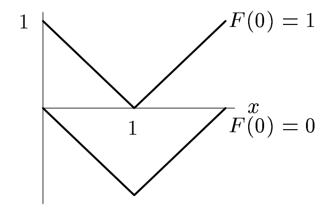
CHAPTER SIX

Solutions for Section 6.1

Exercises

1.



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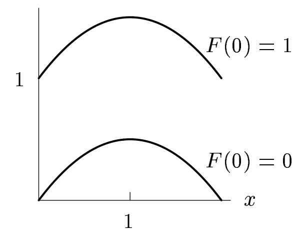

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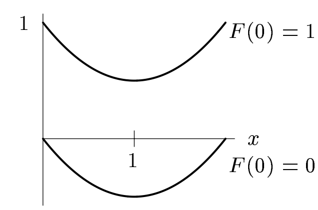

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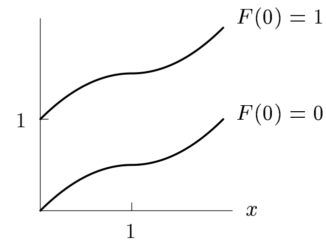


5. By the Fundamental Theorem of Calculus, we know that

$$f(2) - f(0) = \int_0^2 f'(x)dx.$$

Using a left-hand sum, we estimate $\int_0^2 f'(x)dx \approx (10)(2) = 20$. Using a right-hand sum, we estimate $\int_0^2 f'(x)dx \approx (18)(2) = 36$. Averaging, we have

$$\int_0^2 f'(x)dx \approx \frac{20+36}{2} = 28.$$

We know $f(0) = 100$, so

$$f(2) = f(0) + \int_0^2 f'(x)dx \approx 100 + 28 = 128.$$

Similarly, we estimate

$$\int_2^4 f'(x)dx \approx \frac{(18)(2) + (23)(2)}{2} = 41,$$

so

$$f(4) = f(2) + \int_2^4 f'(x)dx \approx 128 + 41 = 169.$$

Similarly,

$$\int_4^6 f'(x)dx \approx \frac{(23)(2) + (25)(2)}{2} = 48,$$

so

$$f(6) = f(4) + \int_4^6 f'(x)dx \approx 169 + 48 = 217.$$

The values are shown in the table.

x	0	2	4	6
$f(x)$	100	128	169	217

6. The change in $f(x)$ between 0 and 2 is equal to $\int_0^2 f'(x)\,dx$. A left-hand estimate for this integral is $(17)(2) = 34$ and a right hand estimate is $(15)(2) = 30$. Our best estimate is the average, 32. The change in $f(x)$ between 0 and 2 is $+32$. Since $f(0) = 50$, we have $f(2) = 82$. We find the other values similarly. The results are shown in Table 6.1.

Table 6.1

x	0	2	4	6
$f(x)$	50	82	107	119

7. (a) The value of the integral is negative since the area below the x-axis is greater than the area above the x-axis. We count boxes: The area below the x-axis includes approximately 11.5 boxes and each box has area $(2)(1) = 2$, so

$$\int_0^5 f(x)dx \approx -23.$$

The area above the x-axis includes approximately 2 boxes, each of area 2, so

$$\int_5^7 f(x)dx \approx 4.$$

So we have

$$\int_0^7 f(x)dx = \int_0^5 f(x)dx + \int_5^7 f(x)dx \approx -23 + 4 = -19.$$

(b) By the Fundamental Theorem of Calculus, we have

$$F(7) - F(0) = \int_0^7 f(x)dx$$

so,

$$F(7) = F(0) + \int_0^7 f(x)dx = 25 + (-19) = 6.$$

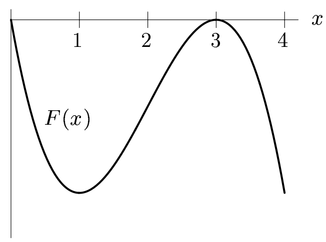
8. Since dP/dt is negative for $t < 3$ and positive for $t > 3$, we know that P is decreasing for $t < 3$ and increasing for $t > 3$. Between each two integer values, the magnitude of the change is equal to the area between the graph dP/dt and the t-axis. For example, between $t = 0$ and $t = 1$, we see that the change in P is -1. Since $P = 2$ at $t = 0$, we must have $P = 1$ at $t = 1$. The other values are found similarly, and are shown in Table 6.2.

Table 6.2

t	1	2	3	4	5
P	1	0	$-1/2$	0	1

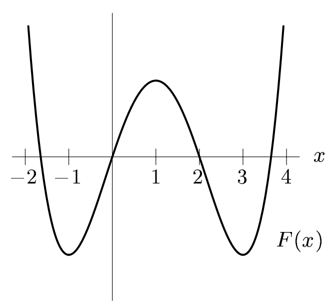
Problems

9. **(a)** Critical points of $F(x)$ are the zeros of f: $x = 1$ and $x = 3$.
(b) $F(x)$ has a local minimum at $x = 1$ and a local maximum at $x = 3$.
(c)

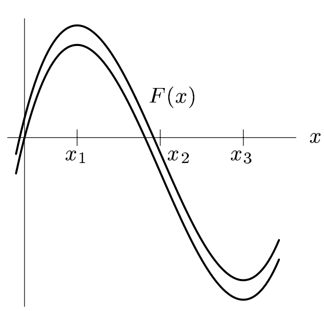


Notice that the graph could also be above or below the x-axis at $x = 3$.

10. **(a)** Critical points of $F(x)$ are $x = -1$, $x = 1$ and $x = 3$.
(b) $F(x)$ has a local minimum at $x = -1$, a local maximum at $x = 1$, and a local minimum at $x = 3$.
(c)

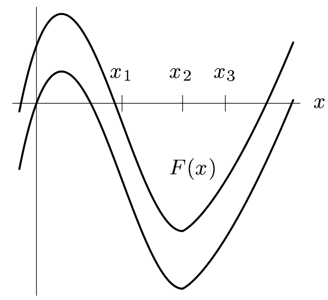


11.



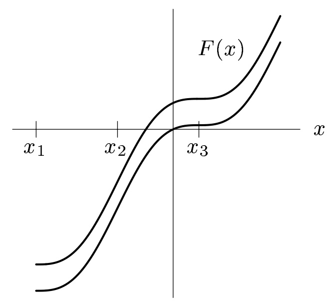
Note that since $f(x_1) = 0$ and $f'(x_1) < 0$, $F(x_1)$ is a local maximum; since $f(x_3) = 0$ and $f'(x_3) > 0$, $F(x_3)$ is a local minimum. Also, since $f'(x_2) = 0$ and f changes from decreasing to increasing about $x = x_2$, F has an inflection point at $x = x_2$.

12.



Note that since $f(x_2) = 0$, $f'(x_2) > 0$, so $F(x_2)$ is a local minimum. Since $f'(x_1) = 0$ and f changes from decreasing to increasing at $x = x_1$, F has an inflection point at $x = x_1$.

13.



Note that since $f(x_1) = 0$, $F(x_1)$ is either a local minimum or a point of inflection; it is impossible to tell which from the graph. Since $f'(x_3) = 0$, and f' changes sign around $x = x_3$, $F(x_3)$ is an inflection point. Also, since $f'(x_2) = 0$ and f changes from increasing to decreasing about $x = x_2$, F has another inflection point at $x = x_2$.

14. Between $t = 0$ and $t = 1$, the particle moves at 10 km/hr for 1 hour. Since it starts at $x = 5$, the particle is at $x = 15$ when $t = 1$. See Figure 6.1. The graph of distance is a straight line between $t = 0$ and $t = 1$ because the velocity is constant then.

Between $t = 1$ and $t = 2$, the particle moves 10 km to the left, ending at $x = 5$. Between $t = 2$ and $t = 3$, it moves 10 km to the right again. See Figure 6.1.

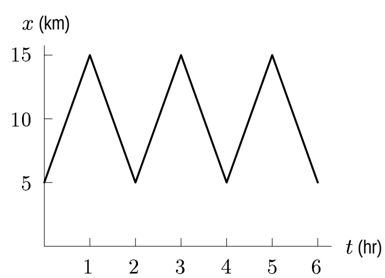


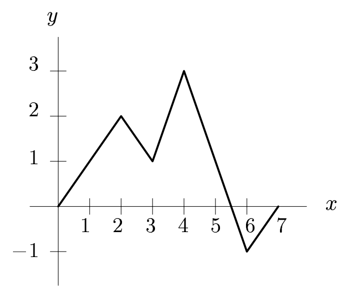
Figure 6.1

As an aside, note that the original velocity graph is not entirely realistic as it suggests the particle reverses direction instantaneously at the end of each hour. In practice this means the reversal of direction occurs over a time interval that is short in comparison to an hour.

15. **(a)** We know that $\int_0^3 f'(x)dx = f(3) - f(0)$ from the Fundamental Theorem of Calculus. From the graph of f' we can see that $\int_0^3 f'(x)dx = 2 - 1 = 1$ by subtracting areas between f' and the x-axis. Since $f(0) = 0$, we find that $f(3) = 1$. Similar reasoning gives $f(7) = \int_0^7 f'(x)dx = 2 - 1 + 2 - 4 + 1 = 0$.

(b) We have $f(0) = 0$, $f(2) = 2$, $f(3) = 1$, $f(4) = 3$, $f(6) = -1$, and $f(7) = 0$. So the graph, beginning at $x = 0$, starts at zero, increases to 2 at $x = 2$, decreases to 1 at $x = 3$, increases to 3 at $x = 4$, then passes through a zero as it decreases to -1 at $x = 6$, and finally increases to 0 at 7. Thus, there are three zeroes: $x = 0$, $x = 5.5$, and $x = 7$.

(c)



16. We can start by finding four points on the graph of $F(x)$. The first one is given: $F(2) = 3$. By the Fundamental Theorem of Calculus, $F(6) = F(2) + \int_2^6 F'(x)dx$. The value of this integral is -7 (the area is 7, but the graph lies below the x-axis), so $F(6) = 3 - 7 = -4$. Similarly, $F(0) = F(2) - 2 = 1$, and $F(8) = F(6) + 4 = 0$. We sketch a graph of $F(x)$ by connecting these points, as shown in Figure 6.2.

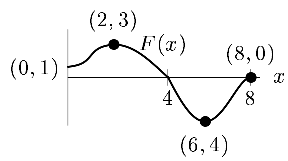
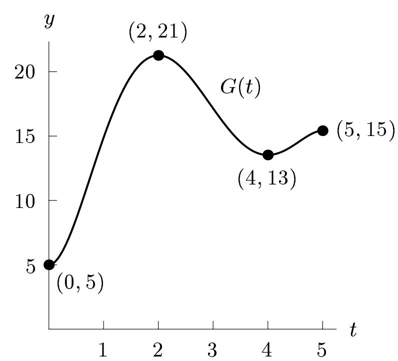


Figure 6.2

17. The critical points are at $(0, 5)$, $(2, 21)$, $(4, 13)$, and $(5, 15)$. A graph is given below.



18. Looking at the graph of g' below, we see that the critical points of g occur when $x = 15$ and $x = 40$, since $g'(x) = 0$ at these values. Inflection points of g occur when $x = 10$ and $x = 20$, because $g'(x)$ has a local maximum or minimum at these values. Knowing these four key points, we sketch the graph of $g(x)$ as follows.

We start at $x = 0$, where $g(0) = 50$. Since g' is negative on the interval $[0, 10]$, the value of $g(x)$ is decreasing there. At $x = 10$ we have

$$\begin{aligned} g(10) &= g(0) + \int_0^{10} g'(x)\,dx \\ &= 50 - (\text{area of shaded trapezoid } T_1) \\ &= 50 - \left(\frac{10 + 20}{2} \cdot 10\right) = -100. \end{aligned}$$

Similarly,

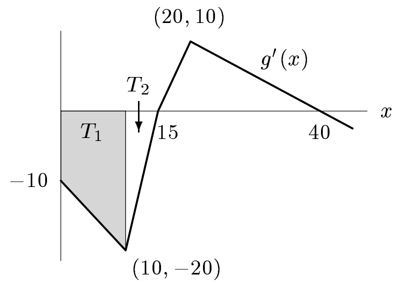
$$\begin{aligned} g(15) &= g(10) + \int_{10}^{15} g'(x)\,dx \\ &= -100 - (\text{area of triangle } T_2) \\ &= -100 - \frac{1}{2}(5)(20) = -150. \end{aligned}$$

Continuing,

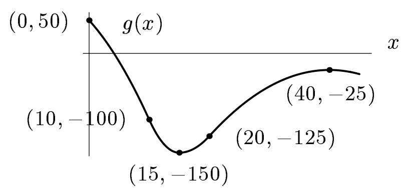
$$g(20) = g(15) + \int_{15}^{20} g'(x)\,dx = -150 + \frac{1}{2}(5)(10) = -125,$$

and

$$g(40) = g(20) + \int_{20}^{40} g'(x)\,dx = -125 + \frac{1}{2}(20)(10) = -25.$$



We now find concavity of $g(x)$ in the intervals $[0, 10]$, $[10, 15]$, $[15, 20]$, $[20, 40]$ by checking whether $g'(x)$ increases or decreases in these same intervals. If $g'(x)$ increases, then $g(x)$ is concave up; if $g'(x)$ decreases, then $g(x)$ is concave down. Thus we finally have our graph of $g(x)$:



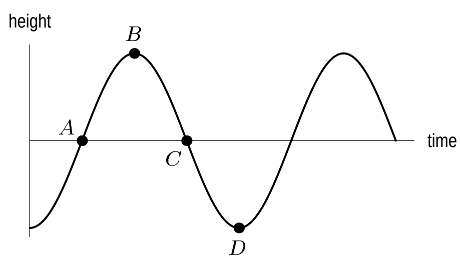
19. Between time $t = 0$ and time $t = B$, the velocity of the cork is always positive, which means the cork is moving upwards. At time $t = B$, the velocity is zero, and so the cork has stopped moving altogether. Since shortly thereafter the velocity of the cork becomes negative, the cork will next begin to move downwards. Thus when $t = B$ the cork has risen as far as it ever will, and is riding on top of the crest of the wave.

From time $t = B$ to time $t = D$, the velocity of the cork is negative, which means it is falling. When $t = D$, the velocity is again zero, and the cork has ceased to fall. Thus when $t = D$ the cork is riding on the bottom of the trough of the wave.

Since the cork is on the crest at time B and in the trough at time D, it is probably midway between crest and trough when the time is midway between B and D. Thus at time $t = C$ the cork is moving through the equilibrium position on its way down. (The equilibrium position is where the cork would be if the water were absolutely calm.) By symmetry, $t = A$ is the time when the cork is moving through the equilibrium position on the way up.

Since acceleration is the derivative of velocity, points where the acceleration is zero would be critical points of the velocity function. Since point A (a maximum) and point C (a minimum) are critical points, the acceleration is zero there.

A possible graph of the height of the cork is shown below. The horizontal axis represents a height equal to the average depth of the ocean at that point (the equilibrium position of the cork).



20. The rate of change is negative for $t < 5$ and positive for $t > 5$, so the concentration of adrenaline decreases until $t = 5$ and then increases. Since the area under the x-axis is greater than the area over the x-axis, the concentration of adrenaline goes down more than it goes up. Thus, the concentration at $t = 8$ is less than the concentration at $t = 0$. See Figure 6.3.

