CHAPTER SEVEN

Solutions for Section 7.1

Exercises

1. **(a)** $\frac{d}{dx}\sin(x^2+1) = 2x\cos(x^2+1)$; $\frac{d}{dx}\sin(x^3+1) = 3x^2\cos(x^3+1)$
(b) (i) $\frac{1}{2}\sin(x^2+1)+C$ (ii) $\frac{1}{3}\sin(x^3+1)+C$
(c) (i) $-\frac{1}{2}\cos(x^2+1)+C$ (ii) $-\frac{1}{3}\cos(x^3+1)+C$

2. **(a)** We substitute $w = 1+x^2$, $dw = 2x\,dx$.

$$\int_{x=0}^{x=1} \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_{w=1}^{w=2} \frac{1}{w}\,dw = \frac{1}{2}\ln|w|\Big|_1^2 = \frac{1}{2}\ln 2.$$

(b) We substitute $w = \cos x$, $dw = -\sin x\,dx$.

$$\int_{x=0}^{x=\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx = -\int_{w=1}^{w=\sqrt{2}/2} \frac{1}{w}\,dw$$
$$= -\ln|w|\Big|_1^{\sqrt{2}/2} = -\ln\frac{\sqrt{2}}{2} = \frac{1}{2}\ln 2.$$

3. We use the substitution $w = 3x$, $dw = 3\,dx$.

$$\int e^{3x}dx = \frac{1}{3}\int e^w dw = \frac{1}{3}e^w + C = \frac{1}{3}e^{3x} + C.$$

Check: $\frac{d}{dx}(\frac{1}{3}e^{3x}+C) = \frac{1}{3}e^{3x}(3) = e^{3x}$.

4. We use the substitution $w = -0.2t$, $dw = -0.2\,dt$.

$$\int 25e^{-0.2t}dt = \frac{25}{-0.2}\int e^w dw = -125e^w + C = -125e^{-0.2t} + C.$$

Check: $\frac{d}{dt}(-125e^{-0.2t}+C) = -125e^{-0.2t}(-0.2) = 25e^{-0.2t}$.

5. We use the substitution $w = 2x$, $dw = 2\,dx$.

$$\int \sin(2x)dx = \frac{1}{2}\int \sin(w)dw = -\frac{1}{2}\cos(w) + C = -\frac{1}{2}\cos(2x) + C.$$

Check: $\frac{d}{dx}(-\frac{1}{2}\cos(2x)+C) = \frac{1}{2}\sin(2x)(2) = \sin(2x)$.

6. We use the substitution $w = t^2$, $dw = 2t\,dt$.

$$\int t\cos(t^2)dt = \frac{1}{2}\int \cos(w)dw = \frac{1}{2}\sin(w) + C = \frac{1}{2}\sin(t^2) + C.$$

Check: $\frac{d}{dt}(\frac{1}{2}\sin(t^2)+C) = \frac{1}{2}\cos(t^2)(2t) = t\cos(t^2)$.

7. We use the substitution $w = y^2+5$, $dw = 2y\,dy$.

$$\int y(y^2+5)^8\,dy = \frac{1}{2}\int (y^2+5)^8(2y\,dy)$$
$$= \frac{1}{2}\int w^8\,dw = \frac{1}{2}\frac{w^9}{9} + C$$
$$= \frac{1}{18}(y^2+5)^9 + C.$$

Check: $\frac{d}{dy}(\frac{1}{18}(y^2+5)^9+C) = \frac{1}{18}[9(y^2+5)^8(2y)] = y(y^2+5)^8$.

8. We use the substitution $w = t^3 - 3$, $dw = 3t^2\,dt$.

$$\int t^2(t^3-3)^{10}\,dt = \frac{1}{3}\int (t^3-3)^{10}(3t^2dt) = \int w^{10}\left(\frac{1}{3}\,dw\right)$$
$$= \frac{1}{3}\frac{w^{11}}{11} + C = \frac{1}{33}(t^3-3)^{11} + C.$$

Check: $\frac{d}{dt}[\frac{1}{33}(t^3-3)^{11}+C] = \frac{1}{3}(t^3-3)^{10}(3t^2) = t^2(t^3-3)^{10}$.

9. In this case, it seems easier not to substitute.

$$\int y^2(1+y)^2\,dy = \int y^2(y^2+2y+1)\,dy = \int (y^4+2y^3+y^2)\,dy$$
$$= \frac{y^5}{5} + \frac{y^4}{2} + \frac{y^3}{3} + C.$$

Check: $\dfrac{d}{dy}\left(\dfrac{y^5}{5}+\dfrac{y^4}{2}+\dfrac{y^3}{3}+C\right) = y^4+2y^3+y^2 = y^2(y+1)^2$.

10. We use the substitution $w = 1+2x^3$, $dw = 6x^2\,dx$.

$$\int x^2(1+2x^3)^2\,dx = \int w^2(\frac{1}{6}\,dw) = \frac{1}{6}(\frac{w^3}{3}) + C = \frac{1}{18}(1+2x^3)^3 + C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{1}{18}(1+2x^2)^3+C\right] = \dfrac{1}{18}[3(1+2x^3)^2(6x^2)] = x^2(1+2x^3)^2$.

11. We use the substitution $w = x^2-4$, $dw = 2x\,dx$.

$$\int x(x^2-4)^{\frac{7}{2}}\,dx = \frac{1}{2}\int (x^2-4)^{\frac{7}{2}}(2xdx) = \frac{1}{2}\int w^{\frac{7}{2}}\,dw$$
$$= \frac{1}{2}(\frac{2}{9}w^{\frac{9}{2}}) + C = \frac{1}{9}(x^2-4)^{\frac{9}{2}} + C.$$

Check: $\dfrac{d}{dx}[\dfrac{1}{9}(x^2-4)^{\frac{9}{2}}+C] = \dfrac{1}{9}\left[\dfrac{9}{2}(x^2-4)^{\frac{7}{2}}\right]2x = x(x^2-4)^{\frac{7}{2}}$.

12. We use the substitution $w = x^2+3$, $dw = 2x\,dx$.

$$\int x(x^2+3)^2\,dx = \int w^2(\frac{1}{2}\,dw) = \frac{1}{2}\frac{w^3}{3} + C = \frac{1}{6}(x^2+3)^3 + C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{1}{6}(x^2+3)^3+C\right] = \dfrac{1}{6}\left[3(x^2+3)^2(2x)\right] = x(x^2+3)^2$.

13. We use the substitution $w = 4-x$, $dw = -dx$.

$$\int \frac{1}{\sqrt{4-x}}\,dx = -\int \frac{1}{\sqrt{w}}\,dw = -2\sqrt{w} + C = -2\sqrt{4-x} + C.$$

Check: $\dfrac{d}{dx}(-2\sqrt{4-x}+C) = -2\cdot\dfrac{1}{2}\cdot\dfrac{1}{\sqrt{4-x}}\cdot -1 = \dfrac{1}{\sqrt{4-x}}$.

14. We use the substitution $w = y+5$, $dw = dy$, to get

$$\int \frac{dy}{y+5} = \int \frac{dw}{w} = \ln|w| + C = \ln|y+5| + C.$$

Check: $\dfrac{d}{dy}(\ln|y+5|+C) = \dfrac{1}{y+5}$.

15. We use the substitution $w = 2t-7$, $dw = 2\,dt$.

$$\int (2t-7)^{73}\,dt = \frac{1}{2}\int w^{73}\,dw = \frac{1}{(2)(74)}w^{74} + C = \frac{1}{148}(2t-7)^{74} + C.$$

Check: $\dfrac{d}{dt}\left[\dfrac{1}{148}(2t-7)^{74}+C\right] = \dfrac{74}{148}(2t-7)^{73}(2) = (2t-7)^{73}$.

16. In this case, it seems easier not to substitute.

$$\int (x^2+3)^2\,dx = \int (x^4+6x^2+9)\,dx = \frac{x^5}{5} + 2x^3 + 9x + C.$$

Check: $\dfrac{d}{dx}\left[\dfrac{x^5}{5} + 2x^3 + 9x + C\right] = x^4 + 6x^2 + 9 = (x^2+3)^2.$

17. We use the substitution $w = \cos\theta + 5,\ dw = -\sin\theta\,d\theta$.

$$\begin{aligned}\int \sin\theta(\cos\theta+5)^7\,d\theta &= -\int w^7\,dw = -\frac{1}{8}w^8 + C\\ &= -\frac{1}{8}(\cos\theta+5)^8 + C.\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{d\theta}\left[-\frac{1}{8}(\cos\theta+5)^8 + C\right] &= -\frac{1}{8}\cdot 8(\cos\theta+5)^7\cdot(-\sin\theta)\\ &= \sin\theta(\cos\theta+5)^7\end{aligned}$$

18. We use the substitution $w = \cos 3t,\ dw = -3\sin 3t\,dt$.

$$\begin{aligned}\int \sqrt{\cos 3t}\sin 3t\,dt &= -\frac{1}{3}\int \sqrt{w}\,dw\\ &= -\frac{1}{3}\cdot\frac{2}{3}w^{\frac{3}{2}} + C = -\frac{2}{9}(\cos 3t)^{\frac{3}{2}} + C.\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{dt}\left[-\frac{2}{9}(\cos 3t)^{\frac{3}{2}} + C\right] &= -\frac{2}{9}\cdot\frac{3}{2}(\cos 3t)^{\frac{1}{2}}\cdot(-\sin 3t)\cdot 3\\ &= \sqrt{\cos 3t}\sin 3t.\end{aligned}$$

19. We use the substitution $w = -x^2$, $dw = -2x\,dx$.

$$\begin{aligned}\int xe^{-x^2}\,dx &= -\frac{1}{2}\int e^{-x^2}(-2x\,dx) = -\frac{1}{2}\int e^w\,dw\\ &= -\frac{1}{2}e^w + C = -\frac{1}{2}e^{-x^2} + C.\end{aligned}$$

Check: $\frac{d}{dx}(-\frac{1}{2}e^{-x^2} + C) = (-2x)(-\frac{1}{2}e^{-x^2}) = xe^{-x^2}$.

20. We use the substitution $w = \sin\theta,\ dw = \cos\theta\,d\theta$.

$$\int \sin^6\theta\cos\theta\,d\theta = \int w^6\,dw = \frac{w^7}{7} + C = \frac{\sin^7\theta}{7} + C.$$

Check: $\dfrac{d}{d\theta}\left[\dfrac{\sin^7\theta}{7} + C\right] = \sin^6\theta\cos\theta.$

21. We use the substitution $w = x^3+1$, $dw = 3x^2\,dx$, to get

$$\int x^2e^{x^3+1}\,dx = \frac{1}{3}\int e^w\,dw = \frac{1}{3}e^w + C = \frac{1}{3}e^{x^3+1} + C.$$

Check: $\dfrac{d}{dx}\left(\dfrac{1}{3}e^{x^3+1} + C\right) = \dfrac{1}{3}e^{x^3+1}\cdot 3x^2 = x^2e^{x^3+1}.$

22. We use the substitution $w = \sin 5\theta,\ dw = 5\cos 5\theta\, d\theta$.

$$\int \sin^6 5\theta \cos 5\theta\, d\theta = \frac{1}{5}\int w^6\, dw = \frac{1}{5}(\frac{w^7}{7}) + C = \frac{1}{35}\sin^7 5\theta + C.$$

Check: $\frac{d}{d\theta}(\frac{1}{35}\sin^7 5\theta + C) = \frac{1}{35}[7\sin^6 5\theta](5\cos 5\theta) = \sin^6 5\theta\cos 5\theta$.
Note that we could also use Problem 20 to solve this problem, substituting $w = 5\theta$ and $dw = 5\, d\theta$ to get:

$$\begin{aligned}\int \sin^6 5\theta\cos 5\theta\, d\theta &= \frac{1}{5}\int \sin^6 w\cos w\, dw \\ &= \frac{1}{5}(\frac{\sin^7 w}{7}) + C = \frac{1}{35}\sin^7 5\theta + C.\end{aligned}$$

23. We use the substitution $w = \sin\alpha,\ dw = \cos\alpha\, d\alpha$.

$$\int \sin^3\alpha\cos\alpha\, d\alpha = \int w^3\, dw = \frac{w^4}{4} + C = \frac{\sin^4\alpha}{4} + C.$$

Check: $\frac{d}{d\alpha}\left(\frac{\sin^4\alpha}{4} + C\right) = \frac{1}{4}\cdot 4\sin^3\alpha\cdot\cos\alpha = \sin^3\alpha\cos\alpha$.

24. We use the substitution $w = \cos 2x,\ dw = -2\sin 2x\, dx$.

$$\begin{aligned}\int \tan 2x\, dx &= \int \frac{\sin 2x}{\cos 2x}\, dx = -\frac{1}{2}\int\frac{dw}{w} \\ &= -\frac{1}{2}\ln|w| + C = -\frac{1}{2}\ln|\cos 2x| + C.\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{dx}\left[-\frac{1}{2}\ln|\cos 2x| + C\right] &= -\frac{1}{2}\cdot\frac{1}{\cos 2x}\cdot -2\sin 2x \\ &= \frac{\sin 2x}{\cos 2x} = \tan 2x.\end{aligned}$$

25. We use the substitution $w = \ln z, dw = \frac{1}{z}\, dz$.

$$\int \frac{(\ln z)^2}{z}\, dz = \int w^2\, dw = \frac{w^3}{3} + C = \frac{(\ln z)^3}{3} + C.$$

Check: $\frac{d}{dz}\left[\frac{(\ln z)^3}{3} + C\right] = 3\cdot\frac{1}{3}(\ln z)^2\cdot\frac{1}{z} = \frac{(\ln z)^2}{z}$.

26. We use the substitution $w = e^t + t,\ dw = (e^t + 1)\, dt$.

$$\int \frac{e^t + 1}{e^t + t}\, dt = \int \frac{1}{w}\, dw = \ln|w| + C = \ln|e^t + t| + C.$$

Check: $\frac{d}{dt}(\ln|e^t + t| + C) = \frac{e^t + 1}{e^t + t}$.

27. We use the substitution $w = y^2 + 4,\ dw = 2y\, dy$.

$$\int \frac{y}{y^2 + 4}\, dy = \frac{1}{2}\int\frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(y^2 + 4) + C.$$

(We can drop the absolute value signs since $y^2 + 4 \geq 0$ for all y.)
Check: $\frac{d}{dy}\left[\frac{1}{2}\ln(y^2 + 4) + C\right] = \frac{1}{2}\cdot\frac{1}{y^2 + 4}\cdot 2y = \frac{y}{y^2 + 4}$.

28. We use the substitution $w = \sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}\,dx$.

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int \cos w(2\,dw) = 2\sin w + C = 2\sin\sqrt{x} + C.$$

Check: $\dfrac{d}{dx}(2\sin\sqrt{x} + C) = 2\cos\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{\sqrt{x}}.$

29. We use the substitution $w = \sqrt{y}$, $dw = \dfrac{1}{2\sqrt{y}}\,dy$.

$$\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy = 2\int e^w\,dw = 2e^w + C = 2e^{\sqrt{y}} + C.$$

Check: $\dfrac{d}{dy}(2e^{\sqrt{y}} + C) = 2e^{\sqrt{y}}\cdot\dfrac{1}{2\sqrt{y}} = \dfrac{e^{\sqrt{y}}}{\sqrt{y}}.$

30. We use the substitution $w = x + e^x$, $dw = (1 + e^x)\,dx$.

$$\int \frac{1 + e^x}{\sqrt{x + e^x}}\,dx = \int \frac{dw}{\sqrt{w}} = 2\sqrt{w} + C = 2\sqrt{x + e^x} + C.$$

Check: $\dfrac{d}{dx}(2\sqrt{x + e^x} + C) = 2\cdot\dfrac{1}{2}(x + e^x)^{-\frac{1}{2}}\cdot(1 + e^x) = \dfrac{1 + e^x}{\sqrt{x + e^x}}.$

31. We use the substitution $w = 2 + e^x$, $dw = e^x\,dx$.

$$\int \frac{e^x}{2 + e^x}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(2 + e^x) + C.$$

(We can drop the absolute value signs since $2 + e^x \geq 0$ for all x.)

Check: $\dfrac{d}{dx}[\ln(2 + e^x) + C] = \dfrac{1}{2 + e^x}\cdot e^x = \dfrac{e^x}{2 + e^x}.$

32. We use the substitution $w = x^2 + 2x + 19$, $dw = 2(x + 1)dx$.

$$\int \frac{(x + 1)dx}{x^2 + 2x + 19} = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(x^2 + 2x + 19) + C.$$

(We can drop the absolute value signs, since $x^2 + 2x + 19 = (x + 1)^2 + 18 > 0$ for all x.)

Check: $\dfrac{1}{dx}[\dfrac{1}{2}\ln(x^2 + 2x + 19)] = \dfrac{1}{2}\dfrac{1}{x^2 + 2x + 19}(2x + 2) = \dfrac{x + 1}{x^2 + 2x + 19}.$

33. We use the substitution $w = 1 + 3t^2$, $dw = 6t\,dt$.

$$\int \frac{t}{1 + 3t^2}\,dt = \int \frac{1}{w}(\frac{1}{6}\,dw) = \frac{1}{6}\ln|w| + C = \frac{1}{6}\ln(1 + 3t^2) + C.$$

(We can drop the absolute value signs since $1 + 3t^2 > 0$ for all t).

Check: $\dfrac{d}{dt}\left[\dfrac{1}{6}\ln(1 + 3t^2) + C\right] = \dfrac{1}{6}\dfrac{1}{1 + 3t^2}(6t) = \dfrac{t}{1 + 3t^2}.$

34. We use the substitution $w = e^x + e^{-x}$, $dw = (e^x - e^{-x})\,dx$.

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx = \int \frac{dw}{w} = \ln|w| + C = \ln(e^x + e^{-x}) + C.$$

(We can drop the absolute value signs since $e^x + e^{-x} > 0$ for all x).

Check: $\dfrac{d}{dx}[\ln(e^x + e^{-x}) + C] = \dfrac{1}{e^x + e^{-x}}(e^x - e^{-x}).$

35. It seems easier not to substitute.

$$\int \frac{(t + 1)^2}{t^2}\,dt = \int \frac{(t^2 + 2t + 1)}{t^2}\,dt$$
$$= \int \left(1 + \frac{2}{t} + \frac{1}{t^2}\right)\,dt = t + 2\ln|t| - \frac{1}{t} + C.$$

Check: $\dfrac{d}{dt}(t + 2\ln|t| - \dfrac{1}{t} + C) = 1 + \dfrac{2}{t} + \dfrac{1}{t^2} = \dfrac{(t + 1)^2}{t^2}.$

36. We use the substitution $w = \sin(x^2),\ dw = 2x\cos(x^2)\,dx$.

$$\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}\,dx = \frac{1}{2}\int w^{-\frac{1}{2}}\,dw = \frac{1}{2}(2w^{\frac{1}{2}}) + C = \sqrt{\sin(x^2)} + C.$$

Check: $\frac{d}{dx}(\sqrt{\sin(x^2)} + C) = \frac{1}{2\sqrt{\sin(x^2)}}[\cos(x^2)]2x = \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}$.

37. Since $d(\sinh x)/dx = \cosh x$, we have

$$\int \cosh x\,dx = \sinh x + C.$$

38. Since $d(\cosh 3t)/dt = 3\sinh 3t$, we have

$$\int \sinh 3t\,dt = \frac{1}{3}\cosh 3t + C.$$

39. Since $d(\cosh z)/dz = \sinh z$, the chain rule shows that

$$\frac{d}{dz}(e^{\cosh z}) = (\sinh z)e^{\cosh z}.$$

Thus,

$$\int (\sinh z)e^{\cosh z}\,dz = e^{\cosh z} + C.$$

40. Since $d(\sinh(2w+1))/dw = 2\cosh(2w+1)$, we have

$$\int \cosh(2w+1)\,dw = \frac{1}{2}\sinh(2w+1) + C.$$

41. The general antiderivative is $\int(\pi t^3 + 4t)\,dt = (\pi/4)t^4 + 2t^2 + C$.

42. Make the substitution $w = 3x$, $dw = 3\,dx$. We have

$$\int \sin 3x\,dx = \frac{1}{3}\int \sin w\,dw = \frac{1}{3}(-\cos w) + C = -\frac{1}{3}\cos 3x + C.$$

43. Make the substitution $w = x^2$, $dw = 2x\,dx$. We have

$$\int 2x\cos(x^2)\,dx = \int \cos w\,dw = \sin w + C = \sin x^2 + C.$$

44. Make the substitution $w = t^3$, $dw = 3t^2\,dt$. The general antiderivative is $\int 12t^2\cos(t^3)\,dt = 4\sin(t^3) + C$.

45. Make the substitution $w = 2 - 5x$, then $dw = -5dx$. We have

$$\int \sin(2-5x)dx = \int \sin w\left(-\frac{1}{5}\right)dw = -\frac{1}{5}(-\cos w) + C = \frac{1}{5}\cos(2-5x) + C.$$

46. Make the substitution $w = \sin x$, $dw = \cos x\,dx$. We have

$$\int e^{\sin x}\cos x\,dx = \int e^w\,dw = e^w + C = e^{\sin x} + C.$$

47. Make the substitution $w = x^2 + 1$, $dw = 2x\,dx$. We have

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(x^2+1) + C.$$

(Notice that since $x^2 + 1 \geq 0$, $|x^2+1| = x^2+1$.)

48. Make the substitution $w = 2x$, then $dw = 2dx$. We have

$$\begin{aligned}\int \frac{1}{3\cos^2 2x}\,dx &= \frac{1}{3}\int \frac{1}{\cos^2 w}\left(\frac{1}{2}\right)\,dw \\ &= \frac{1}{6}\int \frac{1}{\cos^2 w}\,dw = \frac{1}{6}\tan w + C = \frac{1}{6}\tan 2x + C.\end{aligned}$$

49. $\int_0^{\pi} \cos(x+\pi)\,dx = \sin(x+\pi)\Big|_0^{\pi} = \sin(2\pi) - \sin(\pi) = 0 - 0 = 0$

50. We substitute $w = \pi x$. Then $dw = \pi\,dx$.

$$\int_{x=0}^{x=\frac{1}{2}} \cos \pi x\,dx = \int_{w=0}^{w=\pi/2} \cos w (\frac{1}{\pi}\,dw) = \frac{1}{\pi}(\sin w)\Big|_0^{\pi/2} = \frac{1}{\pi}$$

51. $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta = e^{-\cos\theta}\Big|_0^{\pi/2} = e^{-\cos(\pi/2)} - e^{-\cos(0)} = 1 - \frac{1}{e}$

52. $\int_1^2 2xe^{x^2}\,dx = e^{x^2}\Big|_1^2 = e^{2^2} - e^{1^2} = e^4 - e = e(e^3 - 1)$

53. We substitute $w = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then $dw = \frac{1}{3}x^{-\frac{2}{3}}\,dx = \frac{1}{3\sqrt[3]{x^2}}\,dx$.

$$\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx = \int_{x=1}^{x=8} e^w(3\,dw) = 3e^w\Big|_{x=1}^{x=8} = 3e^{\sqrt[3]{x}}\Big|_1^8 = 3(e^2 - e).$$

54. We substitute $w = t + 2$, so $dw = dt$.

$$\int_{t=-1}^{t=e-2} \frac{1}{t+2}dt = \int_{w=1}^{w=e} \frac{dw}{w} = \ln|w|\Big|_1^e = \ln e - \ln 1 = 1.$$

55. We substitute $w = \sqrt{x}$. Then $dw = \frac{1}{2}x^{-1/2}dx$.

$$\int_{x=1}^{x=4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int_{w=1}^{w=2} \cos w(2\,dw)$$
$$= 2(\sin w)\Big|_1^2 = 2(\sin 2 - \sin 1).$$

56. We substitute $w = 1 + x^2$. Then $dw = 2x\,dx$.

$$\int_{x=0}^{x=2} \frac{x}{(1+x^2)^2}\,dx = \int_{w=1}^{w=5} \frac{1}{w^2}\left(\frac{1}{2}\,dw\right) = -\frac{1}{2}\left(\frac{1}{w}\right)\Bigg|_1^5 = \frac{2}{5}.$$

57.

$$\int_{-1}^{3}(x^3 + 5x)\,dx = \frac{x^4}{4}\Bigg|_{-1}^{3} + \frac{5x^2}{2}\Bigg|_{-1}^{3} = 40.$$

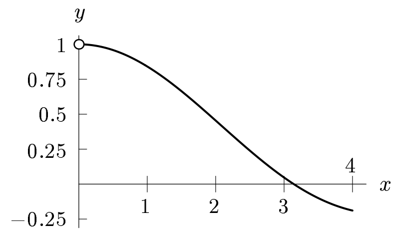
58. $\int_{-1}^{1} \frac{1}{1+y^2}\,dy = \tan^{-1} y\Big|_{-1}^{1} = \frac{\pi}{2}.$

59. $\int_1^3 \frac{1}{x}\,dx = \ln x\Big|_1^3 = \ln 3.$

60. $\int_1^3 \frac{dt}{(t+7)^2} = \frac{-1}{t+7}\Big|_1^3 = \left(-\frac{1}{10}\right) - \left(-\frac{1}{8}\right) = \frac{1}{40}$

61. $\int_{-1}^{2} \sqrt{x+2}\,dx = \frac{2}{3}(x+2)^{3/2}\Big|_{-1}^{2} = \frac{2}{3}\left[(4)^{3/2} - (1)^{3/2}\right] = \frac{2}{3}(7) = \frac{14}{3}$

62. It turns out that $\frac{\sin x}{x}$ cannot be integrated using elementary methods. However, the function is decreasing on [1,2]. One way to see this is to graph the function on a calculator or computer, as has been done below:



So since our function is monotonic, the error for our left- and right-hand sums is less than or equal to $\left|\frac{\sin 2}{2} - \frac{\sin 1}{1}\right| \Delta t \approx 0.61\Delta t$. So with 13 intervals, our error will be less than 0.05. With $n = 13$, the left sum is about 0.674, and the right sum is about 0.644. For more accurate sums, with $n = 100$ the left sum is about 0.6613 and the right sum is about 0.6574. The actual integral is about 0.6593.

