac{2}{3}(-1.413)}{\cos(-1.413) - \frac{2}{3}} \approx -1.502,$$

$$x_6 = -1.502 - \frac{\sin(-1.502) - \frac{2}{3}(-1.502)}{\cos(-1.502) - \frac{2}{3}} \approx -1.496.$$

Now using $x_0 = 0.906$,

$$x_1 = 0.906 - \frac{\sin(0.906) - \frac{2}{3}(0.906)}{\cos(0.906) - \frac{2}{3}} \approx 4.584,$$

$$x_2 = 4.584 - \frac{\sin(4.584) - \frac{2}{3}(4.584)}{\cos(4.584) - \frac{2}{3}} \approx -0.509,$$

$$x_3 = -0.510 - \frac{\sin(-0.509) - \frac{2}{3}(-0.509)}{\cos(-0.509) - \frac{2}{3}} \approx .207,$$

$$x_4 = -1.300 - \frac{\sin(.207) - \frac{2}{3}(.207)}{\cos(.207) - \frac{2}{3}} \approx -0.009,$$

$$x_5 = -1.543 - \frac{\sin(-0.009) - \frac{2}{3}(-0.009)}{\cos(-0.009) - \frac{2}{3}} \approx 0,$$

(ii) Starting with 0.904 and 0.905 yields the same value, but the two paths to get to the root are very different. Starting with 0.906 leads to a different root. Our starting points were near the maximum value of f. Consequently, a small change in x_0 makes a large change in x_1.