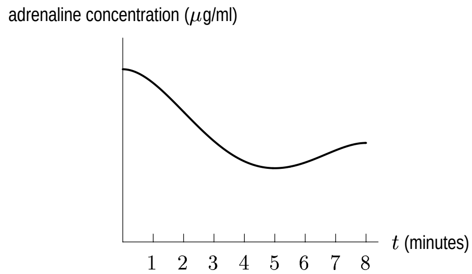


Figure 6.3

21. (a) The total volume emptied must increase with time and cannot decrease. The smooth graph (I) that is always increasing is therefore the volume emptied from the bladder. The jagged graph (II) that increases then decreases to zero is the flow rate.

(b) The total change in volume is the integral of the flow rate. Thus, the graph giving total change (I) shows an antiderivative of the rate of change in graph (II).

22. The graph of $f(x) = 2\sin(x^2)$ is shown in Figure 6.4. We see that there are roots at $x = 1.77$ and $x = 2.51$. These are the critical points of $F(x)$. Looking at the graph, it appears that of the three areas marked, A_1 is the largest, A_2 is next, and A_3 is smallest. Thus, as x increases from 0 to 3, the function $F(x)$ increases (by A_1), decreases (by A_2), and then increases again (by A_3). Therefore, the maximum is attained at the critical point $x = 1.77$.

What is the value of the function at this maximum? We know that $F(1) = 5$, so we need to find the change in F between $x = 1$ and $x = 1.77$. We have

$$\text{Change in } F = \int_1^{1.77} 2\sin(x^2)\,dx = 1.17.$$

We see that $F(1.77) = 5 + 1.17 = 6.17$, so the maximum value of F on this interval is 6.17.

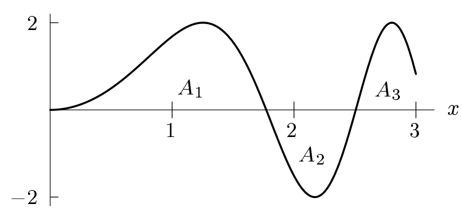
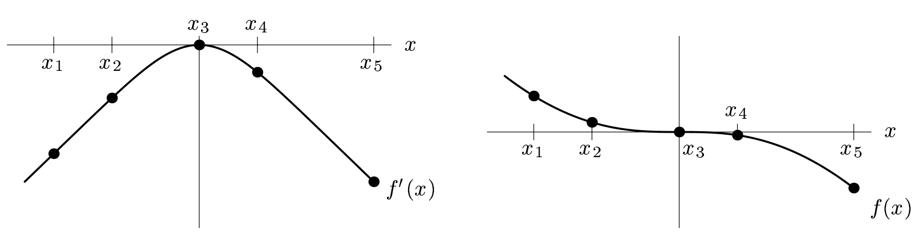


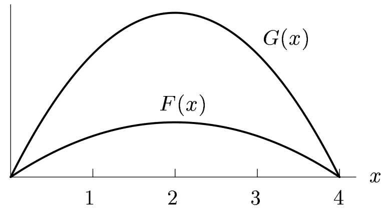
Figure 6.4

23.



(a) $f(x)$ is greatest at x_1.
(b) $f(x)$ is least at x_5.
(c) $f'(x)$ is greatest at x_3..
(d) $f'(x)$ is least at x_5.
(e) $f''(x)$ is greatest at x_1.
(f) $f''(x)$ is least at x_5.

24. Both $F(x)$ and $G(x)$ have roots at $x = 0$ and $x = 4$. Both have a critical point (which is a local maximum) at $x = 2$. However, since the area under $g(x)$ between $x = 0$ and $x = 2$ is larger than the area under $f(x)$ between $x = 0$ and $x = 2$, the y-coordinate of $G(x)$ at 2 will be larger than the y-coordinate of $F(x)$ at 2. See below.



25. **(a)** Suppose $Q(t)$ is the amount of water in the reservoir at time t. Then

$$Q'(t) = \begin{array}{c}\text{Rate at which water}\\ \text{in reservoir is changing}\end{array} = \begin{array}{c}\text{Inflow}\\ \text{rate}\end{array} - \begin{array}{c}\text{Outflow}\\ \text{rate}\end{array}$$

Thus the amount of water in the reservoir is increasing when the inflow curve is above the outflow, and decreasing when it is below. This means that $Q(t)$ is a maximum where the curves cross in July 1993 (as shown in Figure 6.5), and $Q(t)$ is decreasing fastest when the outflow is farthest above the inflow curve, which occurs about October 1993 (see Figure 6.5).

To estimate values of $Q(t)$, we use the Fundamental Theorem which says that the change in the total quantity of water in the reservoir is given by

$$Q(t) - Q(\text{Jan'93}) = \int_{\text{Jan93}}^{t} (\text{inflow rate} - \text{outflow rate})\, dt$$

or

$$Q(t) = Q(\text{Jan'93}) + \int_{\text{Jan93}}^{t} (\text{inflow rate} - \text{outflow rate})\, dt.$$

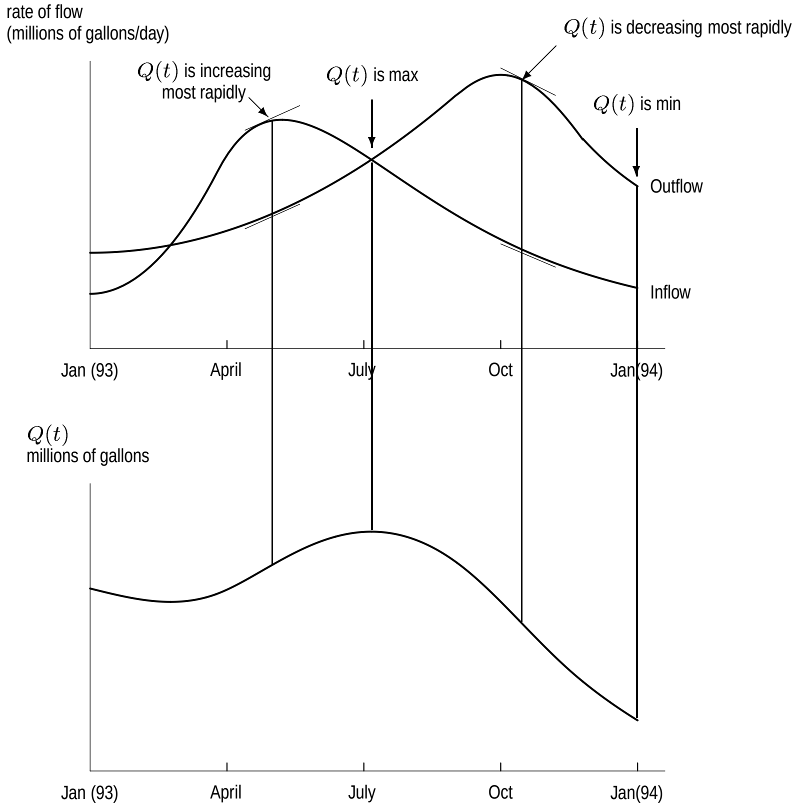


Figure 6.5

(b) See Figure 6.5. Maximum in July 1993. Minimum in Jan 1994.
(c) See Figure 6.5. Increasing fastest in May 1993. Decreasing fastest in Oct 1993.
(d) In order for the water to be the same as Jan '93 the total amount of water which has flowed into the reservoir must be 0. Referring to Figure 6.6, we have

$$\int_{\mathrm{Jan93}}^{\mathrm{July94}} (\text{inflow} - \text{outflow})dt = -A_1 + A_2 - A_3 + A_4 = 0$$

giving $A_1 + A_3 = A_2 + A_4$

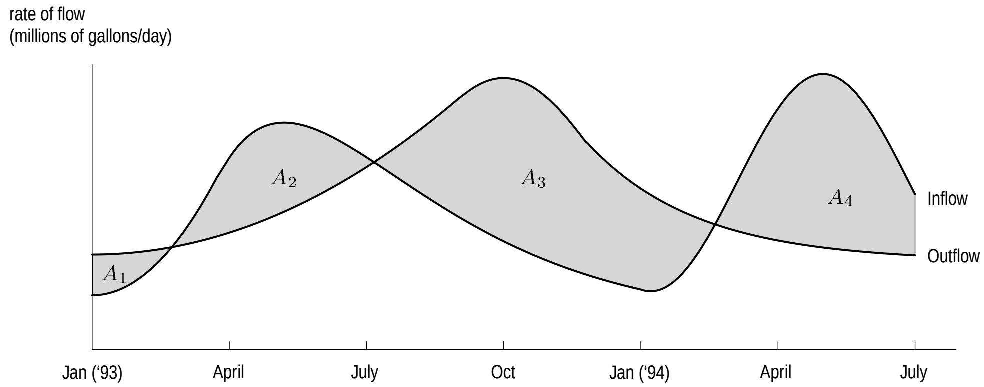


Figure 6.6

Solutions for Section 6.2

Exercises

1. $5x$
2. $\frac{5}{2}x^2$
3. $\frac{1}{3}x^3$
4. $\frac{1}{3}t^3 + \frac{1}{2}t^2$
5. $\sin t$
6. $\frac{2}{3}z^{\frac{3}{2}}$
7. $\ln |z|$
8. $-\frac{1}{t}$
9. $-\frac{1}{2z^2}$
10. e^z
11. $-\cos t$
12. $\frac{2}{3}t^3 + \frac{3}{4}t^4 + \frac{4}{5}t^5$
13. $\frac{t^4}{4} - \frac{t^3}{6} - \frac{t^2}{2}$
14. $\frac{y^5}{5} + \ln |y|$
15. $\sin t + \tan t$
16. $\frac{t^2+1}{t} = t + \frac{1}{t}$, which has antiderivative $\frac{t^2}{2} + \ln |t|$
17. $-\cos 2\theta$
18. $e^t + 5\frac{1}{5}e^{5t} = e^t + e^{5t}$

19. $\frac{1}{3}(t+1)^3$

20. $\dfrac{5^x}{\ln 5}$

21. $\dfrac{5}{2}x^2 - \dfrac{2}{3}x^{\frac{3}{2}}$

22. $F(t) = \int 6t\,dt = 3t^2 + C$

23. $H(x) = \int (x^3 - x)\,dx = \dfrac{x^4}{4} - \dfrac{x^2}{2} + C$

24. $F(x) = \int (x^2 - 4x + 7)\,dx = \dfrac{x^3}{3} - 2x^2 + 7x + C$

25. $R(t) = \int (t^3 + 5t - 1)\,dt = \dfrac{t^4}{4} + \dfrac{5}{2}t^2 - t + C$

26. $F(z) = \int (z + e^z)\,dz = \dfrac{z^2}{2} + e^z + C$

27. $G(t) = \int \sqrt{t}\,dt = \dfrac{2}{3}t^{3/2} + C$

28. $G(x) = \int (\sin x + \cos x)\,dx = -\cos x + \sin x + C$

29. $H(x) = \int (4x^3 - 7)\,dx = x^4 - 7x + C$

30. $P(t) = \int (2 + \sin t)\,dt = 2t - \cos t + C$

31. $P(t) = \int \dfrac{1}{\sqrt{t}}\,dt = 2t^{1/2} + C$

32. $G(x) = \int \dfrac{5}{x^3}\,dx = -\dfrac{5}{2x^2} + C$

33. $F(x) = \int \dfrac{1}{x^2}\,dx = -\dfrac{1}{x} + C$

34. $f(x) = 3$, so $F(x) = 3x + C$. $F(0) = 0$ implies that $3 \cdot 0 + C = 0$, so $C = 0$. Thus $F(x) = 3x$ is the only possibility.

35. $f(x) = 2x$, so $F(x) = x^2 + C$. $F(0) = 0$ implies that $0^2 + C = 0$, so $C = 0$. Thus $F(x) = x^2$ is the only possibility.

36. $f(x) = -7x$, so $F(x) = \frac{-7x^2}{2} + C$. $F(0) = 0$ implies that $-\frac{7}{2} \cdot 0^2 + C = 0$, so $C = 0$. Thus $F(x) = -7x^2/2$ is the only possibility.

37. $f(x) = \frac{1}{4}x$, so $F(x) = \frac{x^2}{8} + C$. $F(0) = 0$ implies that $\frac{1}{8} \cdot 0^2 + C = 0$, so $C = 0$. Thus $F(x) = x^2/8$ is the only possibility.

38. $f(x) = x^2$, so $F(x) = \dfrac{x^3}{3} + C$. $F(0) = 0$ implies that $\dfrac{0^3}{3} + C = 0$, so $C = 0$. Thus $F(x) = \dfrac{x^3}{3}$ is the only possibility.

39. $f(x) = x^{1/2}$, so $F(x) = \frac{2}{3}x^{3/2} + C$. $F(0) = 0$ implies that $\frac{2}{3} \cdot 0^{3/2} + C = 0$, so $C = 0$. Thus $F(x) = \frac{2}{3}x^{3/2}$ is the only possibility.

40. $f(x) = 2 + 4x + 5x^2$, so $F(x) = 2x + 2x^2 + \frac{5}{3}x^3 + C$. $F(0) = 0$ implies that $C = 0$. Thus $F(x) = 2x + 2x^2 + \frac{5}{3}x^3$ is the only possibility.

41. $f(x) = \sin x$, so $F(x) = -\cos x + C$. $F(0) = 0$ implies that $-\cos 0 + C = 0$, so $C = 1$. Thus $F(x) = -\cos x + 1$ is the only possibility.