Problems

63. **(a)** This integral can be evaluated using integration by substitution. We use $w = x^2$, $dw = 2xdx$.

$$\int x \sin x^2 dx = \frac{1}{2}\int \sin(w)dw = -\frac{1}{2}\cos(w) + C = -\frac{1}{2}\cos(x^2) + C.$$

(b) This integral cannot be evaluated using a simple integration by substitution.
(c) This integral cannot be evaluated using a simple integration by substitution.
(d) This integral can be evaluated using integration by substitution. We use $w = 1 + x^2$, $dw = 2xdx$.

$$\int \frac{x}{(1+x^2)^2}dx = \frac{1}{2}\int \frac{1}{w^2}dw = \frac{1}{2}(\frac{-1}{w}) + C = \frac{-1}{2(1+x^2)} + C.$$

(e) This integral cannot be evaluated using a simple integration by substitution.
(f) This integral can be evaluated using integration by substitution. We use $w = 2 + \cos x$, $dw = -\sin x dx$.

$$\int \frac{\sin x}{2+\cos x}dx = -\int \frac{1}{w}dw = -\ln|w| + C = -\ln|2+\cos x| + C.$$

64. **(a)** The Fundamental Theorem gives

$$\int_{-\pi}^{\pi} \cos^2\theta \sin\theta\, d\theta = -\frac{\cos^3\theta}{3}\bigg|_{-\pi}^{\pi} = \frac{-(-1)^3}{3} - \frac{-(-1)^3}{3} = 0.$$

This agrees with the fact that the function $f(\theta) = \cos^2\theta\sin\theta$ is odd and the interval of integration is centered at $x = 0$, thus we must get 0 for the definite integral.

(b) The area is given by

$$\text{Area} = \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta = -\frac{\cos^3\theta}{3}\bigg|_0^{\pi} = \frac{-(-1)^3}{3} - \frac{-(1)^3}{3} = \frac{2}{3}.$$

65. Since $f(x) = 1/(x+1)$ is positive on the interval $x = 0$ to $x = 2$, we have

$$\text{Area} = \int_0^2 \frac{1}{x+1}dx = \ln(x+1)\bigg|_0^2 = \ln 3 - \ln 1 = \ln 3.$$

The area is $\ln 3 \approx 1.0986$.

66. To find the area under the graph of $f(x) = xe^{x^2}$, we need to evaluate the definite integral

$$\int_0^2 xe^{x^2}\, dx.$$

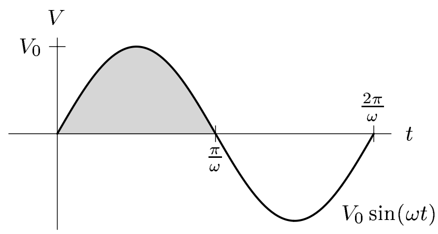
This is done in Example 10, Section 7.1, using the substitution $w = x^2$, the result being

$$\int_0^2 xe^{x^2}\, dx = \frac{1}{2}(e^4 - 1).$$

67. The area under the curve between the given values is given by

$$\text{Area } = \int_2^4 \frac{4}{x}\,dx = 4\ln x\Big|_2^4 = 4(\ln 4 - \ln 2) = 4\ln 2 \approx 2.7726.$$

68.



The period of $V = V_0\sin(\omega t)$ is $2\pi/\omega$, so the area under the first arch is given by

$$\begin{aligned}\text{Area} &= \int_0^{\pi/\omega} V_0\sin(\omega t)\,dt \\ &= -\frac{V_0}{\omega}\cos(\omega t)\Big|_0^{\pi/\omega} \\ &= -\frac{V_0}{\omega}\cos(\pi) + \frac{V_0}{\omega}\cos(0) \\ &= -\frac{V_0}{\omega}(-1) + \frac{V_0}{\omega}(1) = \frac{2V_0}{\omega}.\end{aligned}$$

69. If $f(x) = \dfrac{1}{x+1}$, the average value of f on the interval $0 \le x \le 2$ is defined to be

$$\frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 \frac{dx}{x+1}.$$

We'll integrate by substitution. We let $w = x+1$ and $dw = dx$, and we have

$$\int_{x=0}^{x=2}\frac{dx}{x+1} = \int_{w=1}^{w=3}\frac{dw}{w} = \ln w\Big|_1^3 = \ln 3 - \ln 1 = \ln 3.$$

Thus, the average value of $f(x)$ on $0 \le x \le 2$ is $\frac{1}{2}\ln 3 \approx 0.5493$. See Figure 7.1.

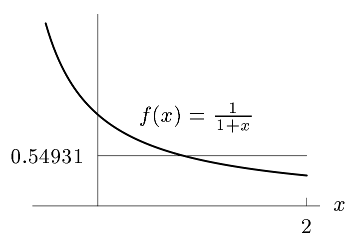


Figure 7.1

70. **(a)** $\displaystyle\int 4x(x^2+1)\,dx = \int (4x^3+4x)\,dx = x^4 + 2x^2 + C.$

(b) If $w = x^2+1$, then $dw = 2x\,dx$.

$$\int 4x(x^2+1)\,dx = \int 2w\,dw = w^2 + C = (x^2+1)^2 + C.$$

(c) The expressions from parts (a) and (b) look different, but they are both correct. Note that $(x^2+1)^2 + C = x^4 + 2x^2 + 1 + C$. In other words, the expressions from parts (a) and (b) differ only by a constant, so they are both correct antiderivatives.

71. **(a)** We first try the substitution $w = \sin\theta,\ dw = \cos\theta\, d\theta$. Then
$\displaystyle\int \sin\theta\cos\theta\, d\theta = \int w\, dw = \frac{w^2}{2} + C = \frac{\sin^2\theta}{2} + C.$

(b) If we instead try the substitution $w = \cos\theta,\ dw = -\sin\theta\, d\theta$, we get
$\displaystyle\int \sin\theta\cos\theta\, d\theta = -\int w\, dw = -\frac{w^2}{2} + C = -\frac{\cos^2\theta}{2} + C.$

(c) Once we note that $\sin 2\theta = 2\sin\theta\cos\theta$, we can also say
$\displaystyle\int \sin\theta\cos\theta\, d\theta = \frac{1}{2}\int \sin 2\theta\, d\theta.$
Substituting $w = 2\theta,\ dw = 2\, d\theta$, the above equals
$\displaystyle\frac{1}{4}\int \sin w\, dw = -\frac{\cos w}{4} + C = -\frac{\cos 2\theta}{4} + C.$

(d) All these answers are correct. Although they have different forms, they differ from each other only in terms of a constant, and thus they are all acceptable antiderivatives. For example, $1 - \cos^2\theta = \sin^2\theta$, so $\frac{\sin^2\theta}{2} = -\frac{\cos^2\theta}{2} + \frac{1}{2}$. Thus the first two expressions differ only by a constant C.
Similarly, $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$, so $-\frac{\cos 2\theta}{4} = -\frac{\cos^2\theta}{2} + \frac{1}{4}$, and thus the second and third expressions differ only by a constant. Of course, if the first two expressions and the last two expressions differ only in the constant C, then the first and last only differ in the constant as well.

72. **(a)** If $w = 2t$, then $dw = 2dt$. When $t = 0$, $w = 0$; when $t = 0.5$, $w = 1$. Thus,

$$\int_0^{0.5} f(2t)\, dt = \int_0^1 f(w)\, \frac{1}{2} dw = \frac{1}{2}\int_0^1 f(w)\, dw = \frac{3}{2}.$$

(b) If $w = 1 - t$, then $dw = -dt$. When $t = 0$, $w = 1$; when $t = 1$, $w = 0$. Thus,

$$\int_0^1 f(1-t)\, dt = \int_1^0 f(w)\,(-dw) = +\int_0^1 f(w)\, dw = 3.$$

(c) If $w = 3 - 2t$, then $dw = -2dt$. When $t = 1$, $w = 1$; when $t = 1.5$, $w = 0$. Thus,

$$\int_1^{1.5} f(3-2t)\, dt = \int_1^0 f(w)\, \left(-\frac{1}{2} dw\right) = +\frac{1}{2}\int_0^1 f(w)\, dw = \frac{3}{2}.$$

73. **(a)** In 1990, we have $P = 5.3e^{0.014(0)} = 5.3$ billion people.
In 2000, we have $P = 5.3e^{0.014(10)} = 6.1$ billion people.

(b) We have

$$\text{Average population} = \frac{1}{10-0}\int_0^{10} 5.3e^{0.014t} dt = \frac{1}{10}\cdot\frac{5.3}{0.014} e^{0.014t}\Big|_0^{10}$$
$$= \frac{1}{10}\left(\frac{5.3}{0.014}(e^{0.14} - e^0)\right) = 5.7.$$

The average population of the world during the 1990s was 5.7 billion people.

74. **(a)** At time $t = 0$, the rate of oil leakage $= r(0) = 50$ thousand liters/minute.
At $t = 60$, rate $= r(60) = 15.06$ thousand liters/minute.

(b) To find the amount of oil leaked during the first hour, we integrate the rate from $t = 0$ to $t = 60$:

$$\text{Oil leaked} = \int_0^{60} 50e^{-0.02t}\, dt = \left(-\frac{50}{0.02}e^{-0.02t}\right)\Big|_0^{60}$$
$$= -2500e^{-1.2} + 2500e^0 = 1747 \text{ thousand liters.}$$

75. **(a)** $E(t) = 1.4e^{0.07t}$

(b)

$$\text{Average Yearly Consumption} = \frac{\text{Total Consumption for the Century}}{100 \text{ years}}$$
$$= \frac{1}{100}\int_0^{100} 1.4e^{0.07t}\, dt$$

$$= (0.014)\left[\frac{1}{0.07}e^{0.07t}\Big|_0^{100}\right]$$
$$= (0.014)\left[\frac{1}{0.07}(e^7 - e^0)\right]$$
$$= 0.2(e^7 - 1) \approx 219 \text{ million megawatt-hours.}$$

(c) We are looking for t such that $E(t) \approx 219$:

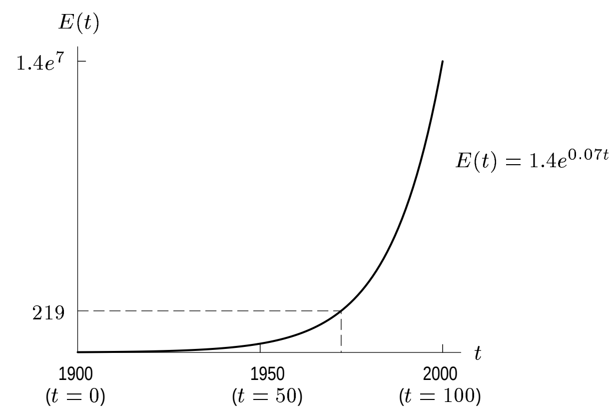
$$1.4e^{0.07t} \approx 219$$
$$e^{0.07t} = 156.4.$$

Taking natural logs,

$$0.07t = \ln 156.4$$
$$t \approx \frac{5.05}{0.07} \approx 72.18.$$

Thus, consumption was closest to the average during 1972.

(d) Between the years 1900 and 2000 the graph of $E(t)$ looks like



From the graph, we can see the t value such that $E(t) = 219$. It lies to the right of $t = 50$, and is thus in the second half of the century.

76. Since $v = \dfrac{dh}{dt}$, it follows that $h(t) = \int v(t)\,dt$ and $h(0) = h_0$. Since

$$v(t) = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) = \frac{mg}{k} - \frac{mg}{k}e^{-\frac{k}{m}t},$$

we have

$$h(t) = \int v(t)\,dt = \frac{mg}{k}\int dt - \frac{mg}{k}\int e^{-\frac{k}{m}t}\,dt.$$

The first integral is simply $\dfrac{mg}{k}t + C$. To evaluate the second integral, make the substitution $w = -\frac{k}{m}t$. Then

$$dw = -\frac{k}{m}\,dt,$$

so

$$\int e^{-\frac{k}{m}t}\,dt = \int e^w\left(-\frac{m}{k}\right)dw = -\frac{m}{k}e^w + C = -\frac{m}{k}e^{-\frac{k}{m}t} + C.$$

Thus

$$h(t) = \int v\,dt = \frac{mg}{k}t - \frac{mg}{k}\left(-\frac{m}{k}e^{-\frac{k}{m}t}\right) + C$$
$$= \frac{mg}{k}t + \frac{m^2g}{k^2}e^{-\frac{k}{m}t} + C.$$

Since $h(0) = h_0$,

$$h_0 = \frac{mg}{k} \cdot 0 + \frac{m^2 g}{k^2} e^0 + C;$$
$$C = h_0 - \frac{m^2 g}{k^2}.$$

Thus

$$h(t) = \frac{mg}{k} t + \frac{m^2 g}{k^2} e^{-\frac{k}{m}t} - \frac{m^2 g}{k^2} + h_0$$
$$h(t) = \frac{mg}{k} t - \frac{m^2 g}{k^2} \left(1 - e^{-\frac{k}{m}t}\right) + h_0.$$

77. Since v is given as the velocity of a falling body, the height h is decreasing, so $v = -\frac{dh}{dt}$, and it follows that $h(t) = -\int v(t)\,dt$ and $h(0) = h_0$. Let $w = e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}$. Then

$$dw = \sqrt{gk}\left(e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}\right)\,dt,$$

so $\dfrac{dw}{\sqrt{gk}} = (e^{t\sqrt{gk}} - e^{-t\sqrt{gk}})\,dt$. Therefore,

$$\begin{aligned}
-\int v(t)dt &= -\int \sqrt{\frac{g}{k}} \left(\frac{e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}\right) dt \\
&= -\sqrt{\frac{g}{k}} \int \frac{1}{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}} \left(e^{t\sqrt{gk}} - e^{-t\sqrt{gk}}\right) dt \\
&= -\sqrt{\frac{g}{k}} \int \left(\frac{1}{w}\right) \frac{dw}{\sqrt{gk}} \\
&= -\sqrt{\frac{g}{gk^2}} \ln|w| + C \\
&= -\frac{1}{k} \ln\left(e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}\right) + C.
\end{aligned}$$

Since

$$h(0) = -\frac{1}{k}\ln(e^0 + e^0) + C = -\frac{\ln 2}{k} + C = h_0,$$

we have $C = h_0 + \dfrac{\ln 2}{k}$. Thus,

$$h(t) = -\frac{1}{k}\ln\left(e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}\right) + \frac{\ln 2}{k} + h_0 = -\frac{1}{k}\ln\left(\frac{e^{t\sqrt{gk}} + e^{-t\sqrt{gk}}}{2}\right) + h_0.$$

78. (a) In the first case, we are given that $R_0 = 1000$ widgets/year. So we have $R = 1000e^{0.15t}$. To determine the total number sold, we need to integrate this rate over the time period from 0 to 10. So the total number of widgets sold is

$$\int_0^{10} 1000e^{0.15t}\,dt = \frac{1000}{0.15}e^{0.15t}\Big|_0^{10} = 6667(e^{1.5} - 1) \approx 23{,}211 \text{ widgets.}$$

In the second case, the total number of widgets sold is

$$\int_0^{10} 150{,}000{,}000e^{0.15t}\,dt = 1{,}000{,}000{,}000e^{0.15t}\Big|_0^{10} \approx 3.5 \text{ billion widgets.}$$

(b) We want to determine T such that

$$\int_0^T 1000e^{0.15t}\,dt \approx \frac{23{,}211}{2}.$$

Evaluating both sides, we get

$$\begin{aligned} 6667(e^{0.15T}-1) &= 11,606 \\ 6667e^{0.15T} &= 18273 \\ e^{0.15T} &= 2.740 \\ 0.15T &= 1.01, \quad \text{so} \quad T = 6.7 \text{ years.} \end{aligned}$$

Similarly, in the second case,

$$\int_0^T 150{,}000{,}000e^{0.15t}\,dt \approx \frac{3,500,000,000}{2}$$

Evaluating both sides, we get

$$\begin{aligned} (1 \text{ billion})(e^{0.15T}-1) &= 1.75 \text{ billion} \\ e^{0.15T} &= 2.75 \\ T &\approx 6.7 \text{ years} \end{aligned}$$

So the half way mark is reached at the same time regardless of the initial rate.

(c) Since half the widgets are sold in the last $3\frac{1}{2}$ years of the decade, if each widget is expected to last $3\frac{1}{2}$ years, their claim could easily be true.

Solutions for Section 7.2

Exercises

1. Let $u = \arctan x$, $v' = 1$. Then $v = x$ and $u' = \dfrac{1}{1+x^2}$. Integrating by parts, we get:

$$\int 1 \cdot \arctan x\,dx = x \cdot \arctan x - \int x \cdot \frac{1}{1+x^2}\,dx.$$

To compute the second integral use the substitution, $z = 1 + x^2$.

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\ln|z| + C = \frac{1}{2}\ln(1+x^2) + C.$$

Thus,

$$\int \arctan x\,dx = x \cdot \arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

2. Let $u = t$ and $v' = e^{5t}$, so $u' = 1$ and $v = \frac{1}{5}e^{5t}$.
Then $\int te^{5t}\,dt = \frac{1}{5}te^{5t} - \int \frac{1}{5}e^{5t}\,dt = \frac{1}{5}te^{5t} - \frac{1}{25}e^{5t} + C$.

3. Let $u = t^2$ and $v' = e^{5t}$, so $u' = 2t$ and $v = \frac{1}{5}e^{5t}$.
Then $\int t^2e^{5t}\,dt = \frac{1}{5}t^2e^{5t} - \frac{2}{5}\int te^{5t}\,dt$.
Using Problem 2, we have $\int t^2e^{5t}\,dt = \frac{1}{5}t^2e^{5t} - \frac{2}{5}(\frac{1}{5}te^{5t} - \frac{1}{25}e^{5t}) + C$
$= \frac{1}{5}t^2e^{5t} - \frac{2}{25}te^{5t} + \frac{2}{125}e^{5t} + C$.

4. Let $u = p$ and $v' = e^{(-0.1)p}$, $u' = 1$. Thus, $v = \int e^{(-0.1)p}\,dp = -10e^{(-0.1)p}$. With this choice of u and v, integration by parts gives:

$$\begin{aligned} \int pe^{(-0.1)p}\,dp &= p(-10e^{(-0.1)p}) - \int (-10e^{(-0.1)p})\,dp \\ &= -10pe^{(-0.1)p} + 10\int e^{(-0.1)p}\,dp \\ &= -10pe^{(-0.1)p} - 100e^{(-0.1)p} + C. \end{aligned}$$

5. Let $u = t$, $v' = \sin t$. Thus, $v = -\cos t$ and $u' = 1$. With this choice of u and v, integration by parts gives:

$$\begin{aligned}\int t \sin t\,dt &= -t\cos t - \int(-\cos t)\,dt \\ &= -t\cos t + \sin t + C.\end{aligned}$$

6. Let $u = \ln y$, $v' = y$. Then, $v = \frac{1}{2}y^2$ and $u' = \frac{1}{y}$. Integrating by parts, we get:

$$\begin{aligned}\int y\ln y\,dy &= \frac{1}{2}y^2\ln y - \int \frac{1}{2}y^2 \cdot \frac{1}{y}\,dy \\ &= \frac{1}{2}y^2\ln y - \frac{1}{2}\int y\,dy \\ &= \frac{1}{2}y^2\ln y - \frac{1}{4}y^2 + C.\end{aligned}$$

7. Let $u = \ln x$ and $v' = x^3$, so $u' = \frac{1}{x}$ and $v = \frac{x^4}{4}$.

Then

$$\int x^3\ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx = \frac{x^4}{4}\ln x - \frac{x^4}{16} + C.$$

8. Let $u = z + 1$, $v' = e^{2z}$. Thus, $v = \frac{1}{2}e^{2z}$ and $u' = 1$. Integrating by parts, we get:

$$\begin{aligned}\int (z+1)e^{2z}\,dz &= (z+1)\cdot\frac{1}{2}e^{2z} - \int \frac{1}{2}e^{2z}\,dz \\ &= \frac{1}{2}(z+1)e^{2z} - \frac{1}{4}e^{2z} + C \\ &= \frac{1}{4}(2z+1)e^{2z} + C.\end{aligned}$$

9. Let $u = t^2$, $v' = \sin t$ implying $v = -\cos t$ and $u' = 2t$. Integrating by parts, we get:

$$\int t^2\sin t\,dt = -t^2\cos t - \int 2t(-\cos t)\,dt.$$

Again, applying integration by parts with $u = t$, $v' = \cos t$, we have:

$$\int t\cos t\,dt = t\sin t + \cos t + C.$$

Thus

$$\int t^2\sin t\,dt = -t^2\cos t + 2t\sin t + 2\cos t + C.$$

10. Let $u = \theta^2$ and $v' = \cos 3\theta$, so $u' = 2\theta$ and $v = \frac{1}{3}\sin 3\theta$.

Then $\int \theta^2\cos 3\theta\,d\theta = \frac{1}{3}\theta^2\sin 3\theta - \frac{2}{3}\int\theta\sin 3\theta\,d\theta$. The integral on the right hand side is simpler than our original integral, but to evaluate it we need to again use integration by parts.

To find $\int\theta\sin 3\theta\,d\theta$, let $u = \theta$ and $v' = \sin 3\theta$, so $u' = 1$ and $v = -\frac{1}{3}\cos 3\theta$.

This gives

$$\int\theta\sin 3\theta\,d\theta = -\frac{1}{3}\theta\cos 3\theta + \frac{1}{3}\int\cos 3\theta\,d\theta = -\frac{1}{3}\theta\cos 3\theta + \frac{1}{9}\sin 3\theta + C.$$

Thus,

$$\int\theta^2\cos 3\theta d\theta = \frac{1}{3}\theta^2\sin 3\theta + \frac{2}{9}\theta\cos 3\theta - \frac{2}{27}\sin 3\theta + C.$$

11. Let $u = \sin\theta$ and $v' = \sin\theta$, so $u' = \cos\theta$ and $v = -\cos\theta$. Then

$$\begin{aligned}\int \sin^2\theta\,d\theta &= -\sin\theta\cos\theta + \int \cos^2\theta\,d\theta \\ &= -\sin\theta\cos\theta + \int (1-\sin^2\theta)\,d\theta \\ &= -\sin\theta\cos\theta + \int 1\,d\theta - \int \sin^2\theta\,d\theta.\end{aligned}$$

By adding $\int \sin^2\theta\,d\theta$ to both sides of the above equation, we find that $2\int \sin^2\theta\,d\theta = -\sin\theta\cos\theta + \theta + C$, so $\int \sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C'$.

12. Let $u = \cos(3\alpha+1)$ and $v' = \cos(3\alpha+1)$, so $u' = -3\sin(3\alpha+1)$, and $v = \frac{1}{3}\sin(3\alpha+1)$. Then

$$\begin{aligned}\int \cos^2(3\alpha+1)\,d\alpha &= \int (\cos(3\alpha+1))\cos(3\alpha+1)\,d\alpha \\ &= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \int \sin^2(3\alpha+1)\,d\alpha \\ &= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \int \left(1-\cos^2(3\alpha+1)\right)\,d\alpha \\ &= \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \alpha - \int \cos^2(3\alpha+1)\,d\alpha.\end{aligned}$$

By adding $\int \cos^2(3\alpha+1)\,d\alpha$ to both sides of the above equation, we find that

$$2\int \cos^2(3\alpha+1)\,d\alpha = \frac{1}{3}\cos(3\alpha+1)\sin(3\alpha+1) + \alpha + C,$$

which gives

$$\int \cos^2(3\alpha+1)\,d\alpha = \frac{1}{6}\cos(3\alpha+1)\sin(3\alpha+1) + \frac{\alpha}{2} + C.$$

13. Let $u = \ln 5q$, $v' = q^5$. Then $v = \frac{1}{6}q^6$ and $u' = \dfrac{1}{q}$. Integrating by parts, we get:

$$\begin{aligned}\int q^5\ln 5q\,dq &= \frac{1}{6}q^6\ln 5q - \int (5\cdot\frac{1}{5q})\cdot\frac{1}{6}q^6\,dq \\ &= \frac{1}{6}q^6\ln 5q - \frac{1}{36}q^6 + C.\end{aligned}$$

14. Let $u = y$ and $v' = (y+3)^{1/2}$, so $u' = 1$ and $v = \frac{2}{3}(y+3)^{3/2}$.
$\int y\sqrt{y+3}\,dy = \frac{2}{3}y(y+3)^{3/2} - \int \frac{2}{3}(y+3)^{3/2}\,dy = \frac{2}{3}y(y+3)^{3/2} - \frac{4}{15}(y+3)^{5/2} + C$.

15. Let $u = (\ln t)^2$ and $v' = 1$, so $u' = \dfrac{2\ln t}{t}$ and $v = t$. Then

$$\int (\ln t)^2\,dt = t(\ln t)^2 - 2\int \ln t\,dt = t(\ln t)^2 - 2t\ln t + 2t + C.$$

(We use the fact that $\displaystyle\int \ln x\,dx = x\ln x - x + C$, a result which can be derived using integration by parts.)

16. Let $u = t+2$ and $v' = \sqrt{2+3t}$, so $u' = 1$ and $v = \frac{2}{9}(2+3t)^{3/2}$. Then

$$\begin{aligned}\int (t+2)\sqrt{2+3t}\,dt &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{2}{9}\int (2+3t)^{3/2}\,dt \\ &= \frac{2}{9}(t+2)(2+3t)^{3/2} - \frac{4}{135}(2+3t)^{5/2} + C.\end{aligned}$$

17. Let $u = \theta + 1$ and $v' = \sin(\theta + 1)$, so $u' = 1$ and $v = -\cos(\theta + 1)$.

$$\begin{aligned}\int (\theta + 1)\sin(\theta + 1)\,d\theta &= -(\theta + 1)\cos(\theta + 1) + \int \cos(\theta + 1)\,d\theta \\ &= -(\theta + 1)\cos(\theta + 1) + \sin(\theta + 1) + C.\end{aligned}$$

18. Let $u = z$, $v' = e^{-z}$. Thus $v = -e^{-z}$ and $u' = 1$. Integration by parts gives:

$$\begin{aligned}\int ze^{-z}\,dz &= -ze^{-z} - \int (-e^{-z})\,dz \\ &= -ze^{-z} - e^{-z} + C \\ &= -(z + 1)e^{-z} + C.\end{aligned}$$

19. Let $u = \ln x$, $v' = x^{-2}$. Then $v = -x^{-1}$ and $u' = x^{-1}$. Integrating by parts, we get:

$$\begin{aligned}\int x^{-2}\ln x\,dx &= -x^{-1}\ln x - \int (-x^{-1})\cdot x^{-1}\,dx \\ &= -x^{-1}\ln x - x^{-1} + C.\end{aligned}$$

20. Let $u = y$ and $v' = \frac{1}{\sqrt{5-y}}$, so $u' = 1$ and $v = -2(5-y)^{1/2}$.

$$\int \frac{y}{\sqrt{5-y}}\,dy = -2y(5-y)^{1/2} + 2\int (5-y)^{1/2}\,dy = -2y(5-y)^{1/2} - \frac{4}{3}(5-y)^{3/2} + C.$$

21. $\displaystyle\int \frac{t+7}{\sqrt{5-t}}\,dt = \int \frac{t}{\sqrt{5-t}}\,dt + 7\int (5-t)^{-1/2}\,dt.$

To calculate the first integral, we use integration by parts. Let $u = t$ and $v' = \frac{1}{\sqrt{5-t}}$, so $u' = 1$ and $v = -2(5-t)^{1/2}$. Then

$$\int \frac{t}{\sqrt{5-t}}\,dt = -2t(5-t)^{1/2} + 2\int (5-t)^{1/2}\,dt = -2t(5-t)^{1/2} - \frac{4}{3}(5-t)^{3/2} + C.$$

We can calculate the second integral directly: $7\int (5-t)^{-1/2} = -14(5-t)^{1/2} + C_1$. Thus

$$\int \frac{t+7}{\sqrt{5-t}}\,dt = -2t(5-t)^{1/2} - \frac{4}{3}(5-t)^{3/2} - 14(5-t)^{1/2} + C_2.$$

22. Let $u = (\ln x)^4$ and $v' = x$, so $u' = \frac{4(\ln x)^3}{x}$ and $v = \frac{x^2}{2}$. Then

$$\int x(\ln x)^4\,dx = \frac{x^2(\ln x)^4}{2} - 2\int x(\ln x)^3\,dx.$$

$\int x(\ln x)^3\,dx$ is somewhat less complicated than $\int x(\ln x)^4\,dx$. To calculate it, we again try integration by parts, this time letting $u = (\ln x)^3$ (instead of $(\ln x)^4$) and $v' = x$. We find

$$\int x(\ln x)^3\,dx = \frac{x^2}{2}(\ln x)^3 - \frac{3}{2}\int x(\ln x)^2\,dx.$$

Once again, express the given integral in terms of a less-complicated one. Using integration by parts two more times, we find that

$$\int x(\ln x)^2\,dx = \frac{x^2}{2}(\ln x)^2 - \int x(\ln x)\,dx$$

and that

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$$

Putting this all together, we have

$$\int x(\ln x)^4\,dx = \frac{x^2}{2}(\ln x)^4 - x^2(\ln x)^3 + \frac{3}{2}x^2(\ln x)^2 - \frac{3}{2}x^2\ln x + \frac{3}{4}x^2 + C.$$

23. Let $u = \arcsin w$ and $v' = 1$, so $u' = \frac{1}{\sqrt{1-w^2}}$ and $v = w$. Then

$$\int \arcsin w\, dw = w \arcsin w - \int \frac{w}{\sqrt{1-w^2}}\, dw = w \arcsin w + \sqrt{1-w^2} + C.$$

24. Let $u = \arctan 7z$ and $v' = 1$, so $u' = \frac{7}{1+49z^2}$ and $v = z$. Now $\int \frac{7z\,dz}{1+49z^2}$ can be evaluated by the substitution $w = 1 + 49z^2$, $dw = 98z\,dz$, so

$$\int \frac{7z\,dz}{1+49z^2} = 7\int \frac{\frac{1}{98}\,dw}{w} = \frac{1}{14}\int \frac{dw}{w} = \frac{1}{14}\ln|w| + C = \frac{1}{14}\ln(1+49z^2) + C$$

So

$$\int \arctan 7z\, dz = z \arctan 7z - \frac{1}{14}\ln(1+49z^2) + C.$$

25. This integral can first be simplified by making the substitution $w = x^2$, $dw = 2x\,dx$. Then

$$\int x \arctan x^2\, dx = \frac{1}{2}\int \arctan w\, dw.$$

To evaluate $\int \arctan w\, dw$, we'll use integration by parts. Let $u = \arctan w$ and $v' = 1$, so $u' = \frac{1}{1+w^2}$ and $v = w$. Then

$$\int \arctan w\, dw = w \arctan w - \int \frac{w}{1+w^2}\, dw = w \arctan w - \frac{1}{2}\ln|1+w^2| + C.$$

Since $1 + w^2$ is never negative, we can drop the absolute value signs. Thus, we have

$$\begin{aligned}\int x \arctan x^2\, dx &= \frac{1}{2}\left(x^2 \arctan x^2 - \frac{1}{2}\ln(1+(x^2)^2) + C\right)\\ &= \frac{1}{2}x^2 \arctan x^2 - \frac{1}{4}\ln(1+x^4) + C.\end{aligned}$$

26. Let $u = x^2$ and $v' = xe^{x^2}$, so $u' = 2x$ and $v = \frac{1}{2}e^{x^2}$. Then

$$\int x^3 e^{x^2}\, dx = \frac{1}{2}x^2 e^{x^2} - \int x e^{x^2}\, dx = \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C.$$

Note that we can also do this problem by substitution and integration by parts. If we let $w = x^2$, so $dw = 2x\,dx$, then $\int x^3 e^{x^2}\, dx = \frac{1}{2}\int we^w\, dw$. We could then perform integration by parts on this integral to get the same result.

27. To simplify matters, let us try the substitution $w = x^3$, $dw = 3x^2\,dx$. Then

$$\int x^5 \cos x^3\, dx = \frac{1}{3}\int w \cos w\, dw.$$

Now we integrate by parts. Let $u = w$ and $v' = \cos w$, so $u' = 1$ and $v = \sin w$. Then

$$\begin{aligned}\frac{1}{3}\int w \cos w\, dw &= \frac{1}{3}[w \sin w - \int \sin w\, dw]\\ &= \frac{1}{3}[w \sin w + \cos w] + C\\ &= \frac{1}{3}x^3 \sin x^3 + \frac{1}{3}\cos x^3 + C\end{aligned}$$

28. $\displaystyle\int_1^5 \ln t\, dt = (t \ln t - t)\Big|_1^5 = 5 \ln 5 - 4 \approx 4.047$

29. $\displaystyle\int_3^5 x \cos x\, dx = (\cos x + x \sin x)\Big|_3^5 = \cos 5 + 5 \sin 5 - \cos 3 - 3 \sin 3 \approx -3.944.$

30. We use integration by parts. Let $u = z$ and $v' = e^{-z}$, so $u' = 1$ and $v = -e^{-z}$.

$$\begin{aligned}\text{Then } \int_0^{10} ze^{-z}\,dz &= -ze^{-z}\Big|_0^{10} + \int_0^{10} e^{-z}\,dz \\ &= -10e^{-10} + (-e^{-z})\Big|_0^{10} \\ &= -11e^{-10} + 1 \\ &\approx 0.9995.\end{aligned}$$

31. $\displaystyle\int_1^3 t\ln t\,dt = \left(\frac{1}{2}t^2\ln t - \frac{1}{2}t\right)\Big|_1^3 = \frac{9}{2}\ln 3 - 2 \approx 2.944.$

32. We use integration by parts. Let $u = \arctan y$ and $v' = 1$, so $u' = \frac{1}{1+y^2}$ and $v = y$. Thus

$$\begin{aligned}\int_0^1 \arctan y\,dy &= (\arctan y)y\Big|_0^1 - \int_0^1 \frac{y}{1+y^2}\,dy \\ &= \frac{\pi}{4} - \frac{1}{2}\ln|1+y^2|\Big|_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.439.\end{aligned}$$

33. $\displaystyle\int_0^5 \ln(1+t)\,dt = ((1+t)\ln(1+t) - (1+t))\Big|_0^5 = 6\ln 6 - 5 \approx 5.751.$

34. We use integration by parts. Let $u = \arcsin z$ and $v' = 1$, so $u' = \dfrac{1}{\sqrt{1-z^2}}$ and $v = z$. Then

$$\int_0^1 \arcsin z\,dz = z\arcsin z\Big|_0^1 - \int_0^1 \frac{z}{\sqrt{1-z^2}}\,dz = \frac{\pi}{2} - \int_0^1 \frac{z}{\sqrt{1-z^2}}\,dz.$$

To find $\displaystyle\int_0^1 \frac{z}{\sqrt{1-z^2}}\,dz$, we substitute $w = 1 - z^2$, so $dw = -2z\,dz$.
Then

$$\int_{z=0}^{z=1} \frac{z}{\sqrt{1-z^2}}\,dz = -\frac{1}{2}\int_{w=1}^{w=0} w^{-\frac{1}{2}}\,dw = \frac{1}{2}\int_{w=0}^{w=1} w^{-\frac{1}{2}}\,dw = w^{\frac{1}{2}}\Big|_0^1 = 1.$$

Thus our final answer is $\frac{\pi}{2} - 1 \approx 0.571$.

35. To simplify the integral, we first make the substitution $z = u^2$, so $dz = 2u\,du$. Then

$$\int_{u=0}^{u=1} u\arcsin u^2\,du = \frac{1}{2}\int_{z=0}^{z=1} \arcsin z\,dz.$$

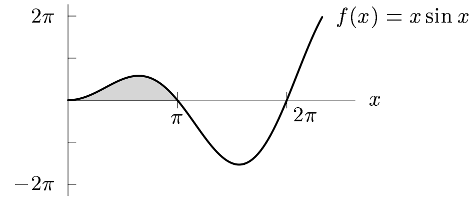
From Problem 34, we know that $\int_0^1 \arcsin z\,dz = \frac{\pi}{2} - 1$. Thus,

$$\int_0^1 u\arcsin u^2\,du = \frac{1}{2}\left(\frac{\pi}{2} - 1\right) \approx 0.285.$$

Problems

36. **(a)** This integral can be evaluated using integration by parts with $u = x$, $v' = \sin x$.
(b) We evaluate this integral using the substitution $w = 1 + x^3$.
(c) We evaluate this integral using the substitution $w = x^2$.
(d) We evaluate this integral using the substitution $w = x^3$.
(e) We evaluate this integral using the substitution $w = 3x + 1$.
(f) This integral can be evaluated using integration by parts with $u = x^2$, $v' = \sin x$.
(g) This integral can be evaluated using integration by parts with $u = \ln x$, $v' = 1$.

37.



The graph of $f(x) = x \sin x$ is shown above. The first positive zero is at $x = \pi$, so, using integration by parts,

$$\begin{aligned}\text{Area } &= \int_0^\pi x \sin x \, dx \\ &= -x \cos x \Big|_0^\pi + \int_0^\pi \cos x \, dx \\ &= -x \cos x \Big|_0^\pi + \sin x \Big|_0^\pi \\ &= -\pi \cos \pi - (-0 \cos 0) + \sin \pi - \sin 0 = \pi.\end{aligned}$$

38. From integration by parts in Problem 11, we obtain

$$\int \sin^2 \theta \, d\theta = -\frac{1}{2} \sin \theta \cos \theta + \frac{1}{2}\theta + C.$$

Using the identity given in the book, we have

$$\int \sin^2 \theta \, d\theta = \int \frac{1 - \cos 2\theta}{2} \, d\theta = \frac{1}{2}\theta - \frac{1}{4} \sin 2\theta + C.$$

Although the answers differ in form, they are really the same, since (by one of the standard double angle formulas) $-\frac{1}{4} \sin 2\theta = -\frac{1}{4}(2 \sin \theta \cos \theta) = -\frac{1}{2} \sin \theta \cos \theta$.

39. Integration by parts: let $u = \cos \theta$ and $v' = \cos \theta$, so $u' = -\sin \theta$ and $v = \sin \theta$.

$$\begin{aligned}\int \cos^2 \theta \, d\theta &= \sin \theta \cos \theta - \int (-\sin \theta)(\sin \theta) \, d\theta \\ &= \sin \theta \cos \theta + \int \sin^2 \theta \, d\theta.\end{aligned}$$

Now use $\sin^2 \theta = 1 - \cos^2 \theta$.

$$\begin{aligned}\int \cos^2 \theta \, d\theta &= \sin \theta \cos \theta + \int (1 - \cos^2 \theta) \, d\theta \\ &= \sin \theta \cos \theta + \int d\theta - \int \cos^2 \theta \, d\theta.\end{aligned}$$

Adding $\int \cos^2 \theta \, d\theta$ to both sides, we have

$$\begin{aligned}2 \int \cos^2 \theta \, d\theta &= \sin \theta \cos \theta + \theta + C \\ \int \cos^2 \theta \, d\theta &= \frac{1}{2} \sin \theta \cos \theta + \frac{1}{2}\theta + C'.\end{aligned}$$

Use the identity $\cos^2 \theta = \frac{1+\cos 2\theta}{2}$.

$$\int \cos^2 \theta \, d\theta = \int \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{1}{2}\theta + \frac{1}{4} \sin 2\theta + C.$$

The only difference is in the two terms $\frac{1}{2} \sin \theta \cos \theta$ and $\frac{1}{4} \sin 2\theta$, but since $\sin 2\theta = 2 \sin \theta \cos \theta$, we have $\frac{1}{4} \sin 2\theta = \frac{1}{4}(2 \sin \theta \cos \theta) = \frac{1}{2} \sin \theta \cos \theta$, so there is no real difference between the formulas.

40. First, let $u = e^x$ and $v' = \sin x$, so $u' = e^x$ and $v = -\cos x$.
Thus $\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx$. To calculate $\int e^x \cos x\,dx$, we again need to use integration by parts. Let $u = e^x$ and $v' = \cos x$, so $u' = e^x$ and $v = \sin x$.
Thus
$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$
This gives
$$\int e^x \sin x\,dx = e^x \sin x - e^x \cos x - \int e^x \sin x\,dx.$$
By adding $\int e^x \sin x\,dx$ to both sides, we obtain
$$2\int e^x \sin x\,dx = e^x(\sin x - \cos x) + C.$$
$$\text{Thus } \int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C.$$
This problem could also be done in other ways; for example, we could have started with $u = \sin x$ and $v' = e^x$ as well.

41. Let $u = e^\theta$ and $v' = \cos\theta$, so $u' = e^\theta$ and $v = \sin\theta$. Then $\int e^\theta \cos\theta\,d\theta = e^\theta \sin\theta - \int e^\theta \sin\theta\,d\theta$.
In Problem 40 we found that $\displaystyle\int e^x \sin x dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.
$$\begin{aligned}\int e^\theta \cos\theta\,d\theta &= e^\theta \sin\theta - \left[\frac{1}{2}e^\theta(\sin\theta - \cos\theta)\right] + C\\ &= \frac{1}{2}e^\theta(\sin\theta + \cos\theta) + C.\end{aligned}$$

42. We integrate by parts. Since in Problem 40 we found that $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x)$, we let $u = x$ and $v' = e^x \sin x$, so $u' = 1$ and $v = \frac{1}{2}e^x(\sin x - \cos x)$.
$$\begin{aligned}\text{Then } \int xe^x \sin x\,dx &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\int e^x(\sin x - \cos x)\,dx\\ &= \frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{2}\int e^x \sin x\,dx + \frac{1}{2}\int e^x \cos x\,dx.\end{aligned}$$
Using Problems 40 and 41, we see that this equals
$$\begin{aligned}&\frac{1}{2}xe^x(\sin x - \cos x) - \frac{1}{4}e^x(\sin x - \cos x) + \frac{1}{4}e^x(\sin x + \cos x) + C\\ &\qquad = \frac{1}{2}xe^x(\sin x - \cos x) + \frac{1}{2}e^x \cos x + C.\end{aligned}$$

43. Again we use Problems 40 and 41. Integrate by parts, letting $u = \theta$ and $v' = e^\theta \cos\theta$, so $u' = 1$ and $v = \frac{1}{2}e^\theta(\sin\theta + \cos\theta)$. Then
$$\begin{aligned}\int \theta e^\theta \cos\theta\,d\theta &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{2}\int e^\theta(\sin\theta + \cos\theta)\,d\theta\\ &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{2}\int e^\theta \sin\theta\,d\theta - \frac{1}{2}\int e^\theta \cos\theta\,d\theta\\ &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{4}e^\theta(\sin\theta - \cos\theta) - \frac{1}{4}(\sin\theta + \cos\theta) + C\\ &= \frac{1}{2}\theta e^\theta(\sin\theta + \cos\theta) - \frac{1}{2}e^\theta \sin\theta + C.\end{aligned}$$

44. We integrate by parts. Since we know what the answer is supposed to be, it's easier to choose u and v'. Let $u = x^n$ and $v' = e^x$, so $u' = nx^{n-1}$ and $v = e^x$. Then
$$\int x^n e^x\,dx = x^n e^x - n\int x^{n-1} e^x\,dx.$$

45. We integrate by parts. Let $u = x^n$ and $v' = \cos ax$, so $u' = nx^{n-1}$ and $v = \frac{1}{a}\sin ax$. Then

$$\begin{aligned}\int x^n \cos ax\,dx &= \frac{1}{a}x^n \sin ax - \int (nx^{n-1})(\frac{1}{a}\sin ax)\,dx \\ &= \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx.\end{aligned}$$

46. We integrate by parts. Let $u = x^n$ and $v' = \sin ax$, so $u' = nx^{n-1}$ and $v = -\frac{1}{a}\cos ax$.

$$\begin{aligned}\text{Then } \int x^n \sin ax\,dx &= -\frac{1}{a}x^n \cos ax - \int (nx^{n-1})(-\frac{1}{a}\cos ax)\,dx \\ &= -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx.\end{aligned}$$

47. We integrate by parts. Since we know what the answer is supposed to be, it's easier to choose u and v'. Let $u = \cos^{n-1} x$ and $v' = \cos x$, so $u' = (n-1)\cos^{n-2} x(-\sin x)$ and $v = \sin x$.

Then

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx \\ &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,dx \\ &= \cos^{n-1} x\sin x - (n-1)\int \cos^n x\,dx + (n-1)\int \cos^{n-2} x\,dx.\end{aligned}$$

Thus, by adding $(n-1)\int \cos^n x\,dx$ to both sides of the equation, we find

$$\begin{aligned}n\int \cos^n x\,dx &= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx, \\ \text{so } \int \cos^n\,dx &= \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.\end{aligned}$$

48. (a) One way to avoid integrating by parts is to take the derivative of the right hand side instead. Since $\int e^{ax}\sin bx\,dx$ is the antiderivative of $e^{ax}\sin bx$,

$$\begin{aligned}e^{ax}\sin bx &= \frac{d}{dx}[e^{ax}(A\sin bx + B\cos bx) + C] \\ &= ae^{ax}(A\sin bx + B\cos bx) + e^{ax}(Ab\cos bx - Bb\sin bx) \\ &= e^{ax}[(aA - bB)\sin bx + (aB + bA)\cos bx].\end{aligned}$$

Thus $aA - bB = 1$ and $aB + bA = 0$. Solving for A and B in terms of a and b, we get

$$A = \frac{a}{a^2+b^2},\ B = -\frac{b}{a^2+b^2}.$$

Thus

$$\int e^{ax}\sin bx = e^{ax}\left(\frac{a}{a^2+b^2}\sin bx - \frac{b}{a^2+b^2}\cos bx\right) + C.$$

(b) If we go through the same process, we find

$$ae^{ax}[(aA - bB)\sin bx + (aB + bA)\cos bx] = e^{ax}\cos bx.$$

Thus $aA - bB = 0$, and $aB + bA = 1$. In this case, solving for A and B yields

$$A = \frac{b}{a^2+b^2},\ B = \frac{a}{a^2+b^2}.$$

Thus $\int e^{ax}\cos bx = e^{ax}(\frac{b}{a^2+b^2}\sin bx + \frac{a}{a^2+b^2}\cos bx) + C$.