42. $\int 5x\,dx = \dfrac{5}{2}x^2 + C.$

43. $\int x^3\,dx = \dfrac{x^4}{4} + C$

44. $\displaystyle\int \sin\theta\, d\theta = -\cos\theta + C$

45. $\displaystyle\int (x^3-2)\, dx = \frac{x^4}{4} - 2x + C$

46. $\displaystyle\int (t^2 + \frac{1}{t^2})\, dt = \frac{t^3}{3} - \frac{1}{t} + C$

47. $\displaystyle\int 4\sqrt{w}\, dw = \frac{8}{3}w^{3/2} + C$

48. $\displaystyle\int (x^2+5x+8)\, dx = \frac{x^3}{3} + \frac{5x^2}{2} + 8x + C$

49. $\displaystyle\int \frac{4}{t^2}\, dt = -\frac{4}{t} + C$

50. $2t^2 + 7t + C$

51. $\sin\theta + C$

52. $5e^z + C$

53. $\displaystyle\frac{x^2}{2} + 2x^{1/2} + C$

54. $-\cos t + C$

55. $\displaystyle\pi x + \frac{x^{12}}{12} + C$

56. $\displaystyle\int \left(t^{3/2} + t^{-3/2}\right)\, dt = \frac{2t^{5/2}}{5} - 2t^{-1/2} + C$

57. $\sin(x+1) + C$

58. $\frac{1}{2}e^{2r} + C$

59. $\displaystyle\int \frac{1}{e^z}\, dz = \int e^{-z}\, dz = -e^{-z} + C$

60. $\displaystyle\int \left(y - \frac{1}{y}\right)^2\, dy = \int \left(y^2 - 2 + \frac{1}{y^2}\right)\, dy = \frac{y^3}{3} - 2y - \frac{1}{y} + C$

61. $\displaystyle\int_0^3 (x^2+4x+3)\, dx = \left(\frac{x^3}{3} + 2x^2 + 3x\right)\Bigg|_0^3 = (9+18+9) - 0 = 36$

62. $\displaystyle\int_1^3 \frac{1}{t}\, dt = \ln|t|\Bigg|_1^3 = \ln|3| - \ln|1| = \ln 3 \approx 1.0986.$

63. $\displaystyle\int_0^{\pi/4} \sin x\, dx = -\cos x\Bigg|_0^{\pi/4} = -\cos\frac{\pi}{4} - (-\cos 0) = -\frac{\sqrt{2}}{2} + 1 = 0.293.$

64. $\displaystyle\int_0^2 3e^x\, dx = 3e^x\Bigg|_0^2 = 3e^2 - 3e^0 = 3e^2 - 3 = 19.167.$

65. $\displaystyle\int_2^5 (x^3 - \pi x^2)\, dx = \left(\frac{x^4}{4} - \frac{\pi x^3}{3}\right)\Bigg|_2^5 = \frac{609}{4} - 39\pi \approx 29.728.$

66. $\displaystyle\int_0^1 \sin\theta\, d\theta = -\cos\theta\Bigg|_0^1 = 1 - \cos 1 \approx 0.460.$

67. Since $\displaystyle\frac{1+y^2}{y} = \frac{1}{y} + y,$

$$\int_1^2 \frac{1+y^2}{y}\, dy = \left(\ln|y| + \frac{y^2}{2}\right)\Bigg|_1^2 = \ln 2 + \frac{3}{2} \approx 2.193.$$

68. $\displaystyle\int_0^2 \left(\frac{x^3}{3} + 2x\right)\, dx = \left(\frac{x^4}{12} + x^2\right)\Bigg|_0^2 = \frac{4}{3} + 4 = 16/3 \approx 5.333.$

69. $\displaystyle\int_0^{\pi/4} (\sin t + \cos t)\,dt = (-\cos t + \sin t)\Big|_0^{\pi/4} = \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (-1+0) = 1.$

70. $\displaystyle\int_{-3}^{-1} \frac{2}{r^3}\,dr = -r^{-2}\Big|_{-3}^{-1} = -1 + \frac{1}{9} = -8/9 \approx -0.889.$

71. $\displaystyle\int_0^1 2e^x\,dx = 2e^x\Big|_0^1 = 2e - 2 \approx 3.437.$

72. Since $(\tan x)' = \dfrac{1}{\cos^2 x}$, $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^2 x}\,dx = \tan x\Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1.$

73. $\displaystyle\int 2^x\,dx = \frac{1}{\ln 2}2^x + C$, since $\dfrac{d}{dx}2^x = \ln 2 \cdot 2^x$, so
$\displaystyle\int_{-1}^1 2^x\,dx = \frac{1}{\ln 2}\left[2^x\Big|_{-1}^1\right] = \frac{3}{2\ln 2} \approx 2.164.$

Problems

74. We have
$$\text{Area} = \int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = 21.$$

75. The graph crosses the x-axis where
$$\begin{aligned} 7 - 8x + x^2 &= 0 \\ (x-7)(x-1) &= 0; \end{aligned}$$
so $x = 1$ and $x = 7$. See Figure 6.7. The parabola opens upward and the region is below the x-axis, so
$$\begin{aligned} \text{Area} &= -\int_1^7 (7 - 8x + x^2)\,dx \\ &= -\left(7x - 4x^2 + \frac{x^3}{3}\right)\Big|_1^7 = 36. \end{aligned}$$

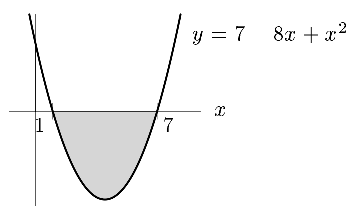
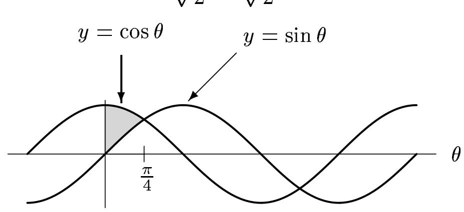


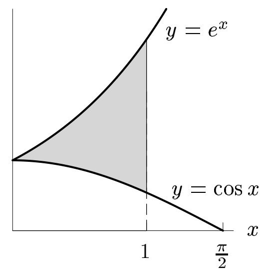
Figure 6.7

76. The graph is shown in the figure below. Since $\cos\theta \ge \sin\theta$ for $0 \le \theta \le \pi/4$, we have
$$\begin{aligned} \text{Area} &= \int_0^{\pi/4} (\cos\theta - \sin\theta)\,d\theta \\ &= (\sin\theta + \cos\theta)\Big|_0^{\pi/4} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1. \end{aligned}$$



77. Since the graph of $y = e^x$ is above the graph of $y = \cos x$ (see the figure below), we have

$$\begin{aligned}\text{Area } &= \int_0^1 (e^x - \cos x)\,dx \\ &= \int_0^1 e^x\,dx - \int_0^1 \cos x\,dx \\ &= e^x\Big|_0^1 - \sin x\Big|_0^1 \\ &= e^1 - e^0 - \sin 1 + \sin 0 \\ &= e - 1 - \sin 1.\end{aligned}$$



78. The area under $f(x) = 8x$ between $x = 1$ and $x = b$ is given by $\int_1^b (8x)dx$. Using the Fundamental Theorem to evaluate the integral:

$$\text{Area } = 4x^2\Big|_1^b = 4b^2 - 4.$$

Since the area is 192, we have

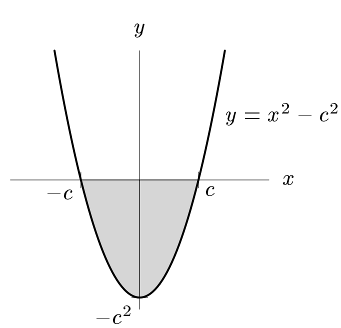
$$\begin{aligned}4b^2 - 4 &= 192 \\ 4b^2 &= 196 \\ b^2 &= 49 \\ b &= \pm 7.\end{aligned}$$

Since b is larger than 1, we have $b = 7$.

79. The graph of $y = x^2 - c^2$ has x-intercepts of $x = \pm c$. See the figure below. The shaded area is given by

$$\begin{aligned}\text{Area } &= -\int_{-c}^{c} (x^2 - c^2)\,dx \\ &= -2\int_0^c (x^2 - c^2)\,dx \\ &= -2\left(\frac{x^3}{3} - c^2x\right)\Big|_0^c = -2\left(\frac{c^3}{3} - c^3\right) = \frac{4}{3}c^3.\end{aligned}$$

We want c to satisfy $(4c^3)/3 = 36$, so $c = 3$.



80. We have

$$\text{Average value} = \frac{1}{10-0}\int_0^{10}(x^2+1)dx = \frac{1}{10}\left(\frac{x^3}{3}+x\right)\Bigg|_0^{10} = \frac{1}{10}\left(\frac{10^3}{3}+10-0\right) = \frac{103}{3}.$$

We see in Figure 6.8 that the average value of $103/3 \approx 34.33$ for $f(x)$ looks right.

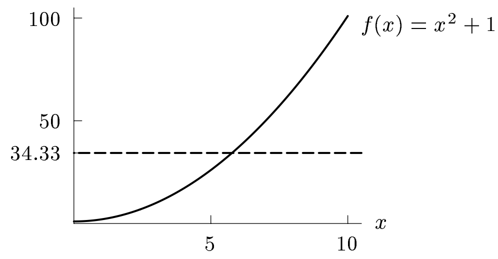


Figure 6.8

81. The average value of $v(x)$ on the interval $1 \le x \le c$ is

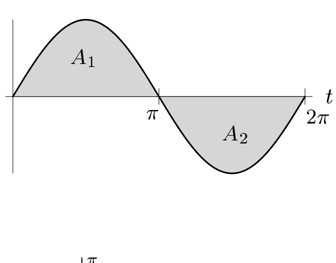
$$\frac{1}{c-1}\int_1^c \frac{6}{x^2}\,dx = \frac{1}{c-1}\left(-\frac{6}{x}\right)\Bigg|_1^c = \frac{1}{c-1}\left(\frac{-6}{c}+6\right) = \frac{6}{c}.$$

Since $\dfrac{1}{c-1}\displaystyle\int_1^c \frac{6}{x^2}dx = 1$, we have $\dfrac{6}{c} = 1$, so $c = 6$.

82. (a) The average value of $f(t) = \sin t$ over $0 \le t \le 2\pi$ is given by the formula

$$\begin{aligned}\text{Average} &= \frac{1}{2\pi-0}\int_0^{2\pi}\sin t\,dt \\ &= \frac{1}{2\pi}(-\cos t)\Big|_0^{2\pi} \\ &= \frac{1}{2\pi}(-\cos 2\pi - (-\cos 0)) = 0.\end{aligned}$$

We can check this answer by looking at the graph of $\sin t$ below. The area below the curve and above the t-axis over the interval $0 \le t \le \pi, A_1$, is the same as the area above the curve but below the t-axis over the interval $\pi \le t \le 2\pi, A_2$. When we take the integral of $\sin t$ over the entire interval $0 \le t \le 2\pi$, we get $A_1 - A_2 = 0$.



(b) Since

$$\int_0^{\pi}\sin t\,dt = -\cos t\Big|_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1)-(-1) = 2,$$

the average value of $\sin t$ on $0 \le t \le \pi$ is given by

$$\text{Average value } = \frac{1}{\pi}\int_0^{\pi}\sin t\,dt = \frac{2}{\pi}.$$

83. Since $C'(x) = 4000 + 10x$ we want to evaluate the indefinite integral

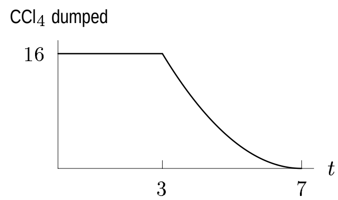
$$\int (4000 + 10x)\, dx = 4000x + 5x^2 + K$$

where K is a constant. Thus $C(x) = 5x^2 + 4000x + K$, and the fixed cost of 1,000,000 riyal means that $C(0) = 1{,}000{,}000 = K$. Therefore, the total cost is

$$C(x) = 5x^2 + 4000x + 1{,}000{,}000.$$

Since $C(x)$ depends on x^2, the square of the depth drilled, costs will increase dramatically when x grows large.

84. **(a)**



(b) 7 years, because $t^2 - 14t + 49 = (t-7)^2$ indicates that the rate of flow was zero after 7 years.

(c)

$$\begin{aligned}
\text{Area under the curve} &= 3(16) + \int_3^7 (t^2 - 14t + 49)\, dt \\
&= 48 + \left(\frac{1}{3}t^3 - 7t^2 + 49t\right)\Bigg|_3^7 \\
&= 48 + \frac{343}{3} - 343 + 343 - 9 + 63 - 147 \\
&= \frac{208}{3} = 69\frac{1}{3} \text{ cubic yards.}
\end{aligned}$$

Solutions for Section 6.3

Exercises

1. $y = \int (x^3 + 5)\, dx = \dfrac{x^4}{4} + 5x + C$

2. $y = \int \left(8x + \dfrac{1}{x}\right)\, dx = 4x^2 + \ln|x| + C$

3. $W = \int 4\sqrt{t}\, dt = \dfrac{8}{3}t^{3/2} + C$

4. $r = \int 3\sin p\, dp = -3\cos p + C$

5. Since $y = x + \sin x - \pi$, we differentiate to see that $dy/dx = 1 + \cos x$, so y satisfies the differential equation. To show that it also satisfies the initial condition, we check that $y(\pi) = 0$:

$$\begin{aligned}
y &= x + \sin x - \pi \\
y(\pi) &= \pi + \sin \pi - \pi = 0.
\end{aligned}$$

6. $y = \int (6x^2 + 4x)\, dx = 2x^3 + 2x^2 + C$. If $y(2) = 10$, then $2(2)^3 + 2(2)^2 + C = 10$ and $C = 10 - 16 - 8 = -14$. Thus, $y = 2x^3 + 2x^2 - 14$.

7. $P = \int 10e^t\, dt = 10e^t + C$. If $P(0) = 25$, then $10e^0 + C = 25$ so $C = 15$. Thus, $P = 10e^t + 15$.

8. $s = \int (-32t + 100)\, dt = -16t^2 + 100t + C$. If $s = 50$ when $t = 0$, then $-16(0)^2 + 100(0) + C = 50$, so $C = 50$. Thus $s = -16t^2 + 100t + 50$.

9. Integrating gives
$$\int \frac{dq}{dz}\,dz = \int (2+\sin z)\,dz = 2z - \cos z + C.$$
If $q = 5$ when $z = 0$, then $2(0) - \cos(0) + C = 5$ so $C = 6$. Thus $q = 2z - \cos z + 6$.

10. We differentiate $y = xe^{-x} + 2$ using the product rule to obtain
$$\begin{aligned}\frac{dy}{dx} &= x\left(e^{-x}(-1)\right) + (1)e^{-x} + 0\\ &= -xe^{-x} + e^{-x}\\ &= (1-x)e^{-x},\end{aligned}$$
and so $y = xe^{-x} + 2$ satisfies the differential equation. We now check that $y(0) = 2$:
$$\begin{aligned} y &= xe^{-x} + 2\\ y(0) &= 0e^0 + 2 = 2.\end{aligned}$$

Problems

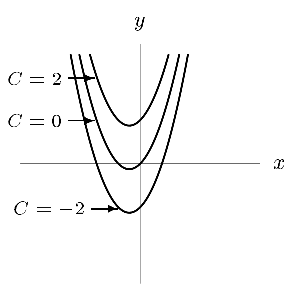
11. **(a)** Acceleration = $a(t) = -9.8$ m/sec^2
Velocity = $v(t) = -9.8t + 40$ m/sec
Height = $h(t) = -4.9t^2 + 40t + 25$ m
.

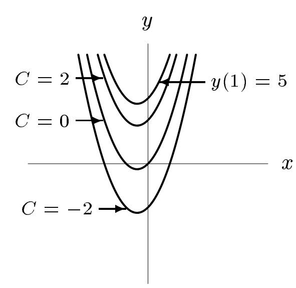
(b) At the highest point,
$$v(t) = -9.8t + 40 = 0,$$
so
$$t = \frac{40}{9.8} = 4.08 \text{ seconds.}$$
At that time, $h(4.08) = 106.6$ m. We see that the tomato reaches a height of 106.6 m, at 4.08 seconds after it is thrown.

(c) The tomato lands when $h(t) = 0$, so
$$-4.9t^2 + 40t + 25 = 0.$$
The solutions are $t = -0.58$ and $t = 8.75$ seconds. We see that it lands 8.75 seconds after it is thrown.

12. **(a)** $y = \int (2x+1)\,dx$, so the solution is $y = x^2 + x + C$.

(b)





(c) At $y(1) = 5$, we have $1^2 + 1 + C = 5$ and so $C = 3$. Thus we have the solution $y = x^2 + x + 3$.