49. Since $f'(x) = 2x$, integration by parts tells us that

$$\begin{aligned}\int_0^{10} f(x)g'(x)\,dx &= f(x)g(x)\Big|_0^{10} - \int_0^{10} f'(x)g(x)\,dx \\ &= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx.\end{aligned}$$

We can use left and right Riemann Sums with $\Delta x = 2$ to approximate $\int_0^{10} xg(x)\,dx$:

$$\begin{aligned}\text{Left sum} &\approx 0\cdot g(0)\Delta x + 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x \\ &= (0(2.3) + 2(3.1) + 4(4.1) + 6(5.5) + 8(5.9))\,2 = 205.6. \\ \text{Right sum} &\approx 2\cdot g(2)\Delta x + 4\cdot g(4)\Delta x + 6\cdot g(6)\Delta x + 8\cdot g(8)\Delta x + 10\cdot g(10)\Delta x \\ &= (2(3.1) + 4(4.1) + 6(5.5) + 8(5.9) + 10(6.1))\,2 = 327.6.\end{aligned}$$

A good estimate for the integral is the average of the left and right sums, so

$$\int_0^{10} xg(x)\,dx \approx \frac{205.6 + 327.6}{2} = 266.6.$$

Substituting values for f and g, we have

$$\begin{aligned}\int_0^{10} f(x)g'(x)\,dx &= f(10)g(10) - f(0)g(0) - 2\int_0^{10} xg(x)\,dx \\ &\approx 10^2(6.1) - 0^2(2.3) - 2(266.6) = 76.8 \approx 77.\end{aligned}$$

50. Using integration by parts we have

$$\begin{aligned}\int_0^1 xf''(x)dx &= xf'(x)\Big|_0^1 - \int_0^1 f'(x)dx \\ &= 1\cdot f'(1) - 0\cdot f'(0) - [f(1) - f(0)] \\ &= 2 - 0 - 5 + 6 = 3.\end{aligned}$$

51. **(a)** We have

$$\begin{aligned}F(a) &= \int_0^a x^2e^{-x}\,dx \\ &= -x^2e^{-x}\Big|_0^a + \int_0^a 2xe^{-x}\,dx \\ &= (-x^2e^{-x} - 2xe^{-x})\Big|_0^a + 2\int_0^a e^{-x}\,dx \\ &= (-x^2e^{-x} - 2xe^{-x} - 2e^{-x})\Big|_0^a \\ &= -a^2e^{-a} - 2ae^{-a} - 2e^{-a} + 2.\end{aligned}$$

(b) $F(a)$ is increasing because x^2e^{-x} is positive, so as a increases, the area under the curve from 0 to a also increases and thus the integral increases.

(c) We have $F'(a) = a^2e^{-a}$, so

$$F''(a) = 2ae^{-a} - a^2e^{-a} = a(2-a)e^{-a}.$$

We see that $F''(a) > 0$ for $0 < a < 2$, so F is concave up on this interval.

52. We have

$$\text{Bioavailability} = \int_0^3 15te^{-0.2t}dt.$$

We first use integration by parts to evaluate the indefinite integral of this function. Let $u = 15t$ and $v' = e^{-0.2t}dt$, so $u' = 15dt$ and $v = -5e^{-0.2t}$. Then,

$$\begin{aligned}\int 15te^{-0.2t}dt &= (15t)(-5e^{-0.2t}) - \int(-5e^{-0.2t})(15dt)\\ &= -75te^{-0.2t} + 75\int e^{-0.2t}dt = -75te^{-0.2} - 375e^{-0.2t} + C.\end{aligned}$$

Thus,

$$\int_0^3 15te^{-0.2t}dt = (-75te^{-0.2t} - 375e^{-0.2t})\Big|_0^3 = -329.29 + 375 = 45.71.$$

The bioavailability of the drug over this time interval is 45.71 (ng/ml)-hours.

53. **(a)** Increasing V_0 increases the maximum value of V, since this maximum is V_0. Increasing ω or ϕ does not affect the maximum of V.

(b) Since

$$\frac{dV}{dt} = -\omega V_0\sin(\omega t + \phi),$$

the maximum of dV/dt is ωV_0. Thus, the maximum of dV/dt is increased if V_0 or ω is increased, and is unaffected if ϕ is increased.

(c) The period of $V = V_0\cos(\omega t + \phi)$ is $2\pi/\omega$, so

$$\text{Average value} = \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega}(V_0\cos(\omega t + \phi))^2\,dt.$$

Substituting $x = \omega t + \phi$, we have $dx = \omega dt$. When $t = 0$, $x = \phi$, and when $t = 2\pi/\omega$, $x = 2\pi + \phi$. Thus,

$$\begin{aligned}\text{Average value} &= \frac{\omega}{2\pi}\int_\phi^{2\pi+\phi} V_0^2(\cos x)^2\frac{1}{\omega}\,dx\\ &= \frac{V_0^2}{2\pi}\int_\phi^{2\pi+\phi}(\cos x)^2\,dx.\end{aligned}$$

Using integration by parts and the fact that $\sin^2 x = 1 - \cos^2 x$, we see that

$$\begin{aligned}\text{Average value} &= \frac{V_0^2}{2\pi}\left[\frac{1}{2}(\cos x\sin x + x)\right]_\phi^{2\pi+\phi}\\ &= \frac{V_0^2}{4\pi}\left[\cos(2\pi+\phi)\sin(2\pi+\phi) + (2\pi+\phi) - \cos\phi\sin\phi - \phi\right]\\ &= \frac{V_0^2}{4\pi}\cdot 2\pi = \frac{V_0^2}{2}.\end{aligned}$$

Thus, increasing V_0 increases the average value; increasing ω or ϕ has no effect.

However, it is not in fact necessary to compute the integral to see that ω does not affect the average value, since all ω's dropped out of the average value expression when we made the substitution $x = \omega t + \phi$.

54. **(a)** We know that $\dfrac{dE}{dt} = r$, so the total energy E used in the first T hours is given by $E = \displaystyle\int_0^T te^{-at}\,dt$. We use integration by parts. Let $u = t$, $v' = e^{-at}$. Then $u' = 1$, $v = -\frac{1}{a}e^{-at}$.

$$\begin{aligned}E &= \int_0^T te^{-at}\,dt\\ &= -\frac{t}{a}e^{-at}\Big|_0^T - \int_0^T\left(-\frac{1}{a}e^{-at}\right)\,dt\end{aligned}$$

$$= -\frac{1}{a}Te^{-aT} + \frac{1}{a}\int_0^T e^{-at}\,dt$$
$$= -\frac{1}{a}Te^{-aT} + \frac{1}{a^2}(1-e^{-aT}).$$

(b)

$$\lim_{T\to\infty} E = -\frac{1}{a}\lim_{T\to\infty}\left(\frac{T}{e^{aT}}\right) + \frac{1}{a^2}\left(1 - \lim_{T\to\infty}\frac{1}{e^{aT}}\right).$$

Since $a > 0$, the second limit on the right hand side in the above expression is 0. In the first limit, although both the numerator and the denominator go to infinity, the denominator e^{aT} goes to infinity more quickly than T does. So in the end the denominator e^{aT} is much greater than the numerator T. Hence $\lim_{T\to\infty}\frac{T}{e^{aT}} = 0$. (You can check this by graphing $y = \frac{T}{e^{aT}}$ on a calculator or computer for some values of a.) Thus $\lim_{T\to\infty} E = \frac{1}{a^2}$.

55. (a) We want to compute C_1, with $C_1 > 0$, such that

$$\int_0^1 (\Psi_1(x))^2\,dx = \int_0^1 (C_1\sin(\pi x))^2\,dx = C_1^2\int_0^1 \sin^2(\pi x)\,dx = 1.$$

We use integration by parts with $u = v' = \sin(\pi x)$.
So $u' = \pi\cos(\pi x)$ and $v = -\frac{1}{\pi}\cos(\pi x)$. Thus

$$\int_0^1 \sin^2(\pi x)\,dx = -\frac{1}{\pi}\sin(\pi x)\cos(\pi x)\Big|_0^1 + \int_0^1 \cos^2(\pi x)\,dx$$
$$= -\frac{1}{\pi}\sin(\pi x)\cos(\pi x)\Big|_0^1 + \int_0^1 (1-\sin^2(\pi x))\,dx.$$

Moving $\int_0^1 \sin^2(\pi x)\,dx$ from the right side to the left side of the equation and solving, we get

$$2\int_0^1 \sin^2(\pi x)\,dx = -\frac{1}{\pi}\sin(\pi x)\cos(\pi x)\Big|_0^1 + \int_0^1 1\,dx = 0 + 1 = 1,$$

so

$$\int_0^1 \sin^2(\pi x)\,dx = \frac{1}{2}.$$

Thus, we have

$$\int_0^1 (\Psi_1(x))^2\,dx = C_1^2\int_0^1 \sin^2(\pi x)\,dx = \frac{C_1^2}{2}.$$

So, to normalize Ψ_1, we take $C_1 > 0$ such that

$$\frac{C_1^2}{2} = 1 \quad\text{so}\quad C_1 = \sqrt{2}.$$

(b) To normalize Ψ_n, we want to compute C_n, with $C_n > 0$, such that

$$\int_0^1 (\Psi_n(x))^2\,dx = C_n^2\int_0^1 \sin^2(n\pi x)\,dx = 1.$$

The solution to part (a) shows us that

$$\int \sin^2(\pi t)\,dt = -\frac{1}{2\pi}\sin(\pi t)\cos(\pi t) + \frac{1}{2}\int 1\,dt.$$

In the integral for Ψ_n, we make the substitution $t = nx$, so $dx = \frac{1}{n}dt$. Since $t = 0$ when $x = 0$ and $t = n$ when $x = 1$, we have

$$\int_0^1 \sin^2(n\pi x)\,dx = \frac{1}{n}\int_0^n \sin^2(\pi t)\,dt$$
$$= \frac{1}{n}\left(-\frac{1}{2\pi}\sin(\pi t)\cos(\pi t)\Big|_0^n + \frac{1}{2}\int_0^n 1\,dt\right)$$
$$= \frac{1}{n}\left(0 + \frac{n}{2}\right) = \frac{1}{2}.$$

Thus, we have

$$\int_0^1 (\Psi_n(x))^2\, dx = C_n^2 \int_0^1 \sin^2(n\pi x)\, dx = \frac{C_n^2}{2}.$$

So to normalize Ψ_n, we take C_n such that

$$\frac{C_n^2}{2} = 1 \quad \text{so} \quad C_n = \sqrt{2}.$$

Solutions for Section 7.3

Exercises

1. $\frac{1}{10}e^{(-3\theta)}(-3\cos\theta + \sin\theta) + C.$
(Let $a = -3, b = 1$ in II-9.)

2. $\frac{1}{6}x^6 \ln x - \frac{1}{36}x^6 + C.$ (Let $n = 5$ in III-13.)

3. The integrand, a polynomial, x^3, multiplied by $\sin 5x$, is in the form of III-15. There are only three successive derivatives of x^3 before 0 is reached (namely, $3x^2$, $6x$, and 6), so there will be four terms. The signs in the terms will be $-++-$, as given in III-15, so we get

$$\int x^3 \sin 5x\, dx = -\frac{1}{5}x^3\cos 5x + \frac{1}{25}\cdot 3x^2 \sin 5x + \frac{1}{125}\cdot 6x\cos 5x - \frac{1}{625}\cdot 6\sin 5x + C.$$

4. Formula III-13 applies only to functions of the form $x^n \ln x$, so we'll have to multiply out and separate into two integrals.

$$\int (x^2+3)\ln x\, dx = \int x^2 \ln x\, dx + 3\int \ln x\, dx.$$

Now we can use formula III-13 on each integral separately, to get

$$\int (x^2+3)\ln x\, dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + 3(x\ln x - x) + C.$$

5. Note that you can't use substitution here: letting $w = x^3+5$ doesn't work, since there is no $dw = 3x^2\, dx$ in the integrand. What will work is simply multiplying out the square: $(x^3+5)^2 = x^6 + 10x^3 + 25$. Then use I-1:

$$\int (x^3+5)^2\, dx = \int x^6\, dx + 10\int x^3\, dx + 25\int 1\, dx = \frac{1}{7}x^7 + 10\cdot\frac{1}{4}x^4 + 25x + C.$$

6. $-\frac{1}{5}\cos^5 w + C$
(Let $x = \cos w$, as suggested in IV-23. Then $-\sin w\, dw = dx$, and $\int \sin w \cos^4 w\, dw = -\int x^4\, dx$.)

7. $-\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C.$
(Use IV-17.)

8. $\frac{1}{\sqrt{3}}\arctan\frac{y}{\sqrt{3}} + C.$
(Let $a = \sqrt{3}$ in V-24).

9. Let $m = 3$ in IV-21.

$$\begin{aligned}\int \frac{1}{\cos^3 x}\, dx &= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\int \frac{1}{\cos x}\, dx \\ &= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{4}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C \text{ by IV-22.}\end{aligned}$$

10. $\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right)e^{2x} + C.$
(Let $a = 2, p(x) = x^3$ in III-14.)

11. $\frac{5}{16}\sin 3\theta \sin 5\theta + \frac{3}{16}\cos 3\theta \cos 5\theta + C.$
(Let $a = 3, b = 5$ in II-12.)

12. $\frac{3}{16}\cos 3\theta \sin 5\theta - \frac{5}{16}\sin 3\theta \cos 5\theta + C.$
(Let $a = 3, b = 5$ in II-10.)

13. $\left(\frac{1}{3}x^2 - \frac{2}{9}x + \frac{2}{27}\right)e^{3x} + C.$
(Let $a = 3, p(x) = x^2$ in III-14.)

14. $\frac{1}{3}e^{x^3} + C.$
(Substitute $w = x^3, dw = 3x^2\,dx$. It isn't necessary to use the table.)

15. $\left(\frac{1}{3}x^4 - \frac{4}{9}x^3 + \frac{4}{9}x^2 - \frac{8}{27}x + \frac{8}{81}\right)e^{3x} + C.$
(Let $a = 3, p(x) = x^4$ in III-14.)

16. Substitute $w = 5u,\ dw = 5\,du$. Then

$$\begin{aligned}\int u^5 \ln(5u)\,du &= \frac{1}{5^6}\int w^5 \ln w\,dw \\ &= \frac{1}{5^6}\left(\frac{1}{6}w^6 \ln w - \frac{1}{36}w^6 + C\right) \\ &= \frac{1}{6}u^6 \ln 5u - \frac{1}{36}u^6 + C.\end{aligned}$$

Or use $\ln 5u = \ln 5 + \ln u$.

$$\begin{aligned}\int u^5 \ln 5u\,du &= \ln 5 \int u^5\,du + \int u^5 \ln u\,du \\ &= \frac{u^6}{6}\ln 5 + \frac{1}{6}u^6 \ln u - \frac{1}{36}u^6 + C \quad \text{(using III-13)} \\ &= \frac{u^6}{6}\ln 5u - \frac{1}{36}u^6 + C.\end{aligned}$$

17. Use long division to reorganize the integral:

$$\int \frac{t^2+1}{t^2-1}\,dt = \int \left(1 + \frac{2}{t^2-1}\right)\,dt = \int dt + \int \frac{2}{(t-1)(t+1)}\,dt.$$

To get this second integral, let $a = 1, b = -1$ in V-26, so

$$\int \frac{t^2+1}{t^2-1}\,dt = t + \ln|t-1| - \ln|t+1| + C.$$

18. Substitute $w = x^2, dw = 2x\,dx$. Then $\int x^3 \sin x^2\,dx = \frac{1}{2}\int w \sin w\,dw$. By III-15, we have

$$\int w \sin w\,dw = -\frac{1}{2}w\cos w + \frac{1}{2}\sin w + C = -\frac{1}{2}x^2\cos x^2 + \frac{1}{2}\sin x^2 + C.$$

19. $\frac{1}{45}(7\cos 2y \sin 7y - 2\sin 2y \cos 7y) + C.$
(Let $a = 2, b = 7$ in II-11.)

20.

$$\begin{aligned}\int y^2 \sin 2y\,dy &= -\frac{1}{2}y^2\cos 2y + \frac{1}{4}(2y)\sin 2y + \frac{1}{8}(2)\cos 2y + C \\ &= -\frac{1}{2}y^2\cos 2y + \frac{1}{2}y\sin 2y + \frac{1}{4}\cos 2y + C.\end{aligned}$$

(Use $a = 2, p(y) = y^2$ in III-15.)

21. $\frac{1}{34}e^{5x}(5\sin 3x - 3\cos 3x) + C.$
(Let $a = 5, b = 3$ in II-8.)

22. Use IV-21 twice to get the exponent down to 1:

$$\int \frac{1}{\cos^5 x}\,dx = \frac{1}{4}\frac{\sin x}{\cos^4 x} + \frac{3}{4}\int \frac{1}{\cos^3 x}\,dx$$
$$\int \frac{1}{\cos^3 x}\,dx = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{2}\int \frac{1}{\cos x}\,dx.$$

Now use IV-22 to get

$$\int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + C.$$

Putting this all together gives

$$\int \frac{1}{\cos^5 x}\,dx = \frac{1}{4}\frac{\sin x}{\cos^4 x} + \frac{3}{8}\frac{\sin x}{\cos^2 x} + \frac{3}{16}\ln\left|\frac{(\sin x)+1}{(\sin x)-1}\right| + C.$$

23. Substitute $w = 2\theta,\ dw = 2\,d\theta$. Then use IV-19, letting $m = 2$.

$$\int \frac{1}{\sin^2 2\theta}\,d\theta = \frac{1}{2}\int \frac{1}{\sin^2 w}\,dw = \frac{1}{2}(-\frac{\cos w}{\sin w}) + C = -\frac{1}{2\tan w} + C = -\frac{1}{2\tan 2\theta} + C.$$

24. Substitute $w = 3\theta,\ dw = 3\,d\theta$. Then use IV-19, letting $m = 3$.

$$\begin{aligned}\int \frac{1}{\sin^3 3\theta}\,d\theta = \frac{1}{3}\int \frac{1}{\sin^3 w}\,dw &= \frac{1}{3}\left[-\frac{1}{2}\frac{\cos w}{\sin^2 w} + \frac{1}{2}\int \frac{1}{\sin w}\,dw\right]\\ &= -\frac{1}{6}\frac{\cos w}{\sin^2 w} + \frac{1}{6}\left[\frac{1}{2}\ln\left|\frac{\cos(w)-1}{\cos(w)+1}\right| + C\right] \text{ by IV-20}\\ &= -\frac{1}{6}\frac{\cos 3\theta}{\sin^2 3\theta} + \frac{1}{12}\ln\left|\frac{\cos(3\theta)-1}{\cos(3\theta)+1}\right| + C.\end{aligned}$$

25. Substitute $w = 7x,\ dw = 7\,dx$. Then use IV-21.

$$\begin{aligned}\int \frac{1}{\cos^4 7x}\,dx = \frac{1}{7}\int \frac{1}{\cos^4 w}\,dw &= \frac{1}{7}\left[\frac{1}{3}\frac{\sin w}{\cos^3 w} + \frac{2}{3}\int \frac{1}{\cos^2 w}\,dw\right]\\ &= \frac{1}{21}\frac{\sin w}{\cos^3 w} + \frac{2}{21}\left[\frac{\sin w}{\cos w} + C\right]\\ &= \frac{1}{21}\frac{\tan w}{\cos^2 w} + \frac{2}{21}\tan w + C\\ &= \frac{1}{21}\frac{\tan 7x}{\cos^2 7x} + \frac{2}{21}\tan 7x + C.\end{aligned}$$

26.

$$\int \frac{1}{x^2+4x+3}\,dx = \int \frac{1}{(x+1)(x+3)}\,dx = \frac{1}{2}(\ln|x+1| - \ln|x+3|) + C.$$

(Let $a = -1$ and $b = -3$ in V-26).

27. Using the advice in IV-23, since both m and n are even and since n is negative, we convert everything to cosines, since $\cos x$ is in the denominator.

$$\begin{aligned}\int \tan^4 x\,dx &= \int \frac{\sin^4 x}{\cos^4 x}\,dx\\ &= \int \frac{(1-\cos^2 x)^2}{\cos^4 x}\,dx\\ &= \int \frac{1}{\cos^4 x}\,dx - 2\int \frac{1}{\cos^2 x}\,dx + \int 1\,dx.\end{aligned}$$

By IV-21

$$\int \frac{1}{\cos^4 x}\,dx = \frac{1}{3}\frac{\sin x}{\cos^3 x} + \frac{2}{3}\int \frac{1}{\cos^2 x}\,dx,$$
$$\int \frac{1}{\cos^2 x}\,dx = \frac{\sin x}{\cos x} + C.$$

Substituting back in, we get

$$\int \tan^4 x\,dx = \frac{1}{3}\frac{\sin x}{\cos^3 x} - \frac{4}{3}\frac{\sin x}{\cos x} + x + C.$$

28.

$$\int \frac{dz}{z(z-3)} = -\frac{1}{3}(\ln|z| - \ln|z-3|) + C.$$

(Let $a = 0, b = 3$ in V-26.)

29.

$$\int \frac{dy}{4-y^2} = -\int \frac{dy}{(y+2)(y-2)} = -\frac{1}{4}(\ln|y-2| - \ln|y+2|) + C.$$

(Let $a = 2, b = -2$ in V-26.)

30. $\arctan(z+2) + C$.
(Substitute $w = z + 2$ and use V-24, letting $a = 1$.)

31.

$$\int \frac{1}{y^2+4y+5}\,dy = \int \frac{1}{1+(y+2)^2}\,dy = \arctan(y+2) + C.$$

(Substitute $w = y + 2$, and let $a = 1$ in V-24).

32.

$$\int \frac{1}{x^2+4x+4}\,dx = \int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2} + C.$$

You need not use the table.

33. We use the Pythagorean Identity to change the integrand in the following manner:

$$\sin^3 x = \left(\sin^2 x\right)\sin x = \left(1-\cos^2 x\right)\sin x = \sin x - \cos^2 x \sin x.$$

Thus, we have

$$\begin{aligned}\int \sin^3 x\,dx &= \int \left(\sin x - \cos^2 x \sin x\right)dx \\ &= \int \sin x\,dx - \int \cos^2 x \sin x\,dx.\end{aligned}$$

The first of these new integrals can be easily found. The second can be found using the substitution $w = \cos x$ so $dw = -\sin x\,dx$. The second integral becomes

$$\begin{aligned}\int \cos^2 x \sin x\,dx &= -\int w^2 dw \\ &= -\frac{1}{3}w^3 + C \\ &= -\frac{1}{3}\cos^3 x + C\end{aligned}$$

and so our final answer is

$$\begin{aligned}\int \sin^3 x\,dx &= \int \sin x\,dx - \int \cos^2 x \sin x\,dx \\ &= -\cos x + (1/3)\cos^3 x + C.\end{aligned}$$

34.

$$\begin{aligned}\int \sin^3 3\theta \cos^2 3\theta \, d\theta &= \int (\sin 3\theta)(\cos^2 3\theta)(1-\cos^2 3\theta)\, d\theta \\ &= \int \sin 3\theta(\cos^2 3\theta - \cos^4 3\theta)\, d\theta.\end{aligned}$$

Using an extension of the tip given in rule IV-23, we let $w = \cos 3\theta,\ dw = -3\sin 3\theta\, d\theta$.

$$\begin{aligned}\int \sin 3\theta(\cos^2 3\theta - \cos^4 3\theta)\, d\theta &= -\frac{1}{3}\int (w^2 - w^4)\, dw \\ &= -\frac{1}{3}(\frac{w^3}{3} - \frac{w^5}{5}) + C \\ &= -\frac{1}{9}(\cos^3 3\theta) + \frac{1}{15}(\cos^5 3\theta) + C.\end{aligned}$$

35. If we make the substitution $w = 2z^2$ then $dw = 4z\, dz$, and the integral becomes:

$$\int z e^{2z^2}\cos(2z^2)\, dz = \frac{1}{4}\int e^w \cos w\, dw$$

Now we can use Formula 9 from the table of integrals to get:

$$\begin{aligned}\frac{1}{4}\int e^w \cos w\, dw &= \frac{1}{4}\left[\frac{1}{2}e^w(\cos w + \sin w) + C\right] \\ &= \frac{1}{8}e^w(\cos w + \sin w) + C \\ &= \frac{1}{8}e^{2z^2}(\cos 2z^2 + \sin 2z^2) + C\end{aligned}$$

Problems

36. Using II-10 in the integral table, if $m \neq \pm n$, then

$$\begin{aligned}\int_{-\pi}^{\pi} \sin m\theta \sin n\theta\, d\theta &= \frac{1}{n^2 - m^2}[m\cos m\theta \sin n\theta - n \sin m\theta \cos n\theta]\Big|_{-\pi}^{\pi} \\ &= \frac{1}{n^2-m^2}[(m\cos m\pi \sin n\pi - n\sin m\pi \cos n\pi) - \\ &\quad (m\cos(-m\pi)\sin(-n\pi) - n\sin(-m\pi)\cos(-n\pi))]\end{aligned}$$

But $\sin k\pi = 0$ for all integers k, so each term reduces to 0, making the whole integral reduce to 0.

37. Using formula II-11, if $m \neq \pm n$, then

$$\int_{-\pi}^{\pi} \cos m\theta \cos n\theta\, d\theta = \frac{1}{n^2-m^2}(n\cos m\theta \sin n\theta - m\sin m\theta \cos n\theta)\Big|_{-\pi}^{\pi}.$$

We see that in the evaluation, each term will have a $\sin k\pi$ term, so the expression reduces to 0.

38. (a)

$$\begin{aligned}\frac{1}{1-0}\int_0^1 V_0\cos(120\pi t)dt &= \frac{V_0}{120\pi}\sin(120\pi t)\Big|_0^1 \\ &= \frac{V_0}{120\pi}[\sin(120\pi) - \sin(0)] \\ &= \frac{V_0}{120\pi}[0-0] = 0.\end{aligned}$$

(b) Let's find the average of V^2 first.

$$\overline{V}^2 = \text{Average of } V^2 = \frac{1}{1-0}\int_0^1 V^2 dt$$
$$= \frac{1}{1-0}\int_0^1 (V_0\cos(120\pi t))^2 dt$$
$$= V_0^2\int_0^1 \cos^2(120\pi t)dt$$

Now, let $120\pi t = x$, and $dt = \dfrac{dx}{120\pi}$. So

$$\overline{V}^2 = \frac{V_0^2}{120\pi}\int_0^{120\pi}\cos^2 x dx.$$
$$= \frac{V_0^2}{120\pi}\left(\frac{1}{2}\cos x\sin x + \frac{1}{2}x\right)\Bigg|_0^{120\pi} \quad \text{II-18}$$
$$= \frac{V_0^2}{120\pi}60\pi = \frac{V_0^2}{2}.$$

So, the average of V^2 is $\dfrac{V_0^2}{2}$ and $\overline{V} = \sqrt{\text{average of } V^2} = \dfrac{V_0}{\sqrt{2}}$.

(c) $V_0 = \sqrt{2}\cdot\overline{V} = 110\sqrt{2} \approx 156$ volts.

39. (a) Since $R(T)$ is the rate or production, we find the total production by integrating:

$$\int_0^N R(t)\,dt = \int_0^N (A + Be^{-t}\sin(2\pi t))\,dt$$
$$= NA + B\int_0^N e^{-t}\sin(2\pi t)\,dt.$$

Let $a = -1$ and $b = 2\pi$ in II-8.

$$= NA + \frac{B}{1+4\pi^2}e^{-t}(-\sin(2\pi t) - 2\pi\cos(2\pi t))\Bigg|_0^N.$$

Since N is an integer (so $\sin 2\pi N = 0$ and $\cos 2\pi N = 1$),

$$\int_0^N R(t)\,dt = NA + B\frac{2\pi}{1+4\pi^2}(1-e^{-N}).$$

Thus the total production is $NA + \frac{2\pi B}{1+4\pi^2}(1-e^{-N})$ over the first N years.

(b) The average production over the first N years is

$$\int_0^N \frac{R(t)\,dt}{N} = A + \frac{2\pi B}{1+4\pi^2}\left(\frac{1-e^{-N}}{N}\right).$$

(c) As $N\to\infty$, $A + \frac{2\pi B}{1+4\pi^2}\frac{1-e^{-N}}{N} \to A$, since the second term in the sum goes to 0. This is why A is called the average!

(d) When t gets large, the term $Be^{-t}\sin(2\pi t)$ gets very small. Thus, $R(t)\approx A$ for most t, so it makes sense that the average of $\int_0^N R(t)\,dt$ is A as $N\to\infty$.

(e) This model is not reasonable for long periods of time, since an oil well has finite capacity and will eventually "run dry." Thus, we cannot expect average production to be close to constant over a long period of time.

40. We want to calculate

$$\int_0^1 C_n\sin(n\pi x)\cdot C_m\sin(m\pi x)\,dx.$$

We use II-11 from the table of integrals with $a = n\pi$, $b = m\pi$. Since $n \neq m$, we see that

$$\begin{aligned}
\int_0^1 \Psi_n(x)\cdot\Psi_m(x)\,dx &= C_nC_m\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx \\
&= \frac{C_nC_m}{m^2\pi^2-n^2\pi^2}\left(n\pi\cos(n\pi x)\sin(m\pi x) - \; m\pi\sin(n\pi x)\cos(m\pi x)\right)\Big|_0^1 \\
&= \frac{C_nC_m}{(m^2-n^2)\pi^2}\Big(n\pi\cos(n\pi)\sin(m\pi) - m\pi\sin(n\pi)\cos(m\pi) \\
&\qquad -n\pi\cos(0)\sin(0) + m\pi\sin(0)\cos(0)\Big) \\
&= 0
\end{aligned}$$

since $\sin(0) = \sin(n\pi) = \sin(m\pi) = 0$.

Solutions for Section 7.4

Exercises

1. Since $25 - x^2 = (5-x)(5+x)$, we take

$$\frac{20}{25-x^2} = \frac{A}{5-x} + \frac{B}{5+x}.$$

So,

$$\begin{aligned}
20 &= A(5+x) + B(5-x) \\
20 &= (A-B)x + 5A + 5B,
\end{aligned}$$

giving

$$\begin{aligned}
A - B &= 0 \\
5A + 5B &= 20.
\end{aligned}$$

Thus $A = B = 2$ and

$$\frac{20}{25-x^2} = \frac{2}{5-x} + \frac{2}{5+x}.$$

2. Since $6x + x^2 = x(6+x)$, we take

$$\frac{x+1}{6x+x^2} = \frac{A}{x} + \frac{B}{6+x}.$$

So,

$$\begin{aligned}
x + 1 &= A(6+x) + Bx \\
x + 1 &= (A+B)x + 6A,
\end{aligned}$$

giving

$$\begin{aligned}
A + B &= 1 \\
6A &= 1.
\end{aligned}$$

Thus $A = 1/6$, and $B = 5/6$ so

$$\frac{x+1}{6x+x^2} = \frac{1/6}{x} + \frac{5/6}{6+x}.$$

3. Since $y^3 - 4y = y(y-2)(y+2)$, we take

$$\frac{8}{y^3-4y} = \frac{A}{y} + \frac{B}{y-2} + \frac{C}{y+2}.$$

So,

$$\begin{aligned} 8 &= A(y-2)(y+2) + By(y+2) + Cy(y-2) \\ 8 &= (A+B+C)y^2 + (2B-2C)y - 4A, \end{aligned}$$

giving

$$\begin{aligned} A+B+C &= 0 \\ 2B-2C &= 0 \\ -4A &= 8. \end{aligned}$$

Thus $A = -2$, $B = C = 1$ so

$$\frac{8}{y^3-4y} = \frac{-2}{y} + \frac{1}{y-2} + \frac{1}{y+2}.$$

4. Since $y^3 - y^2 + y - 1 = (y-1)(y^2+1)$, we take

$$\frac{2y}{y^3-y^2+y-1} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$$

So,

$$\begin{aligned} 2y &= A(y^2+1) + (By+C)(y-1) \\ 2y &= (A+B)y^2 + (C-B)y + A - C, \end{aligned}$$

giving

$$\begin{aligned} A+B &= 0 \\ -B+C &= 2 \\ A-C &= 0. \end{aligned}$$

Thus $A = C = 1$, $B = -1$ so

$$\frac{2y}{y^3-y^2+y-1} = \frac{1}{y-1} + \frac{1-y}{y^2+1}.$$

5. Using the result of Problem 1, we have

$$\int \frac{20}{25-x^2}\,dx = \int \frac{2}{5-x}\,dx + \int \frac{2}{5+x}\,dx = -2\ln|5-x| + 2\ln|5+x| + C.$$

6. Using the result of Problem 2, we have

$$\int \frac{x+1}{6x+x^2}\,dx = \int \frac{1/6}{x}\,dx + \int \frac{5/6}{6+x}\,dx = \frac{1}{6}\left(\ln|x| + 5\ln|6+x|\right) + C.$$

7. Using the result of Problem 3, we have

$$\int \frac{8}{y^3-4y}\,dy = \int \frac{-2}{y}\,dy + \int \frac{1}{y-2}\,dy + \int \frac{1}{y+2}\,dy = -2\ln|y| + \ln|y-2| + \ln|y+2| + C.$$

8. Using the result of Problem 4, we have

$$\int \frac{2y}{y^3-y^2+y-1}\,dy = \int \frac{1}{y-1}\,dy + \int \frac{1-y}{y^2+1}\,dy = \ln|y-1| + \arctan y - \frac{1}{2}\ln\left|y^2+1\right| + C.$$

9. **(a)** Yes, use $x = 3\sin\theta$.
(b) No; better to substitute $w = 9 - x^2$, so $dw = -2x\,dx$.

10. We let
$$\frac{3x^2-8x+1}{x^3-4x^2+x+6} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$
giving
$$\begin{aligned} 3x^2-8x+1 &= A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1) \\ 3x^2-8x+1 &= (A+B+C)x^2 - (2A+5B+C)x - 3A + 6B - 2C \end{aligned}$$
so
$$\begin{aligned} A+B+C &= 3 \\ -2A-5B-C &= -8 \\ -3A+6B-2C &= 1. \end{aligned}$$
Thus, $A = B = C = 1$, so
$$\int \frac{3x^2-8x+1}{x^3-4x^2+x+6}\,dx = \int \frac{dx}{x-2} + \int \frac{dx}{x+1} + \int \frac{dx}{x-3} = \ln|x-2| + \ln|x+1| + \ln|x-3| + K.$$

11. We let
$$\frac{1}{x^3-x^2} = \frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$
giving
$$\begin{aligned} 1 &= Ax(x-1) + B(x-1) + Cx^2 \\ 1 &= (A+C)x^2 + (B-A)x - B \end{aligned}$$
so
$$\begin{aligned} A+C &= 0 \\ B-A &= 0 \\ -B &= 1. \end{aligned}$$
Thus, $A = B = -1$, $C = 1$, so
$$\int \frac{dx}{x^3-x^2} = -\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{dx}{x-1} = -\ln|x| + x^{-1} + \ln|x-1| + K.$$

12. We let
$$\frac{10x+2}{x^3-5x^2+x-5} = \frac{10x+2}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$
giving
$$\begin{aligned} 10x+2 &= A(x^2+1) + (Bx+C)(x-5) \\ 10x+2 &= (A+B)x^2 + (C-5B)x + A - 5C \end{aligned}$$
so
$$\begin{aligned} A+B &= 0 \\ C-5B &= 10 \\ A-5C &= 2. \end{aligned}$$
Thus, $A = 2$, $B = -2$, $C = 0$, so
$$\int \frac{10x+2}{x^3-5x^2+x-5}\,dx = \int \frac{2}{x-5}\,dx - \int \frac{2x}{x^2+1}\,dx = 2\ln|x-5| - \ln\left|x^2+1\right| + K.$$

13. Division gives
$$\frac{x^4+12x^3+15x^2+25x+11}{x^3+12x^2+11x} = x + \frac{4x^2+25x+11}{x^3+12x^2+11x}.$$
Since $x^3+12x^2+11x = x(x+1)(x+11)$, we write
$$\frac{4x^2+25x+11}{x^3+12x^2+11x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+11}$$
giving
$$\begin{aligned}4x^2+25x+11 &= A(x+1)(x+11) + Bx(x+11) + Cx(x+1)\\ 4x^2+25x+11 &= (A+B+C)x^2 + (12A+11B+C)x + 11A\end{aligned}$$
so
$$\begin{aligned}A+B+C &= 4\\ 12A+11B+C &= 25\\ 11A &= 11.\end{aligned}$$
Thus, $A = B = 1$, $C = 2$ so
$$\begin{aligned}\int \frac{x^4+12x^3+15x^2+25x+11}{x^3+12x^2+11x}\,dx &= \int x\,dx + \int \frac{dx}{x} + \int \frac{dx}{x+1} + \int \frac{2dx}{x+11}\\ &= \frac{x^2}{2} + \ln|x| + \ln|x+1| + 2\ln|x+11| + K.\end{aligned}$$

14. Division gives
$$\frac{x^4+3x^3+2x^2+1}{x^2+3x+2} = x^2 + \frac{1}{x^2+3x+2}.$$
Since $x^2+3x+2 = (x+1)(x+2)$, we write
$$\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2},$$
giving
$$\begin{aligned}1 &= A(x+2) + B(x+1)\\ 1 &= (A+B)x + 2A + B\end{aligned}$$
so
$$\begin{aligned}A+B &= 0\\ 2A+B &= 1.\end{aligned}$$
Thus, $A = 1$, $B = -1$ so
$$\begin{aligned}\int \frac{x^4+3x^3+2x^2+1}{x^2+3x+2}\,dx &= \int x^2\,dx + \int \frac{dx}{x+1} - \int \frac{dx}{x+2}\\ &= \frac{x^3}{3} + \ln|x+1| - \ln|x+2| + C.\end{aligned}$$

15. Since $x = (3/2)\sin t$, we have $dx = (3/2)\cos t\,dt$. Substituting into the integral gives
$$\int \frac{1}{\sqrt{9-4x^2}} = \int \frac{1}{\sqrt{9-9\sin^2 t}}\left(\frac{3}{2}\cos t\right)dt = \int \frac{1}{2}dt = \frac{1}{2}t + C = \frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + C.$$

16. Completing the square gives $x^2+4x+5 = 1 + (x+2)^2$. Since $x+2 = \tan t$ and $dx = (1/\cos^2 t)dt$, we have
$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{1+\tan^2 t}\cdot\frac{1}{\cos^2 t}dt = \int dt = t + C = \arctan(x+2) + C.$$

17. Since $x = \sin t + 2$, we have

$$4x - 3 - x^2 = 4(\sin t + 2) - 3 - (\sin t + 2)^2 = 1 - \sin^2 t = \cos^2 t$$

and $dx = \cos t\,dt$, so substitution gives

$$\int \frac{1}{\sqrt{4x - 3 - x^2}} = \int \frac{1}{\sqrt{\cos^2 t}} \cos t\,dt = \int dt = t + C = \arcsin(x - 2) + C.$$

18. **(a)** Substitute $w = x^2 + 10$, so $dw = 2x\,dx$.
(b) Substitute $x = \sqrt{10}\tan\theta$.

Problems

19. Since $x^2 + 6x + 9$ is a perfect square, we write

$$\int \frac{1}{x^2 + 6x + 25} dx = \int \frac{1}{(x^2 + 6x + 9) + 16} dx = \int \frac{1}{(x + 3)^2 + 16} dx.$$

We use the trigonometric substitution $x + 3 = 4\tan\theta$, so $x = 4\tan\theta - 3$.

20. Since $y^2 + 3y + 3 = (y + 3/2)^2 + (3 - 9/4) = (y + 3/2)^2 + 3/4$, we have

$$\int \frac{dy}{y^2 + 3y + 3} = \int \frac{dy}{(y + 3/2)^2 + 3/4}.$$

Substitute $y + 3/2 = \tan\theta$, so $y = (\tan\theta) - 3/2$.

21. Since $x^2 + 2x + 2 = (x + 1)^2 + 1$, we have

$$\int \frac{1}{x^2 + 2x + 2}\,dx = \int \frac{1}{(x + 1)^2 + 1} dx.$$

Substitute $x + 1 = \tan\theta$, so $x = (\tan\theta) - 1$.

22. Since $x^2 + 2x + 2 = (x + 1)^2 + 1$, we have

$$\int \frac{x + 1}{x^2 + 2x + 2}\,dx = \int \frac{x + 1}{(x + 1)^2 + 1} dx.$$

Substitute $w = (x + 1)^2$, so $dw = 2(x + 1)\,dx$.

This integral can also be calculated without completing the square, by substituting $w = x^2 + 2x + 2$, so $dw = 2(x + 1)\,dx$.

23. Since $2z - z^2 = 1 - (z - 1)^2$, we have

$$\int \frac{4}{\sqrt{2z - z^2}}\,dz = 4\int \frac{1}{\sqrt{1 - (z - 1)^2}}\,dz.$$

Substitute $z - 1 = \sin\theta$, so $z = (\sin\theta) + 1$.

24. Since $2z - z^2 = 1 - (z - 1)^2$, we have

$$\int \frac{z - 1}{\sqrt{2z - z^2}}\,dz = \int \frac{z - 1}{\sqrt{1 - (z - 1)^2}}\,dz.$$

Substitute $w = 1 - (z - 1)^2$, so $dw = -2(z - 1)\,dz$.

25. Since $t^2 + 4t + 7 = (t + 2)^2 + 3$, we have

$$\int (t + 2)\sin(t^2 + 4t + 7)\,dt = \int (t + 2)\sin((t + 2)^2 + 3)\,dt.$$

Substitute $w = (t + 2)^2 + 3$, so $dw = 2(t + 2)\,dt$.

This integral can also be computed without completing the square, by substituting $w = t^2 + 4t + 7$, so $dw = (2t + 4)\,dt$.

26. Since $\theta^2 - 4\theta = (\theta - 2)^2 - 4$, we have

$$\int (2 - \theta)\cos(\theta^2 - 4\theta)\,d\theta = \int -(\theta - 2)\cos((\theta - 2)^2 - 4)\,d\theta.$$

Substitute $w = (\theta - 2)^2 - 4$, so $dw = 2(\theta - 2)\,d\theta$.

This integral can also be computed without completing the square, by substituting $w = \theta^2 - 4\theta$, so $dw = (2\theta - 4)\,d\theta$.

27. We write

$$\frac{1}{(x-5)(x-3)} = \frac{A}{x-5} + \frac{B}{x-3},$$

giving

$$\begin{aligned} 1 &= A(x-3) + B(x-5) \\ 1 &= (A+B)x - (3A+5B) \end{aligned}$$

so

$$\begin{aligned} A + B &= 0 \\ -3A - 5B &= 1. \end{aligned}$$

Thus, $A = 1/2$, $B = -1/2$, so

$$\int \frac{1}{(x-5)(x-3)}\,dx = \int \frac{1/2}{x-5}\,dx - \int \frac{1/2}{x-3}\,dx = \frac{1}{2}\ln|x-5| - \frac{1}{2}\ln|x-3| + C.$$

28. Since $2y^2 + 3y + 1 = (2y+1)(y+1)$, we write

$$\frac{y+2}{2y^2+3y+1} = \frac{A}{2y+1} + \frac{B}{y+1},$$

giving

$$\begin{aligned} y + 2 &= A(y+1) + B(2y+1) \\ y + 2 &= (A+2B)y + A + B \end{aligned}$$

so

$$\begin{aligned} A + 2B &= 1 \\ A + B &= 2. \end{aligned}$$

Thus, $A = 3$, $B = -1$, so

$$\int \frac{y+2}{2y^2+3y+1}\,dy = \int \frac{3}{2y+1}\,dy - \int \frac{1}{y+1}\,dy = \frac{3}{2}\ln|2y+1| - \ln|y+1| + C.$$

29. Since $x^3 + x = x(x^2+1)$ cannot be factored further, we write

$$\frac{x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Multiplying by $x(x^2+1)$ gives

$$\begin{aligned} x + 1 &= A(x^2+1) + (Bx+C)x \\ x + 1 &= (A+B)x^2 + Cx + A, \end{aligned}$$

so

$$\begin{aligned} A + B &= 0 \\ C &= 1 \\ A &= 1. \end{aligned}$$

Thus, $A = C = 1$, $B = -1$, and we have

$$\begin{aligned} \int \frac{x+1}{x^3+x}\,dx = \int \left(\frac{1}{x} + \frac{-x+1}{x^2+1}\right) &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1} \\ &= \ln|x| - \frac{1}{2}\ln\left|x^2+1\right| + \arctan x + K. \end{aligned}$$

30. Since $x^2 + x^4 = x^2(1 + x^2)$ cannot be factored further, we write

$$\frac{x-2}{x^2+x^4} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{1+x^2}.$$

Multiplying by $x^2(1+x^2)$ gives

$$\begin{aligned} x - 2 &= Ax(1+x^2) + B(1+x^2) + (Cx+D)x^2 \\ x - 2 &= (A+C)x^3 + (B+D)x^2 + Ax + B, \end{aligned}$$

so

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ A &= 1 \\ B &= -2. \end{aligned}$$

Thus, $A = 1$, $B = -2$, $C = -1$, $D = 2$, and we have

$$\begin{aligned} \int \frac{x-2}{x^2+x^4}\,dx = \int \left(\frac{1}{x} - \frac{2}{x^2} + \frac{-x+2}{1+x^2}\right)\,dx &= \int \frac{dx}{x} - 2\int \frac{dx}{x^2} - \int \frac{x\,dx}{1+x^2} + 2\int \frac{dx}{1+x^2} \\ &= \ln|x| + \frac{2}{x} - \frac{1}{2}\ln\left|1+x^2\right| + 2\arctan x + K. \end{aligned}$$

31. Let $x = 3\sin\theta$ so $dx = 3\cos\theta\,d\theta$, giving

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} 3\cos\theta\,d\theta = \int \frac{(9\sin^2\theta)(3\cos\theta)}{3\cos\theta}\,d\theta = 9\int \sin^2\theta\,d\theta.$$

Integrating by parts and using the identity $\cos^2\theta + \sin^2\theta = 1$ gives

$$\int \sin^2\theta\,d\theta = -\sin\theta\cos\theta + \int \cos^2\theta\,d\theta = -\sin\theta\cos\theta + \int (1-\sin^2\theta)\,d\theta$$

$$\int \sin^2\theta\,d\theta = -\frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2} + C.$$

Since $\sin\theta = x/3$ and $\cos\theta = \sqrt{1-x^2/9} = \sqrt{9-x^2}/3$, and $\theta = \arcsin(x/3)$, we have

$$\begin{aligned} \int \frac{x^2}{\sqrt{9-x^2}}\,dx &= 9\int \sin^2\theta\,d\theta = -\frac{9}{2}\sin\theta\cos\theta + \frac{9}{2}\theta + C \\ &= -\frac{9}{2}\cdot\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C = -\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + C \end{aligned}$$

32. Let $y = 5\tan\theta$ so $dy = (5/\cos^2\theta)\,d\theta$. Since $1 + \tan^2\theta = 1/\cos^2\theta$, we have

$$\int \frac{y^2}{25+y^2}\,dy = \int \frac{25\tan^2\theta}{25(1+\tan^2\theta)}\cdot\frac{5}{\cos^2\theta}\,d\theta = 5\int \tan^2\theta\,d\theta.$$

Using $1 + \tan^2\theta = 1/\cos^2\theta$ again gives

$$\int \frac{y^2}{25+y^2}\,dy = 5\int \tan^2\theta\,d\theta = 5\int \left(\frac{1}{\cos^2\theta} - 1\right)\,d\theta = 5\tan\theta - 5\theta + C.$$

In addition, since $\theta = \arctan(y/5)$, we get

$$\int \frac{y^2}{25+y^2}\,dy = y - 5\arctan\left(\frac{y}{5}\right) + C.$$

33. Let $t = \tan\theta$ so $dt = (1/\cos^2\theta)d\theta$. Since $\sqrt{1+\tan^2\theta} = 1/\cos\theta$, we have

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = \int \frac{1/\cos^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}\,d\theta = \int \frac{\cos\theta}{\tan^2\theta\cos^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta.$$

The last integral can be evaluated by guess-and-check or by substituting $w = \sin\theta$. The result is

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta = -\frac{1}{\sin\theta} + C.$$

Since $t = \tan\theta$ and $1/\cos^2\theta = 1 + \tan^2\theta$, we have

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+t^2}}.$$

In addition, $\tan\theta = \sin\theta/\cos\theta$ so

$$\sin\theta = \tan\theta\cos\theta = \frac{t}{\sqrt{1+t^2}}.$$

Thus

$$\int \frac{dt}{t^2\sqrt{1+t^2}} = -\frac{\sqrt{1+t^2}}{t} + C.$$

34. Since $(4-z^2)^{3/2} = (\sqrt{4-z^2})^3$, we substitute $z = 2\sin\theta$, so $dz = 2\cos\theta\,d\theta$. We get

$$\int \frac{dz}{(4-z^2)^{3/2}} = \int \frac{2\cos\theta\,d\theta}{(4-4\sin^2\theta)^{3/2}} = \int \frac{2\cos\theta\,d\theta}{8\cos^3\theta} = \frac{1}{4}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\tan\theta + C$$