13.
$$\begin{aligned}\frac{dy}{dt} &= k\sqrt{t} = kt^{1/2}\\ y &= \frac{2}{3}kt^{3/2} + C.\end{aligned}$$
Since $y = 0$ when $t = 0$, we have $C = 0$, so
$$y = \frac{2}{3}kt^{3/2}.$$

14. **(a)** To find the height of the balloon, we integrate its velocity with respect to time:

$$\begin{aligned} h(t) &= \int v(t)\,dt \\ &= \int (-32t + 40)\,dt \\ &= -32\frac{t^2}{2} + 40t + C. \end{aligned}$$

Since at $t = 0$, we have $h = 30$, we can solve for C to get $C = 30$, giving us a height of

$$h(t) = -16t^2 + 40t + 30.$$

(b) To find the average velocity between $t = 1.5$ and $t = 3$, we find the total displacement and divide by time.

$$\text{Average velocity} = \frac{h(3) - h(1.5)}{3 - 1.5} = \frac{6 - 54}{1.5} = -32 \text{ ft/sec}.$$

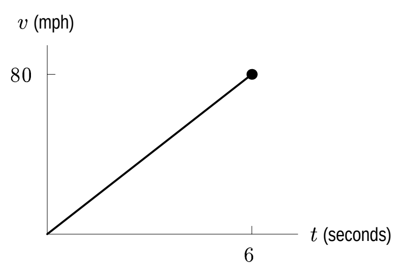
The balloon's average velocity is 32 ft/sec downward.

(c) First, we must find the time when $h(t) = 6$. Solving the equation $-16t^2 + 40t + 30 = 6$, we get

$$\begin{aligned} 6 &= -16t^2 + 40t + 30 \\ 0 &= -16t^2 + 40t + 24 \\ 0 &= 2t^2 - 5t - 3 \\ 0 &= (2t + 1)(t - 3). \end{aligned}$$

Thus, $t = -1/2$ or $t = 3$. Since $t = -1/2$ makes no physical sense, we use $t = 3$ to calculate the balloon's velocity. At $t = 3$, we have a velocity of $v(3) = -32(3) + 40 = -56$ ft/sec. So the balloon's velocity is 56 ft/sec downward at the time of impact.

15. Since the car's acceleration is constant, a graph of its velocity against time t is linear, as shown below.



The acceleration is just the slope of this line:

$$\frac{dv}{dt} = \frac{80 - 0 \text{ mph}}{6 \text{ sec}} = \frac{40}{3} = 13.33\frac{\text{mph}}{\text{sec}}.$$

To convert our units into ft/sec^2,

$$\frac{40}{3} \cdot \frac{\text{mph}}{\text{sec}} \cdot \frac{5280 \text{ ft}}{1 \text{ mile}} \frac{1 \text{ hour}}{3600 \text{ sec}} = 19.55\frac{\text{ft}}{\text{sec}^2}$$

16. Since the acceleration $a = dv/dt$, where v is the velocity of the car, we have

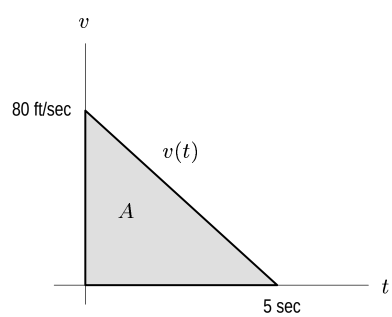
$$\frac{dv}{dt} = -0.6t + 4.$$

Integrating gives

$$v = -0.6\frac{t^2}{2} + 4t + C.$$

The car starts from rest, so $v = 0$ when $t = 0$, and therefore $C = 0$. If x is the distance from the starting point, $v = dx/dt$ and
$$\frac{dx}{dt} = -0.3t^2 + 4t,$$
so
$$x = -\frac{0.3}{3}t^3 + \frac{4}{2}t^2 + C = -0.1t^3 + 2t^2 + C.$$
Since $x = 0$ when $t = 0$, we have $C = 0$, so
$$x = -0.1t^3 + 2t^2.$$
We want to solve for t when $x = 100$:
$$100 = -0.1t^3 + 2t^2.$$
This equation can be rewritten as
$$0.1t^3 - 2t^2 + 100 = 0$$
$$t^3 - 20t^2 + 1000 = 0.$$
The equation can be solved numerically, or by tracing along a graph, or by factoring
$$(t-10)(t^2 - 10t - 100) = 0.$$
The solutions are $t = 10$ and $t = \frac{10\pm\sqrt{500}}{2} = -6.18,\ 16.18$. Since we are told $0 \le t \le 12$, the solution we want is $t = 10$ sec.

17. **(a)**



(b) The total distance is represented by the shaded region A, the area under the graph of $v(t)$.

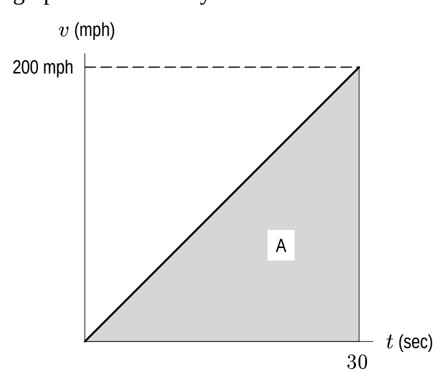
(c) The area A, a triangle, is given by
$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(5\,\text{sec})(80\,\text{ft/sec}) = 200\,\text{ft}.$$

(d) Using integration and the Fundamental Theorem of Calculus, we have $A = \int_0^5 v(t)\,dt$ or $A = s(5) - s(0)$, where $s(t)$ is an antiderivative of $v(t)$.

We have that $a(t)$, the acceleration, is constant: $a(t) = k$ for some constant k. Therefore $v(t) = kt + C$ for some constant C. We have $80 = v(0) = k(0) + C = C$, so that $v(t) = kt + 80$. Putting in $t = 5$, $0 = v(5) = (k)(5) + 80$, or $k = -80/5 = -16$.

Thus $v(t) = -16t + 80$, and an antiderivative for $v(t)$ is $s(t) = -8t^2 + 80t + C$. Since the total distance traveled at $t = 0$ is 0, we have $s(0) = 0$ which means $C = 0$. Finally, $A = \int_0^5 v(t)\,dt = s(5) - s(0) = (-8(5)^2 + (80)(5)) - (-8(0)^2 + (80)(0)) = 200\,\text{ft}$, which agrees with the previous part.

18. Since the acceleration is constant, a graph of the velocity versus time looks like this:



The distance traveled in 30 seconds, which is how long the runway must be, is equal to the area represented by A. We have $A = \frac{1}{2}(\text{base})(\text{height})$. First we convert the required velocity into miles per second.

$$\begin{aligned} 200 \text{ mph} &= \frac{200 \text{ miles}}{\text{hour}}\left(\frac{1 \text{ hour}}{60 \text{ minutes}}\right)\left(\frac{1 \text{ minute}}{60 \text{ seconds}}\right) \\ &= \frac{200}{3600}\ \frac{\text{miles}}{\text{second}} \\ &= \frac{1}{18} \text{ miles/second.} \end{aligned}$$

Therefore $A = \frac{1}{2}(30 \text{ sec})(200 \text{ mph}) = \frac{1}{2}(30 \text{ sec})\left(\frac{1}{18} \text{ miles/sec}\right) = \frac{5}{6}$ miles.

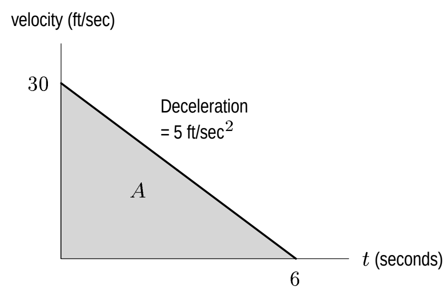
19. **(a)** Since the velocity is constantly decreasing, and $v(6) = 0$, the car stops after 6 seconds.

t (sec)	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$v(t)$ (ft/sec)	30	27.5	25	22.5	20	17.5	15	12.5	10	7.5	5	2.5	0

(b) Over the interval $a \leq t \leq a + \frac{1}{2}$, the left-hand velocity is $v(a)$, and the right-hand velocity is $v(a + \frac{1}{2})$. Since we are considering half-second intervals, $\Delta t = \frac{1}{2}$, and $n = 12$. The left sum is 97.5 ft., and the right sum is 82.5 ft.

(c) Area A in the figure below represents distance traveled.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \cdot 6 \cdot 30 = 90 \text{ ft.}$$

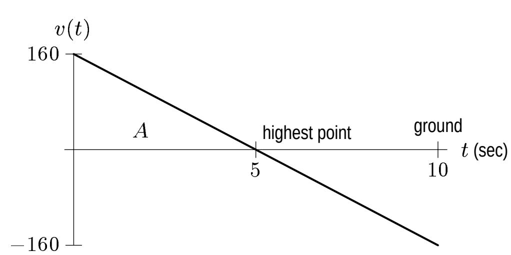


(d) The velocity is constantly decreasing at a rate of 5 ft/sec per second, i.e. after each second the velocity has dropped by 5 units. Therefore $v(t) = 30 - 5t$.

An antiderivative for $v(t)$ is $s(t)$, where $s(t) = 30t - \frac{5}{2}t^2$. Thus by the Fundamental Theorem of Calculus, the distance traveled $= s(6) - s(0) = (30(6) - \frac{5}{2}(6)^2) - (30(0) - \frac{5}{2}(0)^2) = 90$ ft. Since $v(t)$ is decreasing, the left-hand sum in part (b) overestimates the distance traveled, while the right-hand sum underestimates it.

The area A is equal to the average of the left-hand and right-hand sums: $90 \text{ ft} = \frac{1}{2}(97.5 \text{ ft} + 82.5 \text{ ft})$. The left-hand sum is an overestimate of A; the right-hand sum is an underestimate.

20. **(a)**



(b) The highest point is at $t = 5$ seconds. The object hits the ground at $t = 10$ seconds, since by symmetry if the object takes 5 seconds to go up, it takes 5 seconds to come back down.

(c) The maximum height is the distance traveled when going up, which is represented by the area A of the triangle above the time axis.

$$\text{Area} = \frac{1}{2}(160 \text{ ft/sec})(5 \text{ sec}) = 400 \text{ feet.}$$

(d) The slope of the line is -32, so $v(t) = -32t + 160$. Antidifferentiating, we get $s(t) = -16t^2 + 160t + s_0$. $s_0 = 0$, so $s(t) = -16t^2 + 160t$. At $t = 5$, $s(t) = -400 + 800 = 400$ ft.

21. The equation of motion is $y = -\frac{gt^2}{2} + v_0t + y_0 = -16t^2 + 128t + 320$. Taking the first derivative, we get $v = -32t + 128$. The second derivative gives us $a = -32$.

(a) At its highest point, the stone's velocity is zero:
$v = 0 = -32t + 128$, so $t = 4$.

(b) At $t = 4$, the height is $y = -16(4)^2 + 128(4) + 320 = 576$ ft

(c) When the stone hits the beach,

$$y = 0 = -16t^2 + 128t + 320$$
$$0 = -t^2 + 8t + 20 = (10 - t)(2 + t).$$

So $t = 10$ seconds.

(d) Impact is at $t = 10$. The velocity, v, at this time is $v(10) = -32(10) + 128 = -192$ ft/sec. Upon impact, the stone's velocity is 192 ft/sec downward.

22. **(a)** $a(t) = 1.6$, so $v(t) = 1.6t + v_0 = 1.6t$, since the initial velocity is 0.

(b) $s(t) = 0.8t^2 + s_0$, where s_0 is the rock's initial height.

23. **(a)** $s = v_0t - 16t^2$, where v_0 = initial velocity, and $v = s' = v_0 - 32t$. At the maximum height, $v = 0$, so $v_0 = 32t_{\text{max}}$. Plugging into the distance equation yields $100 = 32t_{\text{max}}^2 - 16t_{\text{max}}^2 = 16t_{\text{max}}^2$, so $t_{\text{max}} = \frac{5}{2}$ seconds, from which we get $v_0 = 32\left(\frac{5}{2}\right) = 80$ ft/sec.

(b) This time $g = 5$ ft/sec^2, so $s = v_0t - 2.5t^2 = 80t - 2.5t^2$, and $v = s' = 80 - 5t$. At the highest point, $v = 0$, so $t_{\text{max}} = \frac{80}{5} = 16$ seconds. Plugging into the distance equation yields $s = 80(16) - 2.5(16)^2 = 640$ ft.

24. The height of an object above the ground which begins at rest and falls for t seconds is

$$s(t) = -16t^2 + K,$$

where K is the initial height. Here the flower pot falls from 200 ft, so $K = 200$. To see when the pot hits the ground, solve $-16t^2 + 200 = 0$. The solution is

$$t = \sqrt{\frac{200}{16}} \approx 3.54 \text{ seconds}.$$

Now, velocity is given by $s'(t) = v(t) = -32t$. So, the velocity when the pot hits the ground is

$$v(3.54) \approx -113.1 \text{ ft/sec},$$

which is approximately 77 mph downwards.

25. The first thing we should do is convert our units. We'll bring everything into feet and seconds. Thus, the initial speed of the car is

$$\frac{70 \text{ miles}}{\text{hour}}\left(\frac{1 \text{ hour}}{3600 \text{ sec}}\right)\left(\frac{5280 \text{ feet}}{1 \text{ mile}}\right) \approx 102.7 \text{ ft/sec}.$$

We assume that the acceleration is constant as the car comes to a stop. A graph of its velocity versus time is given in Figure 6.9. We know that the area under the curve represents the distance that the car travels before it comes to a stop, 157 feet. But this area is a triangle, so it is easy to find t_0, the time the car comes to rest. We solve

$$\frac{1}{2}(102.7)t_0 = 157,$$

which gives

$$t_0 \approx 3.06 \text{ sec}.$$

Since acceleration is the rate of change of velocity, the car's acceleration is given by the slope of the line in Figure 6.9. Thus, the acceleration, k, is given by

$$k = \frac{102.7 - 0}{0 - 3.06} \approx -33.56 \text{ ft/sec}^2.$$

Notice that k is negative because the car is slowing down.

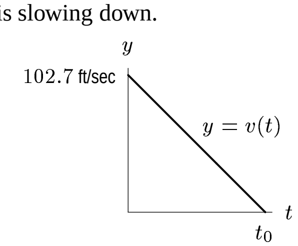
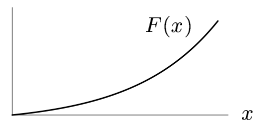


Figure 6.9: Graph of velocity versus time

Solutions for Section 6.4

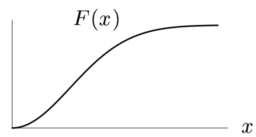
Exercises

1.



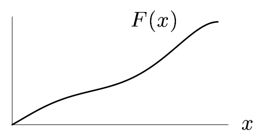
By the Fundamental Theorem, $f(x) = F'(x)$. Since f is positive and increasing, F is increasing and concave up. Since $F(0) = \int_0^0 f(t)dt = 0$, the graph of F must start from the origin.

2.



Since f is always positive, F is always increasing. F has an inflection point where $f' = 0$. Since $F(0) = \int_0^0 f(t)dt = 0$, F goes through the origin.

3.



Since f is always non-negative, F is increasing. F is concave up where f is increasing and concave down where f is decreasing; F has inflection points at the critical points of f. Since $F(0) = \int_0^0 f(t)dt = 0$, the graph of F goes through the origin.

4.

Table 6.3

x	0	0.5	1	1.5	2
$I(x)$	0	0.50	1.09	2.03	3.65

5. Using the Fundamental Theorem, we know that the change in F between $x = 0$ and $x = 0.5$ is given by

$$F(0.5) - F(0) = \int_0^{0.5} \sin t \cos t \, dt \approx 0.115.$$

Since $F(0) = 1.0$, we have $F(0.5) \approx 1.115$. The other values are found similarly, and are given in Table 6.4.

Table 6.4

b	0	0.5	1	1.5	2	2.5	3
$F(b)$	1	1.11492	1.35404	1.4975	1.41341	1.17908	1.00996

6. **(a)** Again using 0.00001 as the lower limit, because the integral is improper, gives $\text{Si}(4) = 1.76$, $\text{Si}(5) = 1.55$.
(b) $\text{Si}(x)$ decreases when the integrand is negative, which occurs when $\pi < x < 2\pi$.

7. If $f'(x) = \sin(x^2)$, then $f(x)$ is of the form

$$f(x) = C + \int_a^x \sin(t^2)\, dt.$$

Since $f(0) = 7$, we take $a = 0$ and $C = 7$, giving

$$f(x) = 7 + \int_0^x \sin(t^2)\, dt.$$

8. If $f'(x) = \dfrac{\sin x}{x}$, then $f(x)$ is of the form

$$f(x) = C + \int_a^x \frac{\sin t}{t}\,dt.$$

Since $f(1) = 5$, we take $a = 1$ and $C = 5$, giving

$$f(x) = 5 + \int_1^x \frac{\sin t}{t}\,dt.$$

9. If $f'(x) = \mathrm{Si}(x)$, then $f(x)$ is of the form

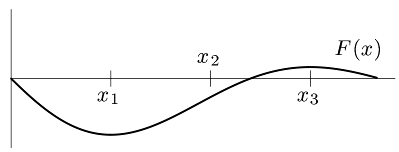
$$f(x) = C + \int_a^x \mathrm{Si}(t)\,dt.$$

Since $f(0) = 2$, we take $a = 0$ and $C = 2$, giving

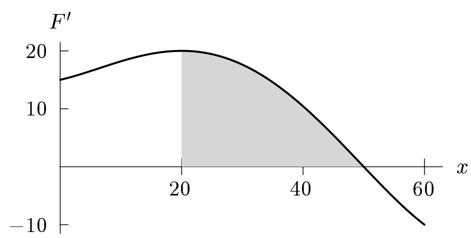
$$f(x) = 2 + \int_0^x \mathrm{Si}(t)\,dt.$$

Problems

10.



11.



We know that $F(x)$ increases for $x < 50$ because the derivative of F is positive there. See figure above. Similarly, $F(x)$ decreases for $x > 50$. Therefore, the graph of F rises until $x = 50$, and then it begins to fall. Thus, the maximum value attained by F is $F(50)$. To evaluate $F(50)$, we use the Fundamental Theorem:

$$F(50) - F(20) = \int_{20}^{50} F'(x)\,dx,$$

which gives

$$F(50) = F(20) + \int_{20}^{50} F'(x)\,dx = 150 + \int_{20}^{50} F'(x)\,dx.$$

The definite integral equals the area of the shaded region under the graph of F', which is roughly 350. Therefore, the greatest value attained by F is $F(50) \approx 150 + 350 = 500$.

12. Since $F'(x) = e^{-x^2}$ and $F(0) = 2$, we have

$$F(x) = F(0) + \int_0^x e^{-t^2}\,dt = 2 + \int_0^x e^{-t^2}\,dt.$$

Substituting $x = 1$ and evaluating the integral numerically gives

$$F(1) = 2 + \int_0^1 e^{-t^2}\,dt = 2.747.$$

13. Since $G'(x) = \cos(x^2)$ and $G(0) = -3$, we have

$$G(x) = G(0) + \int_0^x \cos(t^2)\,dt = -3 + \int_0^x \cos(t^2)\,dt.$$

Substituting $x = -1$ and evaluating the integral numerically gives

$$G(-1) = -3 + \int_0^{-1} \cos(t^2)\,dt = -3.905.$$

14. $\cos(x^2)$.

15. $(1+x)^{200}$.

16. $\arctan(x^2)$.

17. $\displaystyle \frac{d}{dt}\int_t^\pi \cos(z^3)\,dz = \frac{d}{dt}\left(-\int_\pi^t \cos(z^3)\,dz\right) = -\cos(t^3).$

18. $\displaystyle \frac{d}{dx}\int_x^1 \ln t\,dt = \frac{d}{dx}\left(-\int_1^x \ln t\,dt\right) = -\ln x.$

19. Considering $\mathrm{Si}(x^2)$ as the composition of $\mathrm{Si}(u)$ and $u(x) = x^2$, we may apply the chain rule to obtain

$$\begin{aligned}\frac{d}{dx} &= \frac{d(\mathrm{Si}(u))}{du}\cdot\frac{du}{dx}\\ &= \frac{\sin u}{u}\cdot 2x\\ &= \frac{2\sin(x^2)}{x}.\end{aligned}$$

20. **(a)** The definition of g gives $g(0) = \int_0^0 f(t)\,dt = 0$.

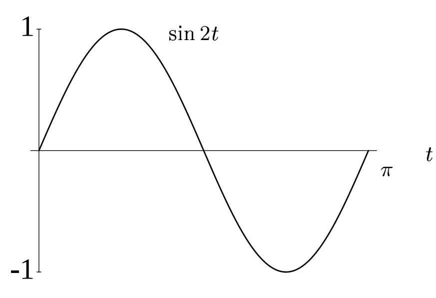
(b) The Fundamental Theorem gives $g'(1) = f(1) = -2$.

(c) The function g is concave upward where g'' is positive. Since $g'' = f'$, we see that g is concave up where f is increasing. This occurs on the interval $1 \le x \le 6$.

(d) The function g decreases from $x = 0$ to $x = 3$ and increases for $3 < x \le 8$, and the magnitude of the increase is more than the magnitude of the decrease. Thus g takes its maximum value at $x = 8$.

21. **(a)** Since $\displaystyle \frac{d}{dt}(\cos(2t)) = -2\sin(2t)$, we have $\displaystyle F(\pi) = \int_0^\pi \sin(2t)\,dt = -\frac{1}{2}\cos(2t)\Big|_0^\pi = -\frac{1}{2}(1-1) = 0.$

(b) $F(\pi) =$ (Area above t-axis) $-$ (Area below t-axis) $= 0$. (The two areas are equal.)



(c) $F(x) \ge 0$ everywhere. $F(x) = 0$ only at integer multiples of π. This can be seen for $x \ge 0$ by noting $F(x) =$ (Area above t-axis) $-$ (Area below t-axis), which is always non-negative and only equals zero when x is an integer multiple of π. For $x > 0$

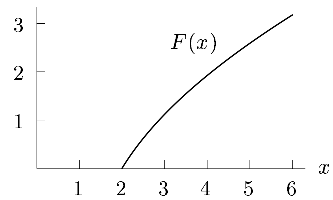
$$\begin{aligned}F(-x) &= \int_0^{-x} \sin 2t\,dt\\ &= -\int_{-x}^0 \sin 2t\,dt\\ &= \int_0^x \sin 2t\,dt = F(x),\end{aligned}$$

since the area from $-x$ to 0 is the negative of the area from 0 to x. So we have $F(x) \ge 0$ for all x.

22. **(a)** $F'(x) = \dfrac{1}{\ln x}$ by the Construction Theorem.

(b) For $x \geq 2$, $F'(x) > 0$, so $F(x)$ is increasing. Since $F''(x) = -\dfrac{1}{x(\ln x)^2} < 0$ for $x \geq 2$, the graph of $F(x)$ is concave down.

(c)



23.

$$\begin{aligned}\frac{d}{dx}[x\,\mathrm{erf}(x)] &= \mathrm{erf}(x)\frac{d}{dx}(x) + x\frac{d}{dx}[\mathrm{erf}(x)]\\ &= \mathrm{erf}(x) + x\frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right)\\ &= \mathrm{erf}(x) + \frac{2}{\sqrt{\pi}}xe^{-x^2}.\end{aligned}$$

24. If we let $f(x) = \mathrm{erf}(x)$ and $g(x) = \sqrt{x}$, then we are looking for $\frac{d}{dx}[f(g(x))]$. By the chain rule, this is the same as $g'(x)f'(g(x))$. Since

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right)\\ &= \frac{2}{\sqrt{\pi}}e^{-x^2}\end{aligned}$$

and $g'(x) = \dfrac{1}{2\sqrt{x}}$, we have

$$f'(g(x)) = \frac{2}{\sqrt{\pi}}e^{-x},$$

and so

$$\frac{d}{dx}[\mathrm{erf}(\sqrt{x})] = \frac{1}{2\sqrt{x}}\frac{2}{\sqrt{\pi}}e^{-x} = \frac{1}{\sqrt{\pi x}}e^{-x}.$$

25. If we let $f(x) = \int_0^x e^{-t^2}\,dt$ and $g(x) = x^3$, then we use the chain rule because we are looking for $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$. Since $f'(x) = e^{-x^2}$, we have

$$\frac{d}{dx}\left(\int_0^{x^3} e^{-t^2}\,dt\right) = f'(x^3)\cdot 3x^2 = e^{-(x^3)^2}\cdot 3x^2 = 3x^2e^{-x^6}.$$

26. We split the integral $\int_x^{x^3} e^{-t^2}\,dt$ into two pieces, say at $t = 1$ (though it could be at any other point):

$$\int_x^{x^3} e^{-t^2}\,dt = \int_1^{x^3} e^{-t^2}\,dt + \int_x^1 e^{-t^2}\,dt = \int_1^{x^3} e^{-t^2}\,dt - \int_1^x e^{-t^2}\,dt.$$

We have used the fact that $\int_x^1 e^{-t^2}\,dt = -\int_1^x e^{-t^2}\,dt$. Differentiating gives

$$\frac{d}{dx}\left(\int_x^{x^3} e^{-t^2}\,dt\right) = \frac{d}{dx}\left(\int_1^{x^3} e^{-t^2}\,dt\right) - \frac{d}{dx}\left(\int_1^x e^{-t^2}\,dt\right)$$

For the first integral, we use the chain rule with $g(x) = x^3$ as the inside function, so the final answer is

$$\frac{d}{dx}\left(\int_x^{x^3} e^{-t^2}\,dt\right) = e^{-(x^3)^2}\cdot 3x^2 - e^{-x^2} = 3x^2e^{-x^6} - e^{-x^2}.$$

Solutions for Section 6.5

Exercises

1. **(a)** The object is thrown from an initial height of $y = 1.5$ meters.
   (b) The velocity is obtained by differentiating, which gives $v = -9.8t + 7$ m/sec. The initial velocity is $v = 7$ m/sec upward.
   (c) The acceleration due to gravity is obtained by differentiating again, giving $g = -9.8$ m/sec^2, or 9.8 m/sec^2 downward.
2. Since height is measured upward, the initial position of the stone is $h(0) = 250$ meters and the initial velocity is $v = -20$ m/sec. The acceleration due to gravity is $g = -9.8$ m/sec^2. Thus, the height at time t is given by $h(t) = -4.9t^2 - 20t + 250$ meters.

Problems

3. The velocity as a function of time is given by: $v = v_0 + at$. Since the object starts from rest, $v_0 = 0$, and the velocity is just the acceleration times time: $v = -32t$. Integrating this, we get position as a function of time: $y = -16t^2 + y_0$, where the last term, y_0, is the initial position at the top of the tower, so $y_0 = 400$ feet. Thus we have a function giving position as a function of time: $y = -16t^2 + 400$.
   To find at what time the object hits the ground, we find t when $y = 0$. We solve $0 = -16t^2 + 400$ for t, getting $t^2 = 400/16 = 25$, so $t = 5$. Therefore the object hits the ground after 5 seconds. At this time it is moving with a velocity $v = -32(5) = -160$ feet/second.
4. In Problem 3 we used the equation $0 = -16t^2 + 400$ to learn that the object hits the ground after 5 seconds. In a more general form this is the equation $y = -\frac{g}{2}t^2 + v_0 t + y_0$, and we know that $v_0 = 0$, $y_0 = 400$ ft. So the moment the object hits the ground is given by $0 = -\frac{g}{2}t^2 + 400$. In Problem 3 we used $g = 32$ ft/sec^2, but in this case we want to find a g that results in the object hitting the ground after only 5/2 seconds. We put in 5/2 for t and solve for g:
$$0 = -\frac{g}{2}\left(\frac{5}{2}\right)^2 + 400, \text{ so } g = \frac{2(400)}{(5/2)^2} = 128 \text{ ft/sec}^2.$$
5. $a(t) = -32$. Since $v(t)$ is the antiderivative of $a(t)$, $v(t) = -32t + v_0$. But $v_0 = 0$, so $v(t) = -32t$. Since $s(t)$ is the antiderivative of $v(t)$, $s(t) = -16t^2 + s_0$, where s_0 is the height of the building. Since the ball hits the ground in 5 seconds, $s(5) = 0 = -400 + s_0$. Hence $s_0 = 400$ feet, so the window is 400 feet high.
6. Let time $t = 0$ be the moment when the astronaut jumps up. If acceleration due to gravity is 5 ft/sec^2 and initial velocity is 10 ft/sec, then the velocity of the astronaut is described by
$$v(t) = 10 - 5t.$$
   Suppose $y(t)$ describes his distance from the surface of the moon. By the Fundamental Theorem,
$$y(t) - y(0) = \int_0^t (10 - 5x)\,dx$$
$$y(t) = 10t - \frac{1}{2}5t^2.$$
   since $y(0) = 0$ (assuming the astronaut jumps off the surface of the moon).
   The astronaut reaches the maximum height when his velocity is 0, i.e. when
$$\frac{dy}{dt} = v(t) = 10 - 5t = 0.$$
   Solving for t, we get $t = 2$ sec as the time at which he reaches the maximum height from the surface of the moon. At this time his height is
$$y(2) = 10(2) - \frac{1}{2}5(2)^2 = 10 \text{ ft}.$$

When the astronaut is at height $y = 0$, he either just landed or is about to jump. To find how long it is before he comes back down, we find when he is at height $y = 0$. Set $y(t) = 0$ to get

$$\begin{aligned}0 &= 10t - \frac{1}{2}5t^2 \\ 0 &= 20t - 5t^2 \\ 0 &= 4t - t^2 \\ 0 &= t(t-4).\end{aligned}$$

So we have $t = 0$ sec (when he jumps off) and $t = 4$ sec (when he lands, which gives the time he spent in the air).

7. Let the acceleration due to gravity equal $-k$ meters/sec^2, for some positive constant k, and suppose the object falls from an initial height of $s(0)$ meters. We have $a(t) = dv/dt = -k$, so that

$$v(t) = -kt + v_0.$$

Since the initial velocity is zero, we have

$$v(0) = -k(0) + v_0 = 0,$$

which means $v_0 = 0$. Our formula becomes

$$v(t) = \frac{ds}{dt} = -kt.$$

This means

$$s(t) = \frac{-kt^2}{2} + s_0.$$

Since

$$s(0) = \frac{-k(0)^2}{2} + s_0,$$

we have $s_0 = s(0)$, and our formula becomes

$$s(t) = \frac{-kt^2}{2} + s(0).$$

Suppose that the object falls for t seconds. Assuming it hasn't hit the ground, its height is

$$s(t) = \frac{-kt^2}{2} + s(0),$$

so that the distance traveled is

$$s(0) - s(t) = \frac{kt^2}{2} \text{ meters},$$

which is proportional to t^2.