Since $\sin\theta = z/2$, we have $\cos\theta = \sqrt{1-(z/2)^2} = (\sqrt{4-z^2})/2$, so

$$\int \frac{dz}{(4-z^2)^{3/2}} = \frac{1}{4}\tan\theta + C = \frac{1}{4}\frac{\sin\theta}{\cos\theta} + C = \frac{1}{4}\frac{z/2}{(\sqrt{4-z^2})/2} + C = \frac{z}{4\sqrt{4-z^2}} + C$$

35. The denominator $x^2 - 3x + 2$ can be factored as $(x-1)(x-2)$. Splitting the integrand into partial fractions with denominators $(x-1)$ and $(x-2)$, we have

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Multiplying by $(x-1)(x-2)$ gives the identity

$$x = A(x-2) + B(x-1)$$

so

$$x = (A+B)x - 2A - B.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} -2A - B &= 0 \\ A + B &= 1. \end{aligned}$$

Solving these equations gives $A = -1$, $B = 2$ and the integral becomes

$$\int \frac{x}{x^2-3x+2}dx = -\int \frac{1}{x-1}dx + 2\int \frac{1}{x-2}dx = -\ln|x-1| + 2\ln|x-2| + C.$$

36. Completing the square, we get

$$x^2 + 4x + 13 = (x+2)^2 + 9.$$

We use the substitution $x + 2 = 3\tan t$, then $dx = (3/\cos^2 t)\,dt$. Since $\tan^2 t + 1 = 1/\cos^2 t$, the integral becomes

$$\int \frac{1}{(x+2)^2+9}\,dx = \int \frac{1}{9\tan^2 t + 9}\cdot\frac{3}{\cos^2 t}\,dt = \int \frac{1}{3}\,dt = \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C.$$

37. Notice that because $\frac{3x}{(x-1)(x-4)}$ is negative for $2 \le x \le 3$,

$$\text{Area} = -\int_2^3 \frac{3x}{(x-1)(x-4)}dx.$$

Using partial fractions gives

$$\frac{3x}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4} = \frac{(A+B)x - B - 4A}{(x-1)(x-4)}.$$

Multiplying through by $(x-1)(x-4)$ gives

$$3x = (A+B)x - B - 4A$$

so $A = -1$ and $B = 4$. Thus

$$-\int_2^3 \frac{3x}{(x-1)(x-4)}dx = -\int_2^3 \left(\frac{-1}{x-1} + \frac{4}{x-4}\right) dx = (\ln|x-1| - 4\ln|x-4|)\Big|_2^3 = 5\ln 2.$$

38. We have

$$\text{Area} = \int_0^1 \frac{3x^2+x}{(x^2+1)(x+1)}dx.$$

Using partial fractions gives

$$\begin{aligned}\frac{3x^2+x}{(x^2+1)(x+1)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1}\\ &= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}\\ &= \frac{(A+C)x^2+(A+B)x+B+C}{(x^2+1)(x+1)}.\end{aligned}$$

Thus

$$3x^2 + x = (A+C)x^2 + (A+B)x + B + C,$$

giving

$$3 = A + C, \quad 1 = A + B, \quad \text{and} \quad 0 = B + C,$$

with solution

$$A = 2, B = -1, C = 1.$$

Thus

$$\begin{aligned}\text{Area} &= \int_0^1 \frac{3x^2+x}{(x^2+1)(x+1)}dx\\ &= \int_0^1 \left(\frac{2x}{x^2+1} - \frac{1}{x^2+1} + \frac{1}{x+1}\right) dx\\ &= \ln(x^2+1) - \arctan x + \ln|x+1|\Big|_0^1\\ &= 2\ln 2 - \pi/4.\end{aligned}$$

39. We have

$$\text{Area} = \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}dx.$$

Let $x = \sin\theta$ so $dx = \cos\theta\, d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. When $x = 0, \theta = 0$. When $x = 1/2, \theta = \pi/6$.

$$\begin{aligned}\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}dx &= \int_0^{\pi/6} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta = \int_0^{\pi/6} \sin^2\theta\, d\theta\\ &= \left(\frac{\theta}{2} - \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{\pi/6} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}.\end{aligned}$$

The integral $\int \sin^2\theta\, d\theta$ is done using parts and the identity $\cos^2\theta + \sin^2\theta = 1$.

40. We have

$$\text{Area} = \int_0^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}}\,dx.$$

Let $x = 2\sin\theta$ so $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$. When $x = 0, \theta = 0$ and when $x = \sqrt{2}, \theta = \pi/4$.

$$\begin{aligned}\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{4-x^2}}dx &= \int_0^{\pi/4} \frac{(2\sin\theta)^3}{\sqrt{4-(2\sin\theta)^2}} 2\cos\theta\,d\theta \\ &= 8\int_0^{\pi/4} \sin^3\theta\,d\theta = 8\int_0^{\pi/4} (\sin\theta - \sin\theta\cos^2\theta)\,d\theta \\ &= 8\left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\Bigg|_0^{\pi/4} = 8\left(\frac{2}{3} - \frac{5}{6\sqrt{2}}\right).\end{aligned}$$

41. We have

$$\text{Area} = \int_0^3 \frac{1}{\sqrt{x^2+9}}dx.$$

Let $x = 3\tan\theta$ so $dx = (3/\cos^2\theta)d\theta$ and

$$\sqrt{x^2+9} = \sqrt{\frac{9\sin^2\theta}{\cos^2\theta} + 9} = \frac{3}{\cos\theta}.$$

When $x = 0, \theta = 0$ and when $x = 3, \theta = \pi/4$. Thus

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{x^2+9}}dx &= \int_0^{\pi/4} \frac{1}{\sqrt{9\tan^2\theta+9}}\frac{3}{\cos^2\theta}d\theta = \int_0^{\pi/4} \frac{1}{3/\cos\theta}\cdot\frac{3}{\cos^2\theta}d\theta = \int_0^{\pi/4}\frac{1}{\cos\theta}d\theta \\ &= \frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right|\Bigg|_0^{\pi/4} = \frac{1}{2}\ln\left|\frac{1/\sqrt{2}+1}{1/\sqrt{2}-1}\right| = \frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{\sqrt{2}-1}\right).\end{aligned}$$

This answer can be simplified to $\ln(1+\sqrt{2})$ by multiplying the numerator and denominator of the fraction by $(\sqrt{2}+1)$ and using the properties of logarithms. The integral $\int(1/\cos\theta)d\theta$ is done using the Table of Integrals.

42. We have

$$\text{Area} = \int_{\sqrt{3}}^3 \frac{1}{x\sqrt{x^2+9}}dx.$$

Let $x = 3\tan\theta$ so $dx = (3/\cos^2\theta)d\theta$ and

$$x\sqrt{x^2+9} = 3\frac{\sin\theta}{\cos\theta}\sqrt{\frac{9\sin^2\theta}{\cos^2\theta}+9} = \frac{9\sin\theta}{\cos^2\theta}.$$

When $x = \sqrt{3}, \theta = \pi/6$ and when $x = 3, \theta = \pi/4$. Thus

$$\begin{aligned}\int_{\sqrt{3}}^3 \frac{1}{x\sqrt{x^2+9}}dx &= \int_{\pi/6}^{\pi/4} \frac{1}{9\sin\theta/\cos^2\theta}\cdot\frac{3}{\cos^2\theta}d\theta = \frac{1}{3}\int_{\pi/6}^{\pi/4}\frac{1}{\sin\theta}d\theta \\ &= \frac{1}{3}\cdot\frac{1}{2}\ln\left|\frac{\cos\theta-1}{\cos\theta+1}\right|\Bigg|_{\pi/6}^{\pi/4} = \frac{1}{6}\left(\ln\left|\frac{1/\sqrt{2}-1}{1/\sqrt{2}+1}\right| - \ln\left|\frac{\sqrt{3}/2-1}{\sqrt{3}/2+1}\right|\right) \\ &= \frac{1}{6}\left(\ln\left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right| + \ln\left|\frac{\sqrt{3}+2}{\sqrt{3}-2}\right|\right).\end{aligned}$$

This answer can be simplified by multiplying the first fraction by $(1-\sqrt{2})$ in numerator and denominator and the second one by $(\sqrt{3}+2)$. This gives

$$\text{Area} = \frac{1}{6}(\ln(3-2\sqrt{2}) + \ln(7+4\sqrt{3})) = \frac{1}{6}\ln((3-2\sqrt{2})(7+4\sqrt{3})).$$

The integral $\int(1/\sin\theta)d\theta$ is done using the Table of Integrals.

43. **(a)** We differentiate:

$$\frac{d}{d\theta}\left(-\frac{1}{\tan\theta}\right) = \frac{1}{\tan^2\theta}\cdot\frac{1}{\cos^2\theta} = \frac{1}{\frac{\sin^2\theta}{\cos^2\theta}}\cdot\frac{1}{\cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Thus,

$$\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{1}{\tan\theta} + C.$$

(b) Let $y = \sqrt{5}\sin\theta$ so $dy = \sqrt{5}\cos\theta\,d\theta$ giving

$$\int \frac{dy}{y^2\sqrt{5-y^2}} = \int \frac{\sqrt{5}\cos\theta}{5\sin^2\theta\sqrt{5-5\sin^2\theta}}\,d\theta = \frac{1}{5}\int \frac{\sqrt{5}\cos\theta}{\sin^2\theta\sqrt{5}\cos\theta}\,d\theta$$
$$= \frac{1}{5}\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{1}{5\tan\theta} + C.$$

Since $\sin\theta = y/\sqrt{5}$, we have $\cos\theta = \sqrt{1-(y/\sqrt{5})^2} = \sqrt{5-y^2}/\sqrt{5}$. Thus,

$$\int \frac{dy}{y^2\sqrt{5-y^2}} = -\frac{1}{5\tan\theta} + C = -\frac{\sqrt{5-y^2}/\sqrt{5}}{5(y/\sqrt{5})} + C = -\frac{\sqrt{5-y^2}}{5y} + C.$$

44. Using partial fractions, we write

$$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$$
$$1 = A(1-x) + B(1+x) = (B-A)x + A + B.$$

So, $B - A = 0$ and $A + B = 1$, giving $A = B = 1/2$. Thus

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\int\left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx = \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right) + C.$$

Using the substitution $x = \sin\theta$, we get $dx = \cos\theta\,d\theta$, we have

$$\int \frac{dx}{1-x^2} = \int \frac{\cos\theta}{1-\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\cos^2\theta}\,d\theta = \int \frac{1}{\cos\theta}\,d\theta.$$

The Table of Integrals Formula IV-22 gives

$$\int \frac{dx}{1-x^2} = \int \frac{1}{\cos\theta}\,d\theta = \frac{1}{2}\ln\left|\frac{(\sin\theta)+1}{(\sin\theta)-1}\right| + C = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C.$$

The properties of logarithms and the fact that $|x-1| = |1-x|$ show that the two results are the same:

$$\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| = \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right).$$

45. **(a)** We want to evaluate the integral

$$T = \int_0^{a/2} \frac{k\,dx}{(a-x)(b-x)}.$$

Using partial fractions, we have

$$\frac{k}{(a-x)(b-x)} = \frac{C}{a-x} + \frac{D}{b-x}$$
$$k = C(b-x) + D(a-x)$$
$$k = -(C+D)x + Cb + Da$$

so

$$0 = -(C+D)$$
$$k = Cb + Da,$$

giving

$$C = -D = \frac{k}{b-a}.$$

Thus, the time is given by

$$\begin{aligned} T = \int_0^{a/2} \frac{k\,dx}{(a-x)(b-x)} &= \frac{k}{b-a} \int_0^{a/2} \left(\frac{1}{a-x} - \frac{1}{b-x} \right) dx \\ &= \frac{k}{b-a} \left(-\ln|a-x| + \ln|b-x| \right) \Big|_0^{a/2} \\ &= \frac{k}{b-a} \ln \left| \frac{b-x}{a-x} \right| \Big|_0^{a/2} \\ &= \frac{k}{b-a} \left(\ln\left(\frac{2b-a}{a}\right) - \ln\left(\frac{b}{a}\right) \right) \\ &= \frac{k}{b-a} \ln\left(\frac{2b-a}{b}\right). \end{aligned}$$

(b) A similar calculation with x_0 instead of $a/2$ leads to the following expression for the time

$$\begin{aligned} T = \int_0^{x_0} \frac{k\,dx}{(a-x)(b-x)} &= \frac{k}{b-a} \ln \left| \frac{b-x}{a-x} \right| \Big|_0^{x_0} \\ &= \frac{k}{b-a} \left(\ln \left| \frac{b-x_0}{a-x_0} \right| - \ln\left(\frac{b}{a}\right) \right). \end{aligned}$$

As $x_0 \to a$, the value of $|a-x_0| \to 0$, so $|b-x_0|/|a-x_0| \to \infty$. Thus, $T \to \infty$ as $x_0 \to a$. In other words, the time taken tends to infinity.

46. (a) We calculate the integral using partial fractions with denominators P and $L-P$:

$$\begin{aligned} \frac{k}{P(L-P)} &= \frac{A}{P} + \frac{B}{L-P} \\ k &= A(L-P) + BP \\ k &= (B-A)P + AL. \end{aligned}$$

Thus,

$$\begin{aligned} B - A &= 0 \\ AL &= k, \end{aligned}$$

so $A = B = k/L$, and the time is given by

$$\begin{aligned} T = \int_{L/4}^{L/2} \frac{k\,dP}{P(L-P)} &= \frac{k}{L} \int_{L/4}^{L/2} \left(\frac{1}{P} + \frac{1}{L-P} \right) dP = \frac{k}{L} (\ln|P| - \ln|L-P|) \Big|_{L/4}^{L/2} \\ &= \frac{k}{L} \left(\ln\left(\frac{L}{2}\right) - \ln\left(\frac{L}{2}\right) - \ln\left(\frac{L}{4}\right) + \ln\left(\frac{3L}{4}\right) \right) \\ &= \frac{k}{L} \ln\left(\frac{3L/4}{L/4}\right) = \frac{k}{L} \ln(3). \end{aligned}$$

(b) A similar calculation gives the following expression for the time:

$$T = \frac{k}{L} (\ln|P| - \ln|L-P|) \Big|_{P_1}^{P_2} = \frac{k}{L} \left(\ln|P_2| - \ln|L-P_2| - \ln|P_1| + \ln|L-P_1| \right).$$

If $P_2 \to L$, then $L - P_2 \to 0$, so $\ln P_2 \to \ln L$, and $\ln(L-P_2) \to -\infty$. Thus the time tends to infinity.

Solutions for Section 7.5

Exercises

1. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.2. We see that this approximation is an underestimate.

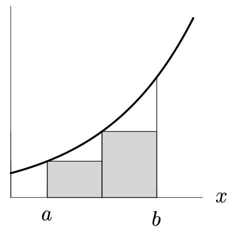


Figure 7.2

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.3. We see that this approximation is an overestimate.

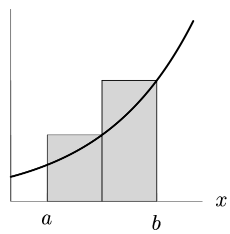


Figure 7.3

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.4. We see that this approximation is an overestimate.

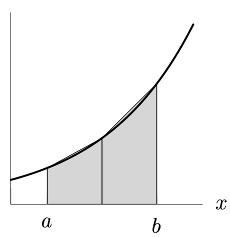


Figure 7.4

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.5. We see from the tangent line interpretation that this approximation is an underestimate

Figure 7.5

2. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.6. We see that this approximation is an overestimate.

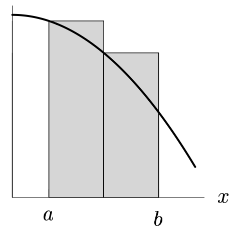


Figure 7.6

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.7. We see that this approximation is an underestimate.

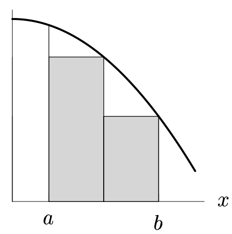


Figure 7.7

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.8. We see that this approximation is an underestimate.

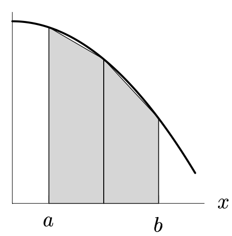


Figure 7.8

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.9. We see from the tangent line interpretation that this approximation is an overestimate.

Figure 7.9

3. (a) The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.10. We see that this approximation is an underestimate.

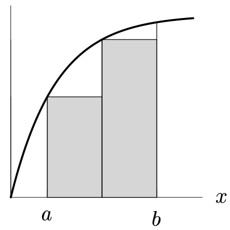


Figure 7.10

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.11. We see that this approximation is an overestimate.

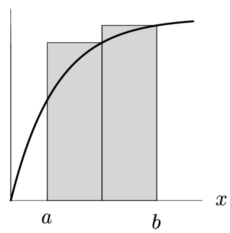


Figure 7.11

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.12. We see that this approximation is an underestimate.

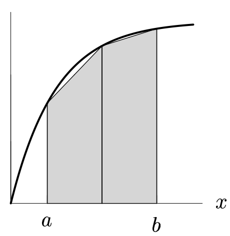


Figure 7.12

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.13. We see from the tangent line interpretation that this approximation is an overestimate.

Figure 7.13

4. **(a)** The approximation LEFT(2) uses two rectangles, with the height of each rectangle determined by the left-hand endpoint. See Figure 7.14. We see that this approximation is an overestimate.

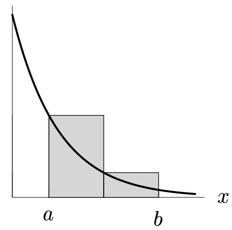


Figure 7.14

(b) The approximation RIGHT(2) uses two rectangles, with the height of each rectangle determined by the right-hand endpoint. See Figure 7.15. We see that this approximation is an underestimate.

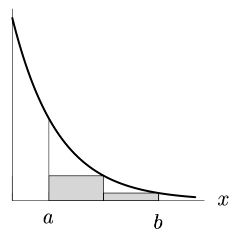


Figure 7.15

(c) The approximation TRAP(2) uses two trapezoids, with the height of each trapezoid given by the secant line connecting the two endpoints. See Figure 7.16. We see that this approximation is an overestimate.

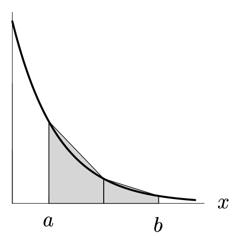


Figure 7.16

(d) The approximation MID(2) uses two rectangles, with the height of each rectangle determined by the height at the midpoint. Alternately, we can view MID(2) as a trapezoid rule where the height is given by the tangent line at the midpoint. Both interpretations are shown in Figure 7.17. We see from the tangent line interpretation that this approximation is an underestimate.

Figure 7.17

5. (a) Since two rectangles are being used, the width of each rectangle is 3. The height is given by the left-hand endpoint so we have

$$\text{LEFT(2)} = f(0)\cdot 3 + f(3)\cdot 3 = 0^2\cdot 3 + 3^2\cdot 3 = 27.$$

(b) Since two rectangles are being used, the width of each rectangle is 3. The height is given by the right-hand endpoint so we have

$$\text{RIGHT(2)} = f(3)\cdot 3 + f(6)\cdot 3 = 3^2\cdot 3 + 6^2\cdot 3 = 135.$$

(c) We know that TRAP is the average of LEFT and RIGHT and so

$$\text{TRAP(2)} = \frac{27+135}{2} = 81.$$

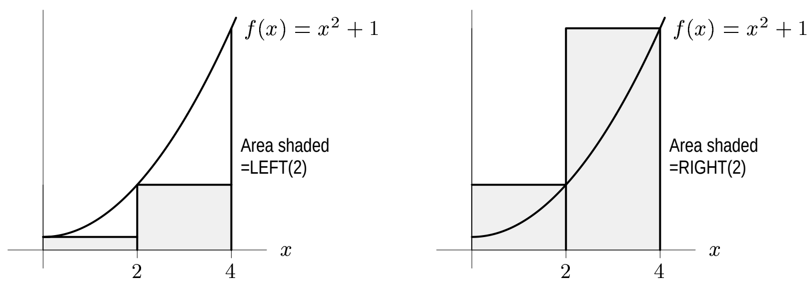
(d) Since two rectangles are being used, the width of each rectangle is 3. The height is given by the height at the midpoint so we have

$$\text{MID(2)} = f(1.5)\cdot 3 + f(4.5)\cdot 3 = (1.5)^2\cdot 3 + (4.5)^2\cdot 3 = 67.5.$$

6. (a)

$$\begin{aligned}\text{LEFT(2)} &= 2\cdot f(0) + 2\cdot f(2)\\ &= 2\cdot 1 + 2\cdot 5\\ &= 12\\ \text{RIGHT(2)} &= 2\cdot f(2) + 2\cdot f(4)\\ &= 2\cdot 5 + 2\cdot 17\\ &= 44\end{aligned}$$

(b)

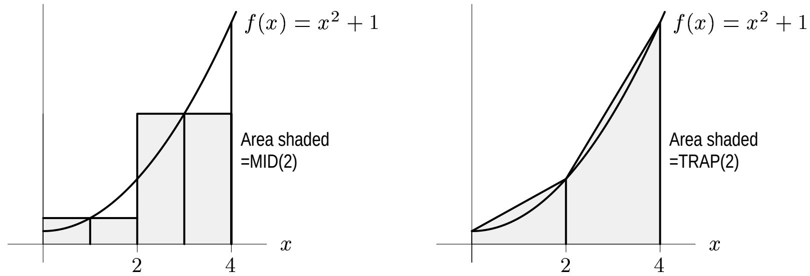


LEFT(2) is an underestimate, while RIGHT(2) is an overestimate.

7. (a)

$$\begin{aligned}\text{MID(2)} &= 2\cdot f(1) + 2\cdot f(3)\\ &= 2\cdot 2 + 2\cdot 10\\ &= 24\\ \text{TRAP(2)} &= \frac{\text{LEFT(2)}+\text{RIGHT(2)}}{2}\\ &= \frac{12+44}{2} \quad \text{(see Problem 6)}\\ &= 28\end{aligned}$$

(b)



MID(2) is an underestimate, since $f(x) = x^2 + 1$ is concave up and a tangent line will be below the curve. TRAP(2) is an overestimate, since a secant line lies above the curve.

Problems

8. **(a)** (i) Let $f(x) = \frac{1}{1+x^2}$. The left-hand Riemann sum is

$$\begin{aligned} &\frac{1}{8}\left(f(0) + f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right) \\ &= \frac{1}{8}\left(\frac{64}{64} + \frac{64}{65} + \frac{64}{68} + \frac{64}{73} + \frac{64}{80} + \frac{64}{89} + \frac{64}{100} + \frac{64}{113}\right) \\ &\approx 8(0.1020) = 0.8160. \end{aligned}$$

(ii) Let $f(x) = \frac{1}{1+x^2}$. The right-hand Riemann sum is

$$\begin{aligned} &\frac{1}{8}\left(f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f(1)\right) \\ &= \frac{1}{8}\left(\frac{64}{65} + \frac{64}{68} + \frac{64}{73} + \frac{64}{80} + \frac{64}{89} + \frac{64}{100} + \frac{64}{113} + \frac{64}{128}\right) \\ &\approx 0.8160 - \frac{1}{16} = 0.7535. \end{aligned}$$

(iii) The trapezoid rule gives us that

$$\text{TRAP}(8) = \frac{\text{LEFT}(8) + \text{RIGHT}(8)}{2} \approx 0.7847.$$

(b) Since $1 + x^2$ is increasing for $x > 0$, so $\dfrac{1}{1+x^2}$ is decreasing over the interval. Thus

$$\text{RIGHT}(8) < \int_0^1 \frac{1}{1+x^2}\,dx < \text{LEFT}(8)$$

$$0.7535 < \frac{\pi}{4} < 0.8160$$

$$3.014 < \pi < 3.264.$$

9. Let $s(t)$ be the distance traveled at time t and $v(t)$ be the velocity at time t. Then the distance traveled during the interval $0 \le t \le 6$ is

$$\begin{aligned} s(6) - s(0) &= s(t)\Big|_0^6 \\ &= \int_0^6 s'(t)\,dt \quad \text{(by the Fundamental Theorem)} \\ &= \int_0^6 v(t)\,dt. \end{aligned}$$

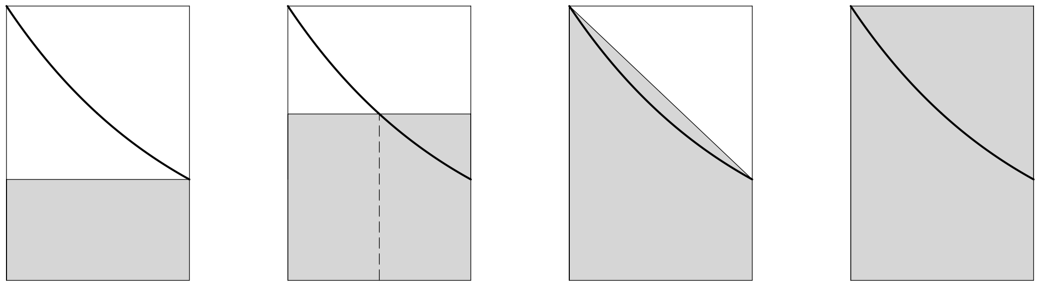
We estimate the distance by estimating this integral.