8. (a) $t = \dfrac{s}{\frac{1}{2}v_{\max}}$, where t is the time it takes for an object to travel the distance s, starting from rest with uniform acceleration a. $v_{\max}$ is the highest velocity the object reaches. Since its initial velocity is 0, the mean of its highest velocity and initial velocity is $\frac{1}{2}v_{\max}$.

(b) By Problem 7, $s = \frac{1}{2}gt^2$, where g is the acceleration due to gravity, so it takes $\sqrt{200/32} = 5/2$ seconds for the body to hit the ground. Since $v = gt$, $v_{\max} = 32(\frac{5}{2}) = 80$ ft/sec. Galileo's statement predicts $(100 \text{ ft})/(40 \text{ ft/sec}) = 5/2$ seconds, and so Galileo's result is verified.

(c) If the acceleration is a constant a, then $s = \frac{1}{2}at^2$, and $v_{\max} = at$. Thus

$$\frac{s}{\frac{1}{2}v_{\max}} = \frac{\frac{1}{2}at^2}{\frac{1}{2}at} = t.$$

9. (a) Since $s(t) = -\frac{1}{2}gt^2$, the distance a body falls in the first second is

$$s(1) = -\frac{1}{2} \cdot g \cdot 1^2 = -\frac{g}{2}.$$

In the second second, the body travels

$$s(2) - s(1) = -\frac{1}{2}\left(g \cdot 2^2 - g \cdot 1^2\right) = -\frac{1}{2}(4g - g) = -\frac{3g}{2}.$$

In the third second, the body travels

$$s(3) - s(2) = -\frac{1}{2}\left(g \cdot 3^2 - g \cdot 2^2\right) = -\frac{1}{2}(9g - 4g) = -\frac{5g}{2},$$

and in the fourth second, the body travels

$$s(4) - s(3) = -\frac{1}{2}\left(g \cdot 4^2 - g \cdot 3^2\right) = -\frac{1}{2}(16g - 9g) = -\frac{7g}{2}.$$

(b) Galileo seems to have been correct. His observation follows from the fact that the differences between consecutive squares are consecutive odd numbers. For, if n is any number, then $n^2 - (n-1)^2 = 2n - 1$, which is the n^{th} odd number (where 1 is the first).

10. If r is the distance from the center of the earth,

$$g = \frac{GM}{r^2},$$

so at 2 meters

$$9.8 = \frac{GM}{(6.4 \times 10^6 + 2)^2}.$$

At 100 meters above the ground,

$$g_{\text{new}} = \frac{GM}{(6.4 \times 10^6 + 100)^2}$$

so

$$\frac{g_{\text{new}}}{9.8} = \frac{GM}{(6.4 \times 10^6 + 100)^2} \Big/ \frac{GM}{(6.4 \times 10^6 + 2)^2}$$

$$g_{\text{new}} = 9.8\left(\frac{6,400,002}{6,400,100}\right)^2 = 9.79969\ldots\ \text{m/sec}^2.$$

Thus, to the first decimal place, the acceleration due to gravity is still 9.8 m/sec^2 at 100 m above the ground.

At 100,000 meters above the ground,

$$g_{\text{new}} = 9.8\left(\frac{6,400,002}{6,500,000}\right)^2 = 9.5\text{m/sec}^2.$$

Solutions for Chapter 6 Review

Exercises

1. $\frac{5}{2}x^2 + 7x + C$

2. $\int (4t + \frac{1}{t})\,dt = 2t^2 + \ln|t| + C$

3. $\int (2 + \cos t)\,dt = 2t + \sin t + C$

4. $\int 7e^x\,dx = 7e^x + C$

5. $\int (3e^x + 2\sin x)\,dx = 3e^x - 2\cos x + C$

6. $\int (x+3)^2\,dx = \int (x^2 + 6x + 9)\,dx = \frac{x^3}{3} + 3x^2 + 9x + C$

7. $\int \frac{8}{\sqrt{x}}\,dx = 16x^{1/2} + C$

8. $3\ln|t| + \frac{2}{t} + C$

9. $e^x + 5x + C$

10. $\frac{2}{5}x^{5/2} - 2\ln|x| + C$

11. $\tan x + C$

12. $\dfrac{1}{\ln 2}2^x + C$, since $\dfrac{d}{dx}(2^x) = (\ln 2) \cdot 2^x$

13. $\displaystyle\int (x+1)^2\,dx = \frac{(x+1)^3}{3} + C.$

Another way to work the problem is to expand $(x+1)^2$ to x^2+2x+1 as follows:

$$\int (x+1)^2\,dx = \int (x^2+2x+1)\,dx = \frac{x^3}{3} + x^2 + x + C.$$

These two answers are the same, since $\dfrac{(x+1)^3}{3} = \dfrac{x^3+3x^2+3x+1}{3} = \dfrac{x^3}{3} + x^2 + x + \dfrac{1}{3}$, which is $\dfrac{x^3}{3} + x^2 + x$, plus a constant.

14. $\displaystyle\int (x+1)^3\,dx = \frac{(x+1)^4}{4} + C.$

Another way to work the problem is to expand $(x+1)^3$ to x^3+3x^2+3x+1:

$$\int (x+1)^3\,dx = \int (x^3+3x^2+3x+1)\,dx = \frac{x^4}{4} + x^3 + \frac{3}{2}x^2 + x + C.$$

It can be shown that these answers are the same by expanding $\dfrac{(x+1)^4}{4}$.

15. $\frac{1}{10}(x+1)^{10} + C$

16. Since $f(x) = \dfrac{x+1}{x} = 1 + \dfrac{1}{x}$, the indefinite integral is $x + \ln|x| + C$

17. Since $f(x) = x + 1 + \dfrac{1}{x}$, the indefinite integral is $\dfrac{1}{2}x^2 + x + \ln|x| + C$

18. $3\sin t + 2t^{3/2} + C$

19. $3\sin x + 7\cos x + C$

20. $2\ln|x| - \pi\cos x + C$

21. $2e^x - 8\sin x + C$

22. $P(t) = \displaystyle\int \frac{1}{t}\,dt = \ln|t| + C$

23. $F(x) = \displaystyle\int \cos x\,dx = \sin x + C$

24. $F(x) = \displaystyle\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$

25. $G(x) = \displaystyle\int \sin x\,dx = -\cos x + C$

26. $F(x) = \displaystyle\int 5e^x\,dx = 5e^x + C$

27. $H(t) = \displaystyle\int \frac{5}{t}\,dt = 5\ln|t| + C$

28. $F(t) = \displaystyle\int \left(t + \frac{1}{t}\right)\,dt = \frac{t^2}{2} + \ln|t| + C$

29. $F(x) = \displaystyle\int (e^x - 1)\,dx = e^x - x + C$

30. $F(x) = \displaystyle\int f(x)\,dx = \int x^2\,dx = \frac{x^3}{3} + C$. If $F(0) = 4$, then $F(0) = 0 + C = 4$ and thus $C = 4$. So $F(x) = \dfrac{x^3}{3} + 4$.

31. We have $F(x) = \dfrac{x^4}{4} + 2x^3 - 4x + C$. Since $F(0) = 4$, we have $4 = 0 + C$, so $C = 4$. So $F(x) = \dfrac{x^4}{4} + 2x^3 - 4x + 4$.

32. $F(x) = \displaystyle\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$. If $F(0) = 4$, then $F(0) = 0 + C = 4$ and thus $C = 4$. So $F(x) = \dfrac{2}{3}x^{3/2} + 4$.

33. $F(x) = \displaystyle\int e^x\,dx = e^x + C$. If $F(0) = 4$, then $F(0) = 1 + C = 4$ and thus $C = 3$. So $F(x) = e^x + 3$.

34. $F(x) = \int \sin x\, dx = -\cos x + C$. If $F(0) = 4$, then $F(0) = -1 + C = 4$ and thus $C = 5$. So $F(x) = -\cos x + 5$.

35. $F(x) = \int \cos x\, dx = \sin x + C$. If $F(0) = 4$, then $F(0) = 0 + C = 4$ and thus $C = 4$. So $F(x) = \sin x + 4$.

36. We have

$$\int_1^3 (6x^2 + 8x + 5)dx = (2x^3 + 4x^2 + 5x)\Big|_1^3 = (54 + 36 + 15) - (2 + 4 + 5) = 94.$$

Problems

37. $\displaystyle\int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 = 9 - 0 = 9.$

38. Since $y = x^3 - x = x(x-1)(x+1)$, the graph crosses the axis at the three points shown in Figure 6.10. The two regions have the same area (by symmetry). Since the graph is below the axis for $0 < x < 1$, we have

$$\begin{aligned} \text{Area } &= 2\left(-\int_0^1 \left(x^3 - x\right)\, dx\right) \\ &= -2\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 = -2\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{2}. \end{aligned}$$

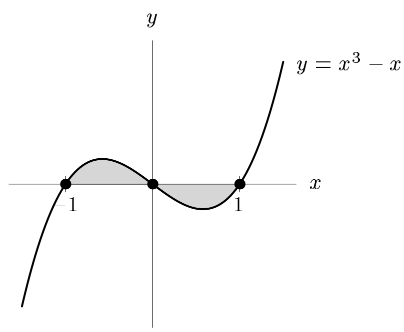


Figure 6.10

39. The area we want (the shaded area in Figure 6.11) is symmetric about the y-axis and so is given by

$$\begin{aligned} \text{Area } &= 2\int_0^{\pi/3} \left(\cos x - \frac{1}{2}\left(\frac{3}{\pi}x\right)^2\right)\, dx \\ &= 2\int_0^{\pi/3} \cos x\, dx - \int_0^{\pi/3} \frac{9}{\pi^2}x^2\, dx \\ &= 2\sin x\Big|_0^{\pi/3} - \frac{9}{\pi^2}\cdot\frac{x^3}{3}\Big|_0^{\pi/3} \\ &= 2\cdot\frac{\sqrt{3}}{2} - \frac{3}{\pi^2}\cdot\frac{\pi^3}{3^3} = \sqrt{3} - \frac{\pi}{9}. \end{aligned}$$

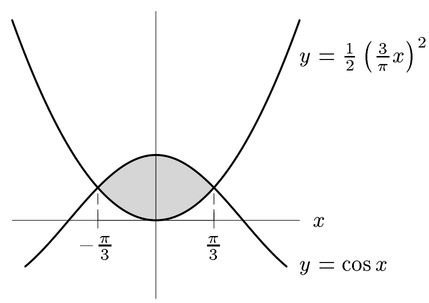


Figure 6.11

40. Since $y < 0$ from $x = 0$ to $x = 1$ and $y > 0$ from $x = 1$ to $x = 3$, we have

$$\begin{aligned}\text{Area} &= -\int_0^1 \left(3x^2 - 3\right)\, dx + \int_1^3 \left(3x^2 - 3\right)\, dx\\ &= -\left(x^3 - 3x\right)\Big|_0^1 + \left(x^3 - 3x\right)\Big|_1^3\\ &= -(-2-0) + (18 - (-2)) = 2 + 20 = 22.\end{aligned}$$

41. **(a)** See Figure 6.12. Since $f(x) > 0$ for $0 < x < 2$ and $f(x) < 0$ for $2 < x < 5$, we have

$$\begin{aligned}\text{Area} &= \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx\\ &= \int_0^2 (x^3 - 7x^2 + 10x)\,dx - \int_2^5 (x^3 - 7x^2 + 10x)\,dx\\ &= \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Big|_0^2 - \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Big|_2^5\\ &= \left[\left(4 - \frac{56}{3} + 20\right) - (0 - 0 + 0)\right] - \left[\left(\frac{625}{4} - \frac{875}{3} + 125\right) - \left(4 - \frac{56}{3} + 20\right)\right]\\ &= \frac{253}{12}.\end{aligned}$$

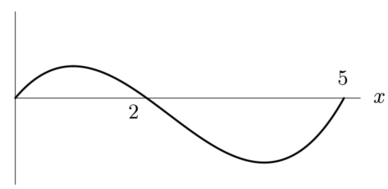


Figure 6.12: Graph of $f(x) = x^3 - 7x^2 + 10x$

(b) Calculating $\int_0^5 f(x)\,dx$ gives

$$\begin{aligned}\int_0^5 f(x)\,dx &= \int_0^5 (x^3 - 7x^2 + 10x)\,dx\\ &= \left(\frac{x^4}{4} - \frac{7x^3}{3} + 5x^2\right)\Big|_0^5\\ &= \left(\frac{625}{4} - \frac{875}{3} + 125\right) - (0 - 0 + 0)\\ &= -\frac{125}{12}.\end{aligned}$$

This integral measures the difference between the area above the x-axis and the area below the x-axis. Since the definite integral is negative, the graph of $f(x)$ lies more below the x-axis than above it. Since the function crosses the axis at $x = 2$,

$$\int_0^5 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^5 f(x)\,dx = \frac{16}{3} - \frac{63}{4} = \frac{-125}{12},$$

whereas

$$\text{Area} = \int_0^2 f(x)\,dx - \int_2^5 f(x)\,dx = \frac{16}{3} + \frac{64}{4} = \frac{253}{12}.$$

42. Since the area under the curve is 6, we have

$$\int_1^b \frac{1}{\sqrt{x}}\,dx = 2x^{1/2}\Big|_1^b = 2b^{1/2} - 2(1) = 6.$$

Thus $b^{1/2} = 4$ and $b = 16$.

43. The graph of $y = c(1-x^2)$ has x-intercepts of $x = \pm 1$. See Figure 6.13. Since it is symmetric about the y-axis, we have

$$\begin{aligned}\text{Area } &= \int_{-1}^{1} c(1-x^2)\,dx = 2c\int_0^1 (1-x^2)\,dx \\ &= 2c\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{4c}{3}.\end{aligned}$$

We want the area to be 1, so

$$\frac{4c}{3} = 1, \quad \text{giving} \quad c = \frac{3}{4}.$$

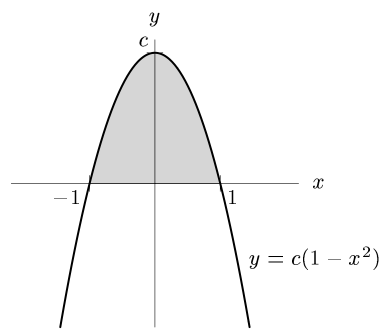
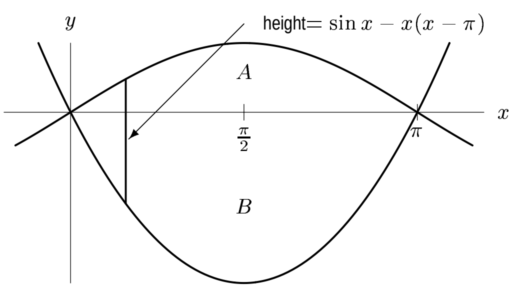


Figure 6.13

44. The curves intersect at $(0,0)$ and $(\pi, 0)$. At any x-coordinate the "height" between the two curves is $\sin x - x(x-\pi)$.