From the table, we find: $\text{LEFT}(6) = 31$, $\text{RIGHT}(6) = 39$, $\text{TRAP}(6) = 35$.

10. Since the function is decreasing, LEFT is an overestimate and RIGHT is an underestimate. Since the graph is concave down, secant lines lie below the graph so TRAP is an underestimate and tangent lines lie above the graph so MID is an overestimate. We can see that MID and TRAP are closer to the exact value than LEFT and RIGHT. In order smallest to largest, we have:
RIGHT(n) $<$ TRAP(n) $<$ Exact value $<$ MID(n) $<$ LEFT(n).

11. For a decreasing function whose graph is concave up, the diagrams below show that RIGHT $<$ MID $<$ TRAP $<$ LEFT. Thus,

(a) $0.664 =$ LEFT, $0.633 =$ TRAP, $0.632 =$ MID, and $0.601 =$ RIGHT.
(b) $0.632 <$ true value < 0.633.



RIGHT $= 0.601$ MID $= 0.632$ TRAP $= 0.633$ LEFT $= 0.664$

12. $f(x)$ is increasing, so RIGHT gives an overestimate and LEFT gives an underestimate.

13. $f(x)$ is concave down, so MID gives an overestimate and TRAP gives an underestimate.

14. $f(x)$ is decreasing and concave up, so LEFT and TRAP give overestimates and RIGHT and MID give underestimates.

15. $f(x)$ is concave up, so TRAP gives an overestimate and MID gives an underestimate.

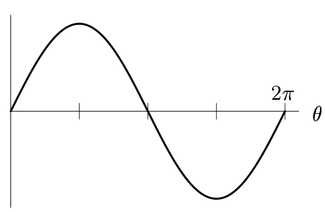
16. **(a)** Since $f(x)$ is closer to horizontal (that is, $|f'| < |g'|$), LEFT and RIGHT will be more accurate with $f(x)$.
(b) Since $g(x)$ has more curvature, MID and TRAP will be more accurate with $f(x)$.

17. **(a)** TRAP(4) gives probably the best estimate of the integral. We cannot calculate MID(4).

$$\text{LEFT(4)} = 3 \cdot 100 + 3 \cdot 97 + 3 \cdot 90 + 3 \cdot 78 = 1095$$
$$\text{RIGHT(4)} = 3 \cdot 97 + 3 \cdot 90 + 3 \cdot 78 + 3 \cdot 55 = 960$$
$$\text{TRAP(4)} = \frac{1095 + 960}{2} = 1027.5.$$

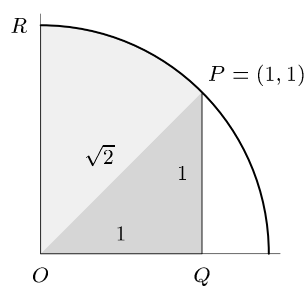
(b) Because there are no points of inflection, the graph is either concave down or concave up. By plotting points, we see that it is concave down. So TRAP(4) is an underestimate.

18. **(a)** $\displaystyle\int_0^{2\pi} \sin\theta\, d\theta = -\cos\theta \Big|_0^{2\pi} = 0.$
(b) $\text{MID}(1)$ is 0 since the midpoint of 0 and 2π is π, and $\sin\pi = 0$. Thus $\text{MID}(1) = 2\pi(\sin\pi) = 0$. The midpoints we use for $\text{MID}(2)$ are $\pi/2$ and $3\pi/2$, and $\sin(\pi/2) = -\sin(3\pi/2)$. Thus $\text{MID}(2) = \pi\sin(\pi/2) + \pi\sin(3\pi/2) = 0$.



(c) $\text{MID}(3) = 0$.
In general, $\text{MID}(n) = 0$ for all n, even though your calculator (because of round-off error) might not return it as such. The reason is that $\sin(x) = -\sin(2\pi - x)$. If we use $\text{MID}(n)$, we will always take sums where we are adding pairs of the form $\sin(x)$ and $\sin(2\pi - x)$, so the sum will cancel to 0. (If n is odd, we will get a $\sin\pi$ in the sum which doesn't pair up with anything — but $\sin\pi$ is already 0.)

19. **(a)**



The graph of $y = \sqrt{2 - x^2}$ is the upper half of a circle of radius $\sqrt{2}$ centered at the origin. The integral represents the area under this curve between the lines $x = 0$ and $x = 1$. From the picture, we see that this area can be split into 2 parts, A_1 and A_2. Notice since $OQ = QP = 1$, $\triangle OQP$ is isosceles. Thus $\angle POQ = \angle ROP = \frac{\pi}{4}$, and A_1 is exactly $\frac{1}{8}$ of the entire circle. Thus the total area is

$$\text{Area} = A_1 + A_2 = \frac{1}{8}\pi(\sqrt{2})^2 + \frac{1 \cdot 1}{2} = \frac{\pi}{4} + \frac{1}{2}.$$

(b) $\text{LEFT}(5) \approx 1.32350, \quad \text{RIGHT}(5) \approx 1.24066, T$
$\text{TRAP}(5) \approx 1.28208, \quad \text{MID}(5) \approx 1.28705$

Exact value ≈ 1.285398163

Left-hand error ≈ -0.03810, Right-hand error ≈ 0.04474,
Trapezoidal error ≈ 0.00332, Midpoint error ≈ -0.001656

Thus right-hand error $>$ trapezoidal error $> 0 >$ midpoint error $>$ left-hand error, and $|$midpt error$| < |$trap error$| < |$left-error$| < |$right-error$|$.

20. We approximate the area of the playing field by using Riemann sums. From the data provided,

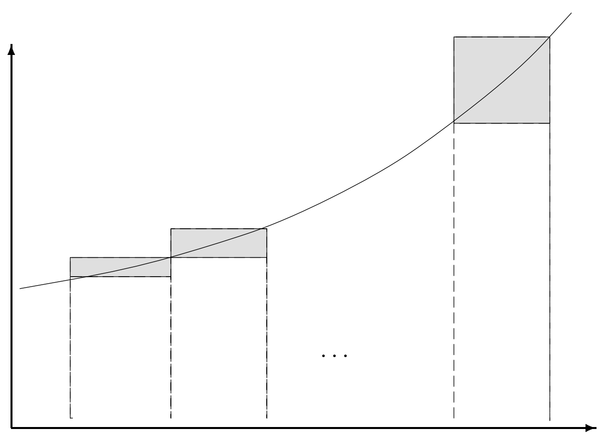
$$\text{LEFT}(10) = \text{RIGHT}(10) = \text{TRAP}(10) = 89{,}000 \text{ square feet.}$$

Thus approximately

$$\frac{89{,}000 \text{ sq. ft.}}{200 \text{ sq. ft./lb.}} = 445 \text{ lbs. of fertilizer}$$

should be necessary.

21.



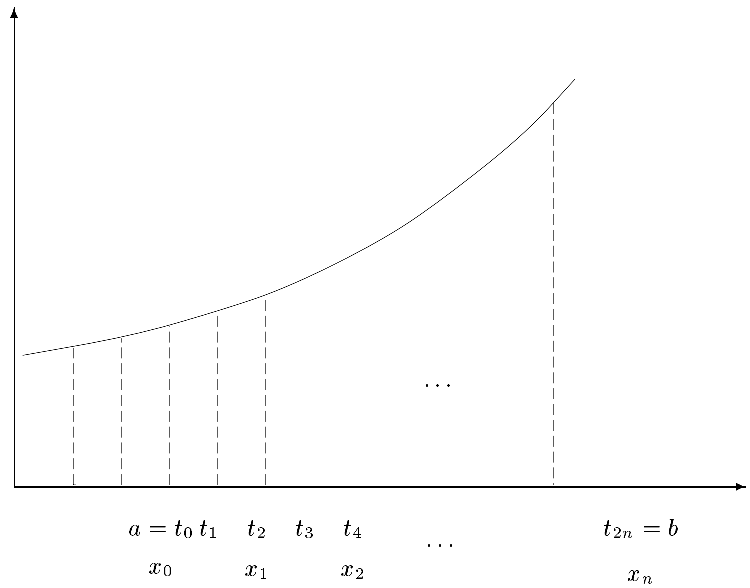
From the diagram, the difference between $\text{RIGHT}(n)$ and $\text{LEFT}(n)$ is the area of the shaded rectangles.
$\text{RIGHT}(n) = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$
$\text{LEFT}(n) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$
Notice that the terms in these two sums are the same, except that $\text{RIGHT}(n)$ contains $f(x_n)\Delta x$ ($= f(b)\Delta x$), and $\text{LEFT}(n)$ contains $f(x_0)\Delta x$ ($= f(a)\Delta x$). Thus

$$\begin{aligned}\text{RIGHT}(n) &= \text{LEFT}(n) + f(x_n)\Delta x - f(x_0)\Delta x \\ &= \text{LEFT}(n) + f(b)\Delta x - f(a)\Delta x\end{aligned}$$

22.

$$\begin{aligned}\text{TRAP}(n) &= \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2} \\ &= \frac{\text{LEFT}(n) + \text{LEFT}(n) + f(b)\Delta x - f(a)\Delta x}{2} \\ &= \text{LEFT}(n) + \frac{1}{2}(f(b) - f(a))\Delta x\end{aligned}$$

23.



Divide the interval $[a, b]$ into n pieces, by $x_0, x_1, x_2, \ldots, x_n$, and also into $2n$ pieces, by $t_0, t_1, t_2, \ldots, t_{2n}$. Then the x's coincide with the even t's, so $x_0 = t_0,\ x_1 = t_2,\ x_2 = t_4,\ \ldots,\ x_n = t_{2n}$ and $\Delta t = \frac{1}{2}\Delta x$.

$$\text{LEFT}(n) = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x$$

Since $\text{MID}(n)$ is obtained by evaluating f at the midpoints $t_1, t_3, t_5, \ldots$ of the x intervals, we get

$$\text{MID}(n) = f(t_1)\Delta x + f(t_3)\Delta x + \cdots + f(t_{2n-1})\Delta x$$

Now

$$\text{LEFT}(2n) = f(t_0)\Delta t + f(t_1)\Delta t + f(t_2)\Delta t + \cdots + f(t_{2n-1})\Delta t.$$

Regroup terms, putting all the even t's first, the odd t's last:

$$\begin{aligned}\text{LEFT}(2n) &= f(t_0)\Delta t + f(t_2)\Delta t + \cdots + f(t_{2n-2})\Delta t + f(t_1)\Delta t + f(t_3)\Delta t + \cdots + f(t_{2n-1})\Delta t \\ &= \underbrace{f(x_0)\frac{\Delta x}{2} + f(x_1)\frac{\Delta x}{2} + \cdots + f(x_{n-1})\frac{\Delta x}{2}}_{\text{LEFT}(n)/2} + \underbrace{f(t_1)\frac{\Delta x}{2} + f(t_3)\frac{\Delta x}{2} + \cdots + f(t_{2n-1})\frac{\Delta x}{2}}_{\text{MID}(n)/2}\end{aligned}$$

So

$$\text{LEFT}(2n) = \frac{1}{2}(\text{LEFT}(n) + \text{MID}(n))$$

24. When $n = 10$, we have $a = 1; b = 2; \Delta x = \frac{1}{10}; f(a) = 1; f(b) = \frac{1}{2}$.
$\text{LEFT}(10) \approx 0.71877, \text{RIGHT}(10) \approx 0.66877, \text{TRAP}(10) \approx 0.69377$
We have
$\text{RIGHT}(10) = \text{LEFT}(10) + f(b)\Delta x - f(a)\Delta x = 0.71877 + \frac{1}{10}(\frac{1}{2}) - \frac{1}{10}(1) = 0.66877$, and $\text{TRAP}(10) = \text{LEFT}(10) + \frac{\Delta x}{2}(f(b) - f(a)) = 0.71877 + \frac{1}{10}\frac{1}{2}(\frac{1}{2} - 1) = 0.69377$,
so the equations are verified.

25. First, we compute:

$$\begin{aligned}(f(b) - f(a))\Delta x &= (f(b) - f(a))\left(\frac{b-a}{n}\right) \\ &= (f(5) - f(2))\left(\frac{3}{n}\right) \\ &= (21 - 13)\left(\frac{3}{n}\right) \\ &= \frac{24}{n}\end{aligned}$$

$\text{RIGHT}(10) = \text{LEFT}(10) + 24 = 3.156 + 2.4 = 5.556.$
$\text{TRAP}(10) = \text{LEFT}(10) + \frac{1}{2}(2.4) = 3.156 + 1.2 = 4.356.$
$\text{LEFT}(20) = \frac{1}{2}(\text{LEFT}(10) + \text{MID}(10)) = \frac{1}{2}(3.156 + 3.242) = 3.199.$
$\text{RIGHT}(20) = \text{LEFT}(20) + 2.4 = 3.199 + 1.2 = 4.399.$
$\text{TRAP}(20) = \text{LEFT}(20) + \frac{1}{2}(1.2) = 3.199 + 0.6 = 3.799.$

Solutions for Section 7.6

Exercises

1. We saw in Problem 5 in Section 7.5 that, for this definite integral, we have LEFT(2) = 27, RIGHT(2) = 135, TRAP(2) = 81, and MID(2) = 67.5. Thus,

$$\text{SIMP}(2) = \frac{2\text{MID}(2) + \text{TRAP}(2)}{3} = \frac{2(67.5) + 81}{3} = 72.$$

Notice that

$$\int_0^6 x^2 dx = \left.\frac{x^3}{3}\right|_0^6 = \frac{6^3}{3} - \frac{0^3}{3} = 72,$$

and so SIMP(2) gives the exact value of the integral in this case.

2. (a) From Problem 7 on page 401, for $\int_0^4 (x^2+1)\,dx$, we have MID(2)$= 24$ and TRAP(2)$= 28$. Thus,

$$\begin{aligned}\text{SIMP}(2) &= \frac{2\text{MID}(2) + \text{TRAP}(2)}{3}\\ &= \frac{2(24) + 28}{3}\\ &= \frac{76}{3}.\end{aligned}$$

(b)

$$\int_0^4 (x^2+1)\,dx = \left.\left(\frac{x^3}{3} + x\right)\right|_0^4 = \left(\frac{64}{3} + 4\right) - (0+0) = \frac{76}{3}$$

(c) Error$= 0$. Simpson's Rule gives the exact answer.

Problems

3. (a)

Table 7.1 *Errors for the left and right rule approximations to* $\int_1^2 \frac{1}{x}\,dx = 0.6931471806\ldots$

n	LEFT(n)	Left error	RIGHT(n)	Right error
2	0.833333	−0.14019	0.583333	0.10981
4	0.759524	−0.06638	0.634524	0.05862
8	0.725372	−0.03222	0.662872	0.03028
16	0.709016	−0.01587	0.677766	0.01538
32	0.701021	−0.00787	0.685396	0.00775
64	0.697069	−0.00392	0.689256	0.00389
128	0.695104	−0.00196	0.691198	0.00195

(b) The left errors are negative and the right errors are positive. This occurs because $f(x) = 1/x$ is decreasing, meaning that the left sums are overestimates and the right sums are underestimates. Doubling n approximately halves the error.

(c)

Table 7.2 *Errors for the trapezoid and midpoint rule approximations to* $\int_1^2 \frac{1}{x}\,dx = 0.6931471806\ldots$

n	TRAP(n)	Trap error	MID(n)	Mid error
2	0.708333	-0.01518	0.685714	0.00743
4	0.697024	-0.00387	0.691220	0.00193
8	0.694122	-0.00097	0.692661	0.00049
16	0.6933912	-0.000244	0.6930252	0.000122
32	0.6932082	-0.000061	0.6931166	0.000031
64	0.6931624	-0.000015	0.6931396	0.000008
128	0.6931510	-0.000004	0.6931453	0.000002

(d) The trapezoid errors are negative because $f(x) = 1/x$ is concave up, and thus, the trapezoids overestimate. The midpoint errors are positive. Doubling n approximately quarters the error.

(e)

Table 7.3 *Errors for Simpson's rule for* $\int_1^2 \frac{1}{x}\,dx = 0.6931471806\ldots$

n	SIMP(n)	error
2	0.69325396825	-0.000106788
4	0.69315453065	-0.000007350
8	0.69314765282	-0.000000472
16	0.69314721029	-0.000000030
32	0.69314718242	-0.000000002

The error is multiplied by approximately $1/16$ when n is doubled.

4. **(a)** $\displaystyle\int_0^2 (x^3 + 3x^2)\,dx = \left(\frac{x^4}{4} + x^3\right)\Bigg|_0^2 = 12.$

(b) $\text{SIMP}(2) = 12.$
$\text{SIMP}(4) = 12.$
$\text{SIMP}(100) = 12.$
$\text{SIMP}(n) = 12$ for all n. Simpson's rule always gives the exact answer if the integrand is a polynomial of degree less than 4.

5. **(a)** $\displaystyle\int_0^4 e^x\,dx = e^x\Big|_0^4 = e^4 - e^0 \approx 53.598\ldots.$

(b) Computing the sums directly, since $\Delta x = 2$, we have
LEFT(2)$= 2\cdot e^0 + 2\cdot e^2 \approx 2(1) + 2(7.389) = 16.778;$ error $= 36.820.$
RIGHT(2)$= 2\cdot e^2 + 2\cdot e^4 \approx 2(7.389) + 2(54.598) = 123.974;$ error $= -70.376.$
TRAP(2)$= \dfrac{16.778 + 123.974}{2} = 70.376;$ error $= 16.778.$
MID(2)$= 2\cdot e^1 + 2\cdot e^3 \approx 2(2.718) + 2(20.086) = 45.608;$ error $= 7.990.$
SIMP(2)$= \dfrac{2(45.608) + 70.376}{3} = 53.864;$ error $= -0.266.$

(c) Similarly, since $\Delta x = 1$, we have LEFT(4)$= 31.193;$ error $= 22.405$
RIGHT(4)$= 84.791;$ error $= -31.193$
TRAP(4)$= 57.992;$ error $= -4.394$
MID(4)$= 51.428;$ error $= 2.170$
SIMP(4)$= 53.616;$ error $= -0.018$

(d) For LEFT and RIGHT, we expect the error to go down by $1/2$, and this is very roughly what we see. For MID and TRAP, we expect the error to go down by $1/4$, and this is approximately what we see. For SIMP, we expect the error to go down by $1/2^4 = 1/16$, and this is approximately what we see.

6. Here, the error in the approximation using $n = 10$ is $4 - 2.346 = 1.654$.

(a) Since the error in the LEFT approximation is proportional to $1/n$, when we triple n from 10 to 30 the error is divided by 3, so the error here is $1.654/3 = 0.551333$, giving $\text{LEFT}(30) = 4 - 0.551333 \approx 3.449$.

(b) The procedure here is identical to part (a), except that the TRAP error is proportional to $1/n^2$, so the error in $\text{TRAP}(30)$ will be $1.654/3^2 = 0.183778$, giving $\text{TRAP}(30) = 4 - 0.183778 \approx 3.816$.

(c) For SIMP, the error will be $1.654/3^4 = 0.0204198$, giving $\text{SIMP}(30) = 4 - 0.0204198 \approx 3.980$.

7. (a) For the left-hand rule, error is approximately proportional to $\frac{1}{n}$. If we let n_p be the number of subdivisions needed for accuracy to p places, then there is a constant k such that

$$\begin{aligned} 5 \times 10^{-5} &= \frac{1}{2} \times 10^{-4} \approx \frac{k}{n_4} \\ 5 \times 10^{-9} &= \frac{1}{2} \times 10^{-8} \approx \frac{k}{n_8} \\ 5 \times 10^{-13} &= \frac{1}{2} \times 10^{-12} \approx \frac{k}{n_{12}} \\ 5 \times 10^{-21} &= \frac{1}{2} \times 10^{-20} \approx \frac{k}{n_{20}} \end{aligned}$$

Thus the ratios $n_4 : n_8 : n_{12} : n_{20} \approx 1 : 10^4 : 10^8 : 10^{16}$, and assuming the computer time necessary is proportional to n_p, the computer times are approximately

4 places:	2 seconds	
8 places:	2×10^4 seconds	$\approx$ 6 hours
12 places:	2×10^8 seconds	$\approx$ 6 years
20 places:	2×10^{16} seconds	$\approx$ 600 million years

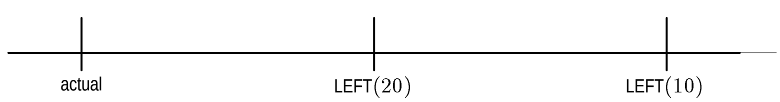
(b) For the trapezoidal rule, error is approximately proportional to $\frac{1}{n^2}$. If we let N_p be the number of subdivisions needed for accuracy to p places, then there is a constant C such that

$$\begin{aligned} 5 \times 10^{-5} &= \frac{1}{2} \times 10^{-4} \approx \frac{C}{{N_4}^2} \\ 5 \times 10^{-9} &= \frac{1}{2} \times 10^{-8} \approx \frac{C}{{N_8}^2} \\ 5 \times 10^{-13} &= \frac{1}{2} \times 10^{-12} \approx \frac{C}{{N_{12}}^2} \\ 5 \times 10^{-21} &= \frac{1}{2} \times 10^{-20} \approx \frac{C}{{N_{20}}^2} \end{aligned}$$

Thus the ratios ${N_4}^2 : {N_8}^2 : {N_{12}}^2 : {N_{20}}^2 \approx 1 : 10^4 : 10^8 : 10^{16}$, and the ratios $N_4 : N_8 : N_{12} : N_{20} \approx 1 : 10^2 : 10^4 : 10^8$. So the computer times are approximately

4 places:	2 seconds	
8 places:	2×10^2 seconds	$\approx$ 3 minutes
12 places:	2×10^4 seconds	$\approx$ 6 hours
20 places:	2×10^8 seconds	$\approx$ 6 years

8. We assume that the error is of the same sign for both $\text{LEFT}(10)$ and $\text{LEFT}(20)$; that is, they are both underestimates or overestimates. Since $\text{LEFT}(20) < \text{LEFT}(10)$, and $\text{LEFT}(20)$ is more accurate, they must both be overestimates.

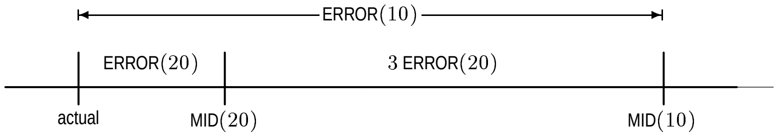


We assume that $\text{LEFT}(10)$ is twice as far from the actual value as $\text{LEFT}(20)$. Thus

$$\begin{aligned} \text{Actual } - \text{LEFT}(20) &= \text{LEFT}(20) - \text{LEFT}(10) \\ \text{Actual } &= 2\,\text{LEFT}(20) - \text{LEFT}(10) \\ &= 0.34289. \end{aligned}$$

Thus the error for $\text{LEFT}(10)$ is 0.04186.