Thus the total area is

$$\begin{aligned}\int_0^{\pi} [\sin x - x(x-\pi)]\,dx = &= \int_0^{\pi} (\sin x - x^2 + \pi x)\,dx \\ &= \left(-\cos x - \frac{x^3}{3} + \frac{\pi x^2}{2}\right)\Big|_0^{\pi} \\ &= \left(1 - \frac{\pi^3}{3} + \frac{\pi^3}{2}\right) - (-1) \\ &= 2 + \frac{\pi^3}{6}.\end{aligned}$$

Another approach is to notice that the area between the two curves is (area A) + (area B).

$$\begin{aligned}\text{Area B } &= -\int_0^{\pi} x(x-\pi)\,dx \text{ since the function is negative on } 0 \le x \le \pi \\ &= -\left(\frac{x^3}{3} - \frac{\pi x^2}{2}\right)\Big|_0^{\pi} = \frac{\pi^3}{2} - \frac{\pi^3}{3} = \frac{\pi^3}{6}; \\ \text{Area A } &= \int_0^{\pi} \sin x\,dx = -\cos x\Big|_0^{\pi} = 2.\end{aligned}$$

Thus the area is $2 + \dfrac{\pi^3}{6}$.

45. See Figure 6.14. The average value of $f(x)$ is given by

$$\text{Average } = \frac{1}{9-0}\int_0^9 \sqrt{x}\,dx = \frac{1}{9}\left(\frac{2}{3}x^{3/2}\Big|_0^9\right) = \frac{1}{9}\left(\frac{2}{3}9^{3/2} - 0\right) = \frac{1}{9}18 = 2.$$

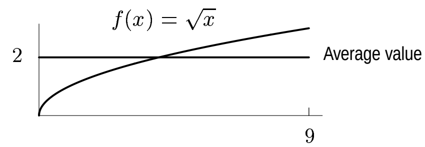


Figure 6.14

46. The total amount of discharge is the integral of the discharge rate from $t = 0$ to $t = 3$:

$$\begin{aligned}\text{Total discharge} &= \int_0^3 (t^2 - 14t + 49)\,dt \\ &= \left(\frac{t^3}{3} - 7t^2 + 49t\right)\Big|_0^3 \\ &= (9 - 63 + 147) - 0 \\ &= 93 \text{ cubic meters.}\end{aligned}$$

47. **(a)** Since $f'(t)$ is positive on the interval $0 < t < 2$ and negative on the interval $2 < t < 5$, the function $f(t)$ is increasing on $0 < t < 2$ and decreasing on $2 < t < 5$. Thus $f(t)$ attains its maximum at $t = 2$. Since the area under the t-axis is greater than the area above the t-axis, the function $f(t)$ decreases more than it increases. Thus, the minimum is at $t = 5$.

(b) To estimate the value of f at $t = 2$, we see that the area under $f'(t)$ between $t = 0$ and $t = 2$ is about 1 box, which has area 5. Thus,

$$f(2) = f(0) + \int_0^2 f'(t)dt \approx 50 + 5 = 55.$$

The maximum value attained by the function is $f(2) \approx 55$.

The area between $f'(t)$ and the t-axis between $t = 2$ and $t = 5$ is about 3 boxes, each of which has an area of 5. Thus

$$f(5) = f(2) + \int_2^5 f'(t)dt \approx 55 + (-15) = 40.$$

The minimum value attained by the function is $f(5) = 40$.

(c) Using part (b), we have $f(5) - f(0) = 40 - 50 = -10$. Alternately, we can use the Fundamental Theorem:

$$f(5) - f(0) = \int_0^5 f'(t)dt \approx 5 - 15 = -10.$$

48. **(a)** Starting at $x = 3$, we are given that $f(3) = 0$. Moving to the left on the interval $2 < x < 3$, we have $f'(x) = -1$, so $f(2) = f(3) - (1)(-1) = 1$. On the interval $0 < x < 2$, we have $f'(x) = 1$, so

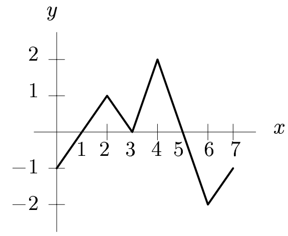
$$f(0) = f(2) + 1(-2) = -1.$$

Moving to the right from $x = 3$, we know that $f'(x) = 2$ on $3 < x < 4$. So $f(4) = f(3) + 2 = 2$. On the interval $4 < x < 6$, $f'(x) = -2$ so

$$f(6) = f(4) + 2(-2) = -2.$$

On the interval $6 < x < 7$, we have $f'(x) = 1$, so

$$f(7) = f(6) + 1 = -2 + 1 = -1.$$



(b) In part (a) we found that $f(0) = -1$ and $f(7) = -1$.

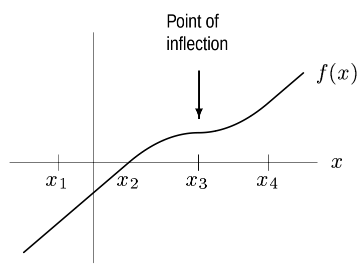
(c) The integral $\int_0^7 f'(x)\,dx$ is given by the sum

$$\int_0^7 f'(x)\,dx = (1)(2) + (-1)(1) + (2)(1) + (-2)(2) + (1)(1) = 0.$$

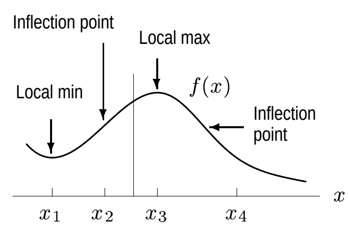
Alternatively, knowing $f(7)$ and $f(0)$ and using the Fundamental Theorem of Calculus, we have

$$\int_0^7 f'(x)\,dx = f(7) - f(0) = -1 - (-1) = 0.$$

49.



50.



51. $F(x)$ represents the net area between $(\sin t)/t$ and the t-axis from $t = \frac{\pi}{2}$ to $t = x$, with area counted as negative for $(\sin t)/t$ below the t-axis. As long as the integrand is positive $F(x)$ is increasing. Therefore, the global maximum of $F(x)$ occurs at $x = \pi$ and is given by the area

$$A_1 = \int_{\pi/2}^{\pi} \frac{\sin t}{t}\,dt.$$

At $x = \pi/2$, $F(x) = 0$. Figure 6.15 shows that the area A_1 is larger than the area A_2. Thus $F(x) > 0$ for $\frac{\pi}{2} < x \le \frac{3\pi}{2}$. Therefore the global minimum is $F(\frac{\pi}{2}) = 0$.

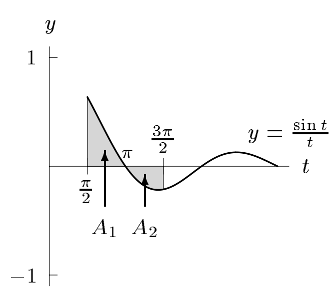


Figure 6.15

52. Since B is the graph of a decreasing function, the graph of its derivative should fall below the x-axis. Thus, f' could be C and f could be B. Since the graph of B is above the x-axis and represents a decreasing function, the function $\int_0^x f(t)\,dt$ should be increasing and concave down. Thus, A could be the graph of $\int_0^x f(t)\,dt$.

53. A function whose derivative is e^{x^2} is of the form

$$f(x) = C + \int_a^x e^{t^2}\,dt \qquad \text{for some value of } C.$$

(a) To ensure that the function goes through the point $(0, 3)$, we take $a = 0$ and $C = 3$:

$$f(x) = 3 + \int_0^x e^{t^2}\,dt.$$

(b) To ensure that the function goes through $(-1, 5)$, we take $a = -1$ and $C = 5$:

$$f(x) = 5 + \int_{-1}^x e^{t^2}\,dt.$$

54. We know the height is given by

$$s = -25t^2 + 72t + 40,$$

so the velocity is given by

$$v = -50t + 72$$

and the acceleration is given by

$$a = -50.$$

The acceleration due to gravity is -50 ft/sec^2 downward. Since $v(0) = 72$, the object was thrown at 72 ft/sec. Since $s(0) = 40$, the object was thrown from a height of 40 ft.

55. The graph of $h(t)$ must slope downwards most steeply when $h'(t)$ has its minimum. The graph of $h(t)$ should have its minimum about two-thirds of the way through the time interval (when the graph of $h'(t)$ intersects the x-axis), and have its final value about half-way between its maximum and minimum values. A possible graph of $h(t)$ is given in Figure 6.16. The placement of the horizontal axis below the graph is arbitrary.

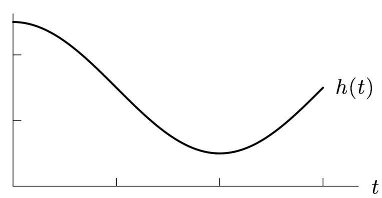


Figure 6.16

56. Let v be the velocity and s be the position of the particle at time t. We know that $a = dv/dt$, so acceleration is the slope of the velocity graph. Similarly, velocity is the slope of the position graph. Graphs of v and s are shown in Figures 6.17 and 6.18, respectively.

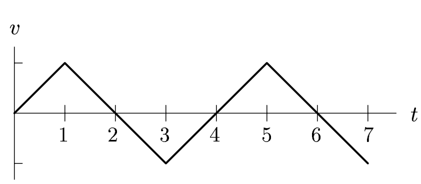


Figure 6.17: Velocity against time

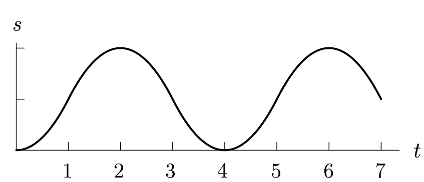


Figure 6.18: Position against time

57. (a) Since 6 sec = $1/10$ min,

$$\text{Angular acceleration } = \frac{2500-1100}{1/10} = 14{,}000 \text{ revs/min}^2.$$

(b) We know angular acceleration is the derivative of angular velocity. Since

$$\text{Angular acceleration } = 14{,}000,$$

we have

$$\text{Angular velocity } = 14{,}000t + C.$$

Measuring time from the moment at which the angular velocity is 1100 revs/min, we have $C = 1100$. Thus,

$$\text{Angular velocity } = 14{,}000t + 1100.$$

Thus the total number of revolutions performed during the period from $t = 0$ to $t = 1/10$ min is given by

$$\begin{array}{c}\text{Number of}\\\text{revolutions}\end{array} = \int_0^{1/10} (14000t + 1100)dt = 7000t^2 + 1100t\Big|_0^{1/10} = 180 \text{ revolutions}.$$

58. (a) Since the rotor is slowing down at a constant rate,

$$\text{Angular acceleration } = \frac{260-350}{1.5} = -60 \text{ revs/min}^2.$$

Units are revolutions per minute per minute, or revs/min^2.

(b) To decrease from 350 to 0 revs/min at a deceleration of 60 revs/min^2,

$$\text{Time needed } = \frac{350}{60} \approx 5.83 \text{ min}.$$

(c) We know angular acceleration is the derivative of angular velocity. Since

$$\text{Angular acceleration } = -60 \text{ revs/min}^2,$$

we have

$$\text{Angular velocity } = -60t + C.$$

Measuring time from the moment when angular velocity is 350 revs/min, we get $C = 350$. Thus

$$\text{Angular velocity } = -60t + 350.$$

So, the total number of revolutions made between the time the angular speed is 350 revs/min and stopping is given by:

$$\begin{aligned}\text{Number of revolutions} &= \int_0^{5.83} (\text{Angular velocity})\, dt\\ &= \int_0^{5.83} (-60t + 350)dt = -30t^2 + 350t\Big|_0^{5.83}\\ &= 1020.83 \text{ revolutions}.\end{aligned}$$

59. (a) Using $g = -32$ ft/sec^2, we have

t (sec)	0	1	2	3	4	5
$v(t)$ (ft/sec)	80	48	16	-16	-48	-80

(b) The object reaches its highest point when $v = 0$, which appears to be at $t = 2.5$ seconds. By symmetry, the object should hit the ground again at $t = 5$ seconds.

(c) Left sum $= 80(1) + 48(1) + 16(\frac{1}{2}) = 136$ ft , which is an overestimate.
Right sum $= 48(1) + 16(1) + (-16)(\frac{1}{2}) = 56$ ft , which is an underestimate.
Note that we used a smaller third rectangle of width $1/2$ to end our sum at $t = 2.5$.

(d) We have $v(t) = 80 - 32t$, so antidifferentiation yields $s(t) = 80t - 16t^2 + s_0$.
But $s_0 = 0$, so $s(t) = 80t - 16t^2$.
At $t = 2.5$, $s(t) = 100$ ft., so 100 ft. is the highest point.

60. The velocity of the car decreases at a constant rate, so we can write: $dv/dt = -a$. Integrating this gives $v = -at + C$. The constant of integration C is the velocity when $t = 0$, so $C = 60$ mph $= 88$ ft/sec, and $v = -at + 88$. From this equation we can see the car comes to rest at time $t = 88/a$.

Integrating the expression for velocity we get $s = -\frac{a}{2}t^2 + 88t + C$, where C is the initial position, so $C = 0$. We can use fact that the car comes to rest at time $t = 88/a$ after traveling 200 feet. Start with

$$s = -\frac{a}{2}t^2 + 88t,$$

and substitute $t = 88/a$ and $s = 200$:

$$\begin{aligned} 200 &= -\frac{a}{2}\left(\frac{88}{a}\right)^2 + 88\left(\frac{88}{a}\right) = \frac{88^2}{2a} \\ a &= \frac{88^2}{2(200)} = 19.36 \text{ ft/sec}^2 \end{aligned}$$

61. (a) In the beginning, both birth and death rates are small; this is consistent with a very small population. Both rates begin climbing, the birth rate faster than the death rate, which is consistent with a growing population. The birth rate is then high, but it begins to decrease as the population increases.

(b)

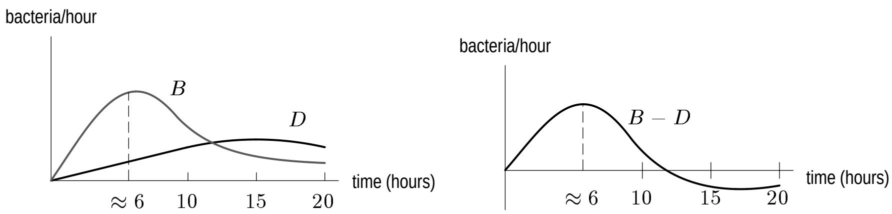


Figure 6.19: Difference between B and D is greatest at $t \approx 6$

The bacteria population is growing most quickly when $B - D$, the rate of change of population, is maximal; that happens when B is farthest above D, which is at a point where the slopes of both graphs are equal. That point is $t \approx 6$ hours.

(c) Total number born by time t is the area under the B graph from $t = 0$ up to time t. See Figure 6.20.

Total number alive at time t is the number born minus the number that have died, which is the area under the B graph minus the area under the D graph, up to time t. See Figure 6.21.

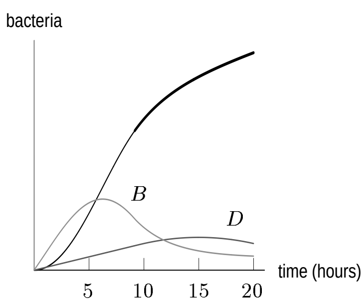


Figure 6.20: Number born by time t is $\int_0^t B(x)\,dx$

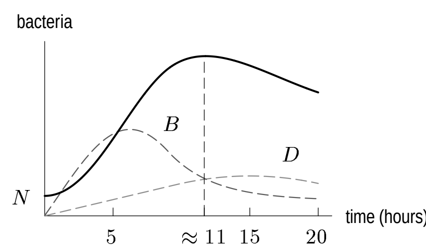
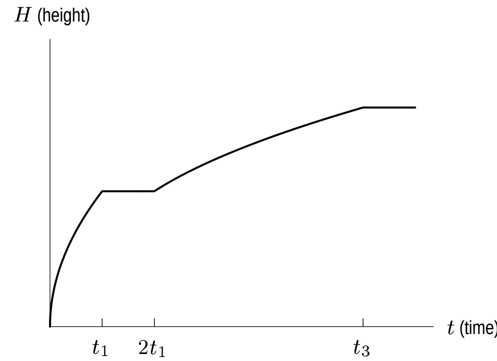


Figure 6.21: Number alive at time t is $\int_0^t (B(x) - D(x))\,dx$

From Figure 6.21, we see that the population is at a maximum when $B = D$, that is, after about 11 hours. This stands to reason, because $B - D$ is the rate of change of population, so population is maximized when $B - D = 0$, that is, when $B = D$.

62.



Suppose t_1 is the time to fill the left side to the top of the middle ridge. Since the container gets wider as you go up, the rate dH/dt decreases with time. Therefore, for $0 \le t \le t_1$, graph is concave down.

At $t = t_1$, water starts to spill over to right side and so depth of left side doesn't change. It takes as long for the right side to fill to the ridge as the left side, namely t_1. Thus the graph is horizontal for $t_1 \le t \le 2t_1$.

For $t \ge 2t_1$, water level is above the central ridge. The graph is climbing because the depth is increasing, but at a slower rate than for $t \le t_1$ because the container is wider. The graph is concave down because width is increasing with depth. Time t_3 represents the time when container is full.

63.
- For $[0, t_1]$, the acceleration is constant and positive and the velocity is positive so the displacement is positive. Thus, the work done is positive.
- For $[t_1, t_2]$, the acceleration, and therefore the force, is zero. Therefore, the work done is zero.
- For $[t_2, t_3]$, the acceleration is negative and thus the force is negative. The velocity, and thus the displacement, is positive; therefore the work done is negative.
- For $[t_3, t_4]$, the acceleration (and thus the force) and the velocity (and thus the displacement) are negative. Thus, the work done is positive.
- For $[t_2, t_4]$, the acceleration and thus the force is constant and negative. Velocity both positive and negative; total displacement is 0. Since force is constant, work is 0.

CAS Challenge Problems

64. **(a)** We have $\Delta x = \dfrac{(b-a)}{n}$ and $x_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$, so, since $f(x_i) = {x_i}^3$,

$$\text{Riemann sum } = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[a + i\left(\frac{b-a}{n}\right)\right]^3\left(\frac{b-a}{n}\right).$$

(b) A CAS gives

$$\sum_{i=1}^{n}\left[a + \frac{i(b-a)}{n}\right]^3\frac{(b-a)}{n} = -\frac{(a-b)(a^3(n-1)^2 + (a^2b+ab^2)(n^2-1) + b^3(n+1)^3)}{4n^2}.$$

Taking the limit as $n \to \infty$ gives

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[a + i\left(\frac{b-a}{n}\right)\right]^3\left(\frac{b-a}{n}\right) = -\frac{(a+b)(a-b)(a^2+b^2)}{4}.$$

(c) The answer to part (b) simplifies to $\dfrac{b^4}{4} - \dfrac{a^4}{4}$. Since $\dfrac{d}{dx}\left(\dfrac{x^4}{4}\right) = x^3$, the Fundamental Theorem of Calculus says that

$$\int_a^b x^3 dx = \left.\frac{x^4}{4}\right|_a^b = \frac{b^4}{4} - \frac{a^4}{4}.$$

65. (a) A CAS gives

$$\int e^{2x}\,dx = \frac{1}{2}e^{2x} \qquad \int e^{3x}\,dx = \frac{1}{3}e^{3x} \qquad \int e^{3x+5}\,dx = \frac{1}{3}e^{3x+5}.$$

(b) The three integrals in part (a) obey the rule

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}.$$

(c) Checking the formula by calculating the derivative

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) &= \frac{1}{a}\frac{d}{dx}e^{ax+b} && \text{by the constant multiple rule}\\ &= \frac{1}{a}e^{ax+b}\frac{d}{dx}(ax+b) && \text{by the chain rule}\\ &= \frac{1}{a}e^{ax+b}\cdot a = e^{ax+b}.\end{aligned}$$

66. (a) A CAS gives

$$\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) \qquad \int \sin(4x)\,dx = -\frac{1}{4}\cos(4x) \qquad \int \sin(3x-2)\,dx = -\frac{1}{3}\cos(3x-2).$$

(b) The three integrals in part (a) obey the rule

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b).$$

(c) Checking the formula by calculating the derivative

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{a}\cos(ax+b)\right) &= -\frac{1}{a}\frac{d}{dx}\cos(ax+b) && \text{by the constant multiple rule}\\ &= -\frac{1}{a}(-\sin(ax+b))\frac{d}{dx}(ax+b) && \text{by the chain rule}\\ &= -\frac{1}{a}(-\sin(ax+b))\cdot a = \sin(ax+b).\end{aligned}$$

67. (a) A CAS gives

$$\begin{aligned}\int \frac{x-2}{x-1}\,dx &= x - \ln|x-1|\\ \int \frac{x-3}{x-1}\,dx &= x - 2\ln|x-1|\\ \int \frac{x-1}{x-2}\,dx &= x + \ln|x-2|\end{aligned}$$

Although the absolute values are needed in the answer, some CASs may not include them.

(b) The three integrals in part (a) obey the rule

$$\int \frac{x-a}{x-b}\,dx = x + (b-a)\ln|x-b|.$$

(c) Checking the formula by calculating the derivative

$$\begin{aligned}\frac{d}{dx}\left(x + (b-a)\ln|x-b|\right) &= 1 + (b-a)\frac{1}{x-b} && \text{by the sum and constant multiple rules}\\ &= \frac{(x-b)+(b-a)}{x-b} = \frac{x-a}{x-b}\end{aligned}$$

68. **(a)** A CAS gives

$$\int \frac{1}{(x-1)(x-3)}\,dx = \frac{1}{2}(\ln|x-3| - \ln|x-1|)$$
$$\int \frac{1}{(x-1)(x-4)}\,dx = \frac{1}{3}(\ln|x-4| - \ln|x-1|)$$
$$\int \frac{1}{(x-1)(x+3)}\,dx = \frac{1}{4}(\ln|x+3| - \ln|x-1|).$$

Although the absolute values are needed in the answer, some CASs may not include them.

(b) The three integrals in part (a) obey the rule

$$\int \frac{1}{(x-a)(x-b)}\,dx = \frac{1}{b-a}(\ln|x-b| - \ln|x-a|).$$

(c) Checking the formula by calculating the derivative

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{b-a}(\ln|x-b| - \ln|x-a|)\right) &= \frac{1}{b-a}\left(\frac{1}{x-b} - \frac{1}{x-a}\right)\\ &= \frac{1}{b-a}\left(\frac{(x-a)-(x-b)}{(x-a)(x-b)}\right)\\ &= \frac{1}{b-a}\left(\frac{b-a}{(x-a)(x-b)}\right) = \frac{1}{(x-a)(x-b)}.\end{aligned}$$

CHECK YOUR UNDERSTANDING

1. True. A function can have only one derivative.

2. True. Check by differentiating $\frac{d}{dx}(2(x+1)^{3/2}) = 2\cdot\frac{3}{2}(x+1)^{1/2} = 3\sqrt{x+1}$.

3. True. Any antiderivative of $3x^2$ is obtained by adding a constant to x^3.

4. True. Any antiderivative of $1/x$ is obtained by adding a constant to $\ln|x|$.

5. False. Differentiating using the product and chain rules gives

$$\frac{d}{dx}\left(\frac{-1}{2x}e^{-x^2}\right) = \frac{1}{2x^2}e^{-x^2} + e^{-x^2}.$$

6. False. It is not true in general that $\int xf(x)dx = x\int f(x)dx$, so this statement is false for many functions $f(x)$. For example, if $f(x)=1$, then $\int xf(x)\,dx = x^2/2 + C$, but $x\int f(x)\,dx = x(x+C)$.

7. True. Adding a constant to an antiderivative gives another antiderivative.

8. True. If $F(x)$ is an antiderivative of $f(x)$, then $F'(x) = f(x)$, so $dy/dx = f(x)$. Therefore, $y = F(x)$ is a solution to this differential equation.

9. True. If $y = F(x)$ is a solution to the differential equation $dy/dx = f(x)$, then $F'(x) = f(x)$, so $F(x)$ is an antiderivative of $f(x)$.

10. True. If acceleration is $a(t) = k$ for some constant k, $k \neq 0$, then we have

$$\text{Velocity} = v(t) = \int a(t)dt = \int kdt = kt + C_1,$$

for some constant C_1. We integrate again to find position as a function of time:

$$\text{Position} = s(t) = \int v(t)dt = \int (kt + C_1)dt = \frac{kt^2}{2} + C_1t + C_2,$$

for some constant C_2. Since $k \neq 0$, this is a quadratic polynomial.

11. True, by the Second Fundamental Theorem of Calculus.

12. True. We see that

$$F(5) - F(3) = \int_0^5 f(t)dt - \int_0^3 f(t)dt = \int_3^5 f(t)dt.$$

13. False. If f is positive then F is increasing, but if f is negative then F is decreasing.

14. True. Since F and G are both antiderivatives of f, they must differ by a constant. In fact, we can see that the constant C is equal to $\int_0^2 f(t)dt$ since

$$F(x) = \int_0^x f(t)dt = \int_2^x f(t)dt + \int_0^2 f(t)dt = G(x) + C.$$

15. False. Since F and G are both antiderivatives of f, we know that they differ by a constant, but they are not necessarily equal. For example, if $f(t) = 1$ then $F(x) = \int_0^x 1\,dt = x$ but $G(x) = \int_2^x 1\,dt = x - 2$.

16. True, since $\int_0^x (f(t) + g(t))dt = \int_0^x f(t)dt + \int_0^x g(t)dt$.

PROJECTS FOR CHAPTER SIX

1. (a) If the poorest $p\%$ of the population has exactly $p\%$ of the goods, then $F(x) = x$.

(b) Any such F is increasing. For example, the poorest 50% of the population includes the poorest 40%, and so the poorest 50% must own more than the poorest 40%. Thus $F(0.4) \le F(0.5)$, and so, in general, F is increasing. In addition, it is clear that $F(0) = 0$ and $F(1) = 1$.

The graph of F is concave up by the following argument. Consider $F(0.05) - F(0.04)$. This is the fraction of resources the fifth poorest percent of the population has. Similarly, $F(0.20) - F(0.19)$ is the fraction of resources that the twentieth poorest percent of the population has. Since the twentieth poorest percent owns more than the fifth poorest percent, we have

$$F(0.05) - F(0.04) \le F(0.20) - F(0.19).$$

More generally, we can see that

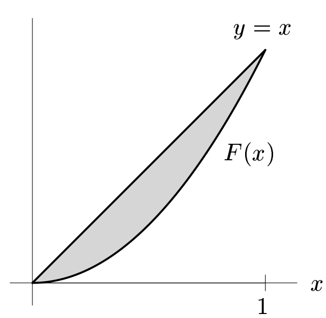
$$F(x_1 + \Delta x) - F(x_1) \le F(x_2 + \Delta x) - F(x_2)$$

for any x_1 smaller than x_2 and for any increment Δx. Dividing this inequality by Δx and taking the limit as $\Delta x \to 0$, we get

$$F'(x_1) \le F'(x_2).$$

So, the derivative of F is an increasing function, i.e. F is concave up.

(c) G is twice the shaded area below in the following figure. If the resource is distributed evenly, then G is zero. The larger G is, the more unevenly the resource is distributed. The maximum possible value of G is 1.



2. (a) In Figure 6.22, the area of the shaded region is $F(M)$. Thus, $F(M) = \int_0^M y(t)\,dt$ and, by the Fundamental Theorem, $F'(M) = y(M)$.

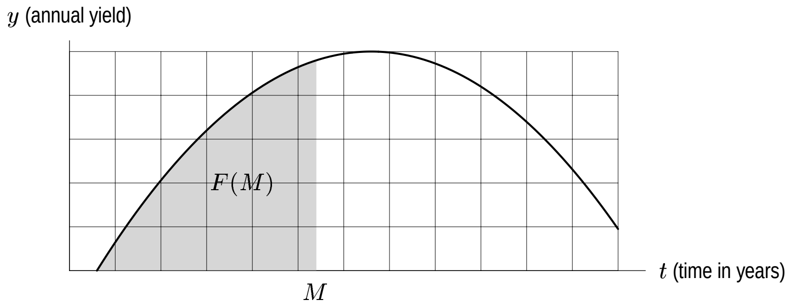


Figure 6.22

(b) Figure 6.23 is a graph of $F(M)$. Note that the graph of y looks like the graph of a quadratic function. Thus, the graph of F looks like a cubic.

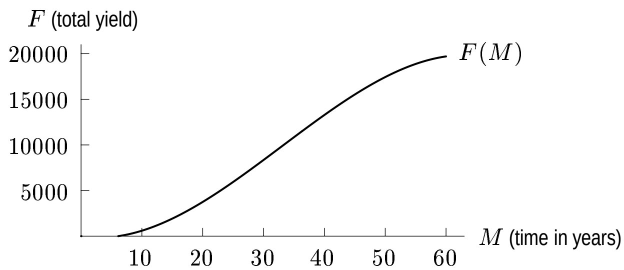


Figure 6.23

(c) We have

$$a(M) = \frac{1}{M}F(M) = \frac{1}{M}\int_0^M y(t)\,dt.$$

(d) If the function $a(M)$ takes on its maximum at some point M, then $a'(M) = 0$. Since

$$a(M) = \frac{1}{M}F(M),$$

differentiating using the quotient rule gives

$$a'(M) = \frac{MF'(M) - F(M)}{M^2} = 0,$$

so $MF'(M) = F(M)$. Since $F'(M) = y(M)$, the condition for a maximum may be written as

$$My(M) = F(M)$$

or as

$$y(M) = a(M).$$

To estimate the value of M which satisfies $My(M) = F(M)$, use the graph of $y(t)$. Notice that $F(M)$ is the area under the curve from 0 to M, and that $My(M)$ is the area of a rectangle of base M and height $y(M)$. Thus, we want the area under the curve to be equal to the area of the rectangle, or $A = B$ in Figure 6.24. This happens when $M \approx 50$ years. In other words, the orchard should be cut down after about 50 years.

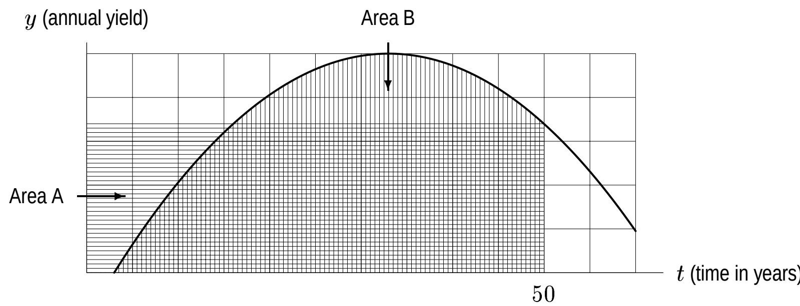


Figure 6.24