9. Since the midpoint rule is sensitive to f'', the simplifying assumption should be that f'' does not change sign in the interval of integration. Thus $\text{MID}(10)$ and $\text{MID}(20)$ will both be overestimates or will both be underestimates. Since the larger number, $\text{MID}(10)$ is less accurate than the smaller number, they must both be overestimates. Then the information that $\text{ERROR}(10) = 4 \times \text{ERROR}(20)$ means that the the value of the integral and the two sums are arranged as follows:

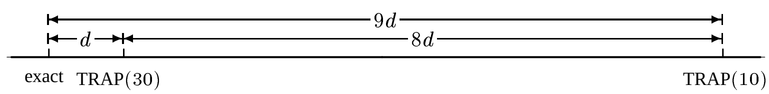


Thus

$$3 \times \text{ERROR}(20) = \text{MID}(10) - \text{MID}(20) = 35.619 - 35.415 = 0.204,$$

so $\text{ERROR}(20) = 0.068$ and $\text{ERROR}(10) = 4 \times \text{ERROR}(20) = 0.272$.

10. Since TRAP(n) seems to be decreasing as n increases, we can assume that TRAP(10) and TRAP(30) are both overestimates. We know that the error in the trapezoid rule is approximately proportional to $1/n^2$. In going from $n = 10$ to $n = 30$, n is multiplied by 3 and so we expect the error to go down roughly by a factor of $1/3^2$, or $1/9$. Therefore, if we let $d = |\text{error}(30)|$, then we have $9d \approx |\text{error}(10)|$.



We see from the figure above that the difference between TRAP(10) and TRAP(30) is $8d$, so

$$\begin{aligned} 8d &= \text{TRAP(10)}-\text{TRAP(30)} \\ 8d &= 12.676 - 10.420 \\ d &= 0.282. \end{aligned}$$

Since d is the magnitude of the error for TRAP(30), and since the exact value is less than TRAP(30), we have

$$\begin{aligned} \text{Exact} &\approx \text{TRAP(30)} - d \\ &= 10.420 - 0.282 \\ &= 10.138. \end{aligned}$$

The exact value[1] of the integral is about 10.138.

11. **(a)** If $f(x) = 1$, then

$$\int_a^b f(x)\,dx = (b-a).$$

Also,

$$\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left(\frac{1}{2} + 2 + \frac{1}{2}\right) = (b-a).$$

So the equation holds for $f(x) = 1$.

If $f(x) = x$, then

$$\int_a^b f(x)\,dx = \left.\frac{x^2}{2}\right|_a^b = \frac{b^2-a^2}{2}.$$

Also,

$$\frac{h}{3}\left(\frac{f(a)}{2} + 2f(m) + \frac{f(b)}{2}\right) = \frac{b-a}{3}\left(\frac{a}{2} + 2\frac{a+b}{2} + \frac{b}{2}\right)$$

[1]This method of improving numerical estimates is essentially equivalent to Richardson's h^2 extrapolation, also called extrapolation to the limit. See, for instance, *Survey of Numerical Analysis*, ed. John Todd, (New York: McGraw-Hill, 1962).

$$\begin{aligned}
&= \frac{b-a}{3}\left(\frac{a}{2}+a+b+\frac{b}{2}\right)\\
&= \frac{b-a}{3}\left(\frac{3}{2}b+\frac{3}{2}a\right)\\
&= \frac{(b-a)(b+a)}{2}\\
&= \frac{b^2-a^2}{2}.
\end{aligned}$$

So the equation holds for $f(x)=x$.

If $f(x)=x^2$, then $\int_a^b f(x)\,dx = \left.\frac{x^3}{3}\right|_a^b = \frac{b^3-a^3}{3}$. Also,

$$\begin{aligned}
\frac{h}{3}\left(\frac{f(a)}{2}+2f(m)+\frac{f(b)}{2}\right) &= \frac{b-a}{3}\left(\frac{a^2}{2}+2\left(\frac{a+b}{2}\right)^2+\frac{b^2}{2}\right)\\
&= \frac{b-a}{3}\left(\frac{a^2}{2}+\frac{a^2+2ab+b^2}{2}+\frac{b^2}{2}\right)\\
&= \frac{b-a}{3}\left(\frac{2a^2+2ab+2b^2}{2}\right)\\
&= \frac{b-a}{3}\left(a^2+ab+b^2\right)\\
&= \frac{b^3-a^3}{3}.
\end{aligned}$$

So the equation holds for $f(x)=x^2$.

(b) For any quadratic function, $f(x)=Ax^2+Bx+C$, the "Facts about Sums and Constant Multiples of Integrands" give us:

$$\int_a^b f(x)\,dx = \int_a^b (Ax^2+Bx+C)\,dx = A\int_a^b x^2\,dx + B\int_a^b x\,dx + C\int_a^b 1\,dx.$$

Now we use the results of part (a) to get:

$$\begin{aligned}
\int_a^b f(x)\,dx &= A\frac{h}{3}\left(\frac{a^2}{2}+2m^2+\frac{b^2}{2}\right)+B\frac{h}{3}\left(\frac{a}{2}+2m+\frac{b}{2}\right)+C\frac{h}{3}\left(\frac{1}{2}+2\cdot 1+\frac{1}{2}\right)\\
&= \frac{h}{3}\left(\frac{Aa^2+Ba+C}{2}+2(Am^2+Bm+C)+\frac{Ab^2+Bb+C}{2}\right)\\
&= \frac{h}{3}\left(\frac{f(a)}{2}+2f(m)+\frac{f(b)}{2}\right)
\end{aligned}$$

12. (a) Suppose $q_i(x)$ is the quadratic function approximating $f(x)$ on the subinterval $[x_i, x_{i+1}]$, and m_i is the midpoint of the interval, $m_i = (x_i + x_{i+1})/2$. Then, using the equation in Problem 11, with $a = x_i$ and $b = x_{i+1}$ and $h = \Delta x = x_{i+1} - x_i$:

$$\int_{x_i}^{x_{i+1}} f(x)dx \approx \int_{x_i}^{x_{i+1}} q_i(x)dx = \frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2}+2q_i(m_i)+\frac{q_i(x_{i+1})}{2}\right).$$

(b) Summing over all subintervals gives

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}} q_i(x)dx = \sum_{i=0}^{n-1}\frac{\Delta x}{3}\left(\frac{q_i(x_i)}{2}+2q_i(m_i)+\frac{q_i(x_{i+1})}{2}\right).$$

Splitting the sum into two parts:

$$\begin{aligned}
&= \frac{2}{3}\sum_{i=0}^{n-1} q_i(m_i)\Delta x + \frac{1}{3}\sum_{i=0}^{n-1}\frac{q_i(x_i)+q_i(x_{i+1})}{2}\Delta x\\
&= \frac{2}{3}\,\mathrm{MID}(n) + \frac{1}{3}\,\mathrm{TRAP}(n)\\
&= \mathrm{SIMP}(n).
\end{aligned}$$

Solutions for Section 7.7

Exercises

1. (a) See Figure 7.18. The area extends out infinitely far along the positive x-axis.

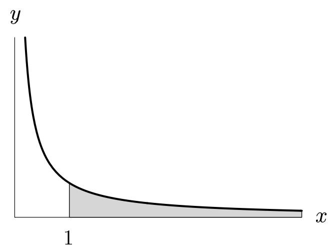


Figure 7.18

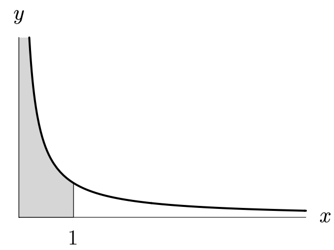


Figure 7.19

(b) See Figure 7.19. The area extends up infinitely far along the positive y-axis.

2. We have

$$\int_0^\infty e^{-0.4x}dx = \lim_{b\to\infty}\int_0^b e^{-0.4x}dx = \lim_{b\to\infty}(-2.5e^{-0.4x})\Big|_0^b = \lim_{b\to\infty}(-2.5e^{-0.4b}+2.5).$$

As $b\to\infty$, we know $e^{-0.4b}\to 0$ and so we see that the integral converges to 2.5. See Figure 7.20. The area continues indefinitely out to the right.

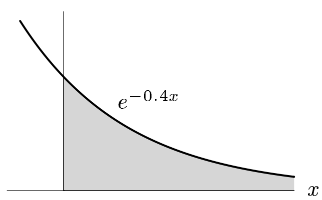


Figure 7.20

3. (a) We use a calculator or computer to evaluate the integrals.
When $b = 5$, we have $\int_0^5 xe^{-x}dx = 0.9596$.
When $b = 10$, we have $\int_0^{10} xe^{-x}dx = 0.9995$.
When $b = 20$, we have $\int_0^{20} xe^{-x}dx = 0.99999996$.

(b) It appears from the answers to part (a) that $\int_0^\infty xe^{-x}dx = 1.0$.

4. (a) See Figure 7.21. The total area under the curve is shaded.

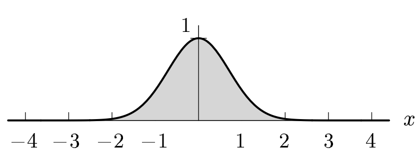


Figure 7.21

(b) When $a = 1$, we use a calculator or computer to see that $\int_{-1}^1 e^{-x^2}dx = 1.49365$.
Similarly, we have:
When $a = 2$, the value of the integral is 1.76416.
When $a = 3$, the value of the integral is 1.77241.
When $a = 4$, the value of the integral is 1.77245.
When $a = 5$, the value of the integral is 1.77245.

(c) It appears that the integral $\int_{-\infty}^\infty e^{-x^2}dx$ converges to approximately 1.77245.

5. We have

$$\int_1^\infty \frac{1}{5x+2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{5x+2}dx = \lim_{b\to\infty}\left(\frac{1}{5}\ln(5x+2)\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{1}{5}\ln(5b+2) - \frac{1}{5}\ln(7)\right).$$

As $b \leftarrow \infty$, we know that $\ln(5b+2) \to \infty$, and so this integral diverges.

6. We have

$$\int_1^\infty \frac{1}{(x+2)^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{(x+2)^2}dx = \lim_{b\to\infty}\left(\frac{-1}{x+2}\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{-1}{b+2} - \frac{-1}{3}\right) = 0 + \frac{1}{3} = \frac{1}{3}.$$

This integral converges to $1/3$.

7. We have

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty}\left(\frac{-1}{2}e^{-x^2}\right)\Big|_0^b = \lim_{b\to\infty}\left(\frac{-1}{2}e^{-b^2} - \frac{-1}{2}\right) = 0 + \frac{1}{2} = \frac{1}{2}.$$

This integral converges to $1/2$.

8.

$$\begin{aligned}\int_1^\infty e^{-2x}\,dx &= \lim_{b\to\infty}\int_1^b e^{-2x}\,dx = \lim_{b\to\infty} -\frac{e^{-2x}}{2}\Big|_1^b \\ &= \lim_{b\to\infty}(-e^{-2b}/2 + e^{-2}/2) = 0 + e^{-2}/2 = e^{-2}/2,\end{aligned}$$

where the first limit is 0 because $\lim_{x\to\infty} e^{-x} = 0$.

9. Using integration by parts with $u = x$ and $v' = e^{-x}$, we find that

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}dx = -(1+x)e^{-x}$$

so

$$\begin{aligned}\int_0^\infty \frac{x}{e^x}\,dx &= \lim_{b\to\infty}\int_0^b \frac{x}{e^x}\,dx \\ &= \lim_{b\to\infty} -1(1+x)e^{-x}\Big|_0^b \\ &= \lim_{b\to\infty}\left[1-(1+b)e^{-b}\right] \\ &= 1.\end{aligned}$$

10.

$$\int_1^\infty \frac{x}{4+x^2} = \lim_{b\to\infty}\int_1^b \frac{x}{4+x^2}\,dx = \lim_{b\to\infty}\frac{1}{2}\ln|4+x^2|\Big|_1^b = \lim_{b\to\infty}\frac{1}{2}\ln|4+b^2| - \frac{1}{2}\ln 5.$$

As $b \to \infty$, $\ln|4+b^2| \to \infty$, so the limit diverges.

11.

$$\begin{aligned}\int_{-\infty}^0 \frac{e^x}{1+e^x}\,dx &= \lim_{b\to-\infty}\int_b^0 \frac{e^x}{1+e^x}\,dx \\ &= \lim_{b\to-\infty}\ln|1+e^x|\Big|_b^0 \\ &= \lim_{b\to-\infty}[\ln|1+e^0| - \ln|1+e^b|] \\ &= \ln(1+1) - \ln(1+0) = \ln 2.\end{aligned}$$

12. First, we note that $1/(z^2+25)$ is an even function. Therefore,

$$\int_{-\infty}^{\infty}\frac{dz}{z^2+25}=\int_{-\infty}^{0}\frac{dz}{z^2+25}+\int_{0}^{\infty}\frac{dz}{z^2+25}=2\int_{0}^{\infty}\frac{dz}{z^2+25}.$$

We'll now evaluate this improper integral by using a limit:

$$\int_{0}^{\infty}\frac{dz}{z^2+25}=\lim_{b\to\infty}\left(\frac{1}{5}\arctan(b/5)-\frac{1}{5}\arctan(0)\right)=\frac{1}{5}\cdot\frac{\pi}{2}=\frac{\pi}{10}.$$

So the original integral is twice that, namely $\pi/5$.

13. This is an improper integral because $\sqrt{16-x^2}=0$ at $x=4$. So

$$\begin{aligned}\int_0^4\frac{dx}{\sqrt{16-x^2}}&=\lim_{b\to 4^-}\int_0^b\frac{dx}{\sqrt{16-x^2}}\\&=\lim_{b\to 4^-}(\arcsin x/4)\Big|_0^b\\&=\lim_{b\to 4^-}[\arcsin(b/4)-\arcsin(0)]=\pi/2-0=\pi/2.\end{aligned}$$

14.

$$\begin{aligned}\int_{\pi/4}^{\pi/2}\frac{\sin x}{\sqrt{\cos x}}\,dx&=\lim_{b\to\pi/2^-}\int_{\pi/4}^{b}\frac{\sin x}{\sqrt{\cos x}}\,dx\\&=\lim_{b\to\pi/2^-}-\int_{\pi/4}^{b}(\cos x)^{-1/2}(-\sin x)\,dx\\&=\lim_{b\to\pi/2^-}-2(\cos x)^{1/2}\Big|_{\pi/4}^{b}\\&=\lim_{b\to\pi/2^-}[-2(\cos b)^{1/2}+2(\cos\pi/4)^{1/2}]\\&=2\left(\frac{\sqrt{2}}{2}\right)^{\frac{1}{2}}=2^{\frac{3}{4}}.\end{aligned}$$

15. This integral is improper because $1/v$ is undefined at $v=0$. To evaluate it, we must split the region of integration up into two pieces, from 0 to 1 and from -1 to 0. But notice,

$$\int_0^1\frac{1}{v}\,dv=\lim_{b\to 0^+}\int_b^1\frac{1}{v}\,dv=\lim_{b\to 0^+}\left(\ln v\Big|_b^1\right)=-\ln b.$$

As $b\to 0^+$, this goes to infinity and the integral diverges, so our original integral also diverges.

16.

$$\lim_{a\to 0^+}\int_a^1\frac{x^4+1}{x}\,dx=\lim_{a\to 0^+}\left(\frac{x^4}{4}+\ln x\right)\Big|_a^1=\lim_{a\to 0^+}[1/4-(a^4/4+\ln a)],$$

which diverges as $a\to 0$, since $\ln a\to-\infty$.

17.

$$\begin{aligned}\int_1^{\infty}\frac{1}{x^2+1}\,dx&=\lim_{b\to\infty}\int_1^b\frac{1}{x^2+1}\,dx\\&=\lim_{b\to\infty}\arctan(x)\Big|_1^b\\&=\lim_{b\to\infty}[\arctan(b)-\arctan(1)]\\&=\pi/2-\pi/4=\pi/4.\end{aligned}$$

18.

$$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x^2+1}}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x^2+1}}\,dx \\ &= \lim_{b\to\infty} \ln|x+\sqrt{x^2+1}|\Big|_1^b \\ &= \lim_{b\to\infty} \ln(b+\sqrt{b^2+1}) - \ln(1+\sqrt{2}).\end{aligned}$$

As $b\to\infty$, this limit does not exist, so the integral diverges.

19. We use V-26 with $a=4$ and $b=-4$:

$$\begin{aligned}\int_0^4 \frac{1}{u^2-16}\,du &= \lim_{b\to 4^-}\int_0^b \frac{1}{u^2-16}\,du \\ &= \lim_{b\to 4^-}\int_0^b \frac{1}{(u-4)(u+4)}\,du \\ &= \lim_{b\to 4^-} \frac{(\ln|u-4| - \ln|u+4|)}{8}\Big|_0^b \\ &= \lim_{b\to 4^-} \frac{1}{8}\left(\ln|b-4| + \ln 4 - \ln|b+4| - \ln 4\right).\end{aligned}$$

As $b\to 4^-$, $\ln|b-4|\to-\infty$, so the limit does not exist and the integral diverges.

20.

$$\begin{aligned}\int_1^\infty \frac{y}{y^4+1}\,dy &= \lim_{b\to\infty}\frac{1}{2}\int_1^b \frac{2y}{(y^2)^2+1}\,dy \\ &= \lim_{b\to\infty}\frac{1}{2}\arctan(y^2)\Big|_1^b \\ &= \lim_{b\to\infty}\frac{1}{2}[\arctan(b^2) - \arctan 1] \\ &= (1/2)[\pi/2-\pi/4] = \pi/8.\end{aligned}$$

21. With the substitution $w=\ln x$, $dw=\frac{1}{x}dx$,

$$\int \frac{dx}{x\ln x} = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|\ln x| + C$$

so

$$\begin{aligned}\int_2^\infty \frac{dx}{x\ln x} &= \lim_{b\to\infty}\int_2^b \frac{dx}{x\ln x} \\ &= \lim_{b\to\infty}\ln|\ln x|\Big|_2^b \\ &= \lim_{b\to\infty}[\ln|\ln b| - \ln|\ln 2|].\end{aligned}$$

As $b\to\infty$, the limit goes to ∞ and hence the integral diverges.

22. With the substitution $w=\ln x$, $dw=\frac{1}{x}dx$,

$$\int \frac{\ln x}{x}\,dx = \int w\,dw = \frac{1}{2}w^2 + C = \frac{1}{2}(\ln x)^2 + C$$

so

$$\int_0^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{\ln x}{x}\,dx = \lim_{a\to 0^+}\frac{1}{2}[\ln(x)]^2\Big|_a^1 = \lim_{a\to 0^+} -\frac{1}{2}[\ln(a)]^2.$$

As $a\to 0^+$, $\ln a\to-\infty$, so the integral diverges.

23. This is a proper integral; use V-26 in the integral table with $a = 4$ and $b = -4$.

$$\begin{aligned}\int_{16}^{20} \frac{1}{y^2-16}\,dy &= \int_{16}^{20} \frac{1}{(y-4)(y+4)}\,dy \\ &= \left.\frac{\ln|y-4| - \ln|y+4|}{8}\right|_{16}^{20} \\ &= \frac{\ln 16 - \ln 24 - (\ln 12 - \ln 20)}{8} \\ &= \frac{\ln 320 - \ln 288}{8} = \frac{1}{8}\ln(10/9) = 0.01317.\end{aligned}$$

24. As in Problem 21, $\int \frac{dx}{x\ln x} = \ln|\ln x| + C$, so

$$\begin{aligned}\int_1^2 \frac{dx}{x\ln x} &= \lim_{b\to 1^+}\int_b^2 \frac{dx}{x\ln x} \\ &= \lim_{b\to 1^+} \ln|\ln x|\Big|_b^2 \\ &= \lim_{b\to 1^+} \ln(\ln 2) - \ln(\ln b).\end{aligned}$$

As $b \to 1^+$, $\ln(\ln b) \to -\infty$, so the integral diverges.

25. Using the substitution $w = -x^{\frac{1}{2}}$, $-2dw = x^{-\frac{1}{2}}\,dx$,

$$\int e^{-x^{\frac{1}{2}}} x^{-\frac{1}{2}}\,dx = -2\int e^w\,dw = -2e^{-x^{\frac{1}{2}}} + C.$$

So

$$\begin{aligned}\int_0^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\,dx &= \lim_{b\to 0^+}\int_b^{\pi} \frac{1}{\sqrt{x}} e^{-\sqrt{x}}\,dx \\ &= \lim_{b\to 0^+} -2e^{-\sqrt{x}}\Big|_b^{\pi} \\ &= 2 - 2e^{-\sqrt{\pi}}.\end{aligned}$$

26. Letting $w = \ln x$, $dw = \frac{1}{x}dx$,

$$\int \frac{dx}{x(\ln x)^2} = \int w^{-2}dw = -w^{-1} + C = -\frac{1}{\ln x} + C,$$

so

$$\begin{aligned}\int_3^{\infty} \frac{dx}{x(\ln x)^2} &= \lim_{b\to\infty}\int_3^b \frac{dx}{x(\ln x)^2} \\ &= \lim_{b\to\infty}\left(-\frac{1}{\ln b} + \frac{1}{\ln 3}\right) \\ &= \frac{1}{\ln 3}.\end{aligned}$$

27.

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx &= \lim_{b\to 2^-}\int_0^b \frac{1}{\sqrt{4-x^2}}\,dx \\ &= \lim_{b\to 2^-} \arcsin\frac{x}{2}\Big|_0^b \\ &= \lim_{b\to 2^-} \arcsin\frac{b}{2} = \arcsin 1 = \frac{\pi}{2}.\end{aligned}$$

28. $\displaystyle\int_4^{\infty} \frac{dx}{(x-1)^2} = \lim_{b\to\infty}\int_4^b \frac{dx}{(x-1)^2} = \lim_{b\to\infty} -\frac{1}{(x-1)}\Big|_4^b = \lim_{b\to\infty}\left[-\frac{1}{b-1} + \frac{1}{3}\right] = \frac{1}{3}.$

29. $\displaystyle\int \frac{dx}{x^2-1} = \int \frac{dx}{(x-1)(x+1)} = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C = \frac{1}{2}\left(\ln\frac{|x-1|}{|x+1|}\right) + C$, so

$$\begin{aligned}\int_4^\infty \frac{dx}{x^2-1} &= \lim_{b\to\infty}\int_4^b \frac{dx}{x^2-1} \\ &= \lim_{b\to\infty} \frac{1}{2}\left(\ln\frac{|x-1|}{|x+1|}\right)\Big|_4^b \\ &= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{b-1}{b+1}\right) - \frac{1}{2}\ln\frac{3}{5}\right] \\ &= -\frac{1}{2}\ln\frac{3}{5} = \frac{1}{2}\ln\frac{5}{3}.\end{aligned}$$

30.

$$\begin{aligned}\int_7^\infty \frac{dy}{\sqrt{y-5}} &= \lim_{b\to\infty}\int_7^b \frac{dy}{\sqrt{y-5}} \\ &= \lim_{b\to\infty} 2\sqrt{y-5}\Big|_7^b \\ &= \lim_{b\to\infty}(2\sqrt{b-5} - 2\sqrt{2}).\end{aligned}$$

As $b\to\infty$, this limit goes to ∞, so the integral diverges.

31. The integrand is undefined at $y=-3$ and $y=3$. To consider the limits one at a time, divide the integral at $y=0$;

$$\begin{aligned}\int_0^3 \frac{y\,dy}{\sqrt{9-y^2}} &= \lim_{b\to 3^-}\int_0^b \frac{y}{\sqrt{9-y^2}}\,dy = \lim_{b\to 3^-}\left(-(9-y^2)^{1/2}\right)\Big|_0^b \\ &= \lim_{b\to 3^-}\left(3-(9-b^2)^{1/2}\right) = 3.\end{aligned}$$

A similar argument shows that

$$\begin{aligned}\int_{-3}^0 \frac{y\,dy}{\sqrt{9-y^2}} &= \lim_{b\to -3^+}\int_b^0 \frac{y}{\sqrt{9-y^2}}\,dy = \lim_{b\to -3^+}\left(-(9-y^2)^{1/2}\right)\Big|_b^0 \\ &= \lim_{b\to -3^+}\left(-3+(9-b^2)^{1/2}\right) = -3.\end{aligned}$$

Thus the original integral converges to a value of 0:

$$\int_{-3}^3 \frac{y\,dy}{\sqrt{9-y^2}} = \int_{-3}^0 \frac{y\,dy}{\sqrt{9-y^2}} + \int_0^3 \frac{y\,dy}{\sqrt{9-y^2}} = -3+3=0.$$

32. The integrand is undefined at $\theta=4$, so we must split the integral there.

$$\int_4^6 \frac{d\theta}{(4-\theta)^2} = \lim_{a\to 4^+}\int_a^6 \frac{d\theta}{(4-\theta)^2} = \lim_{a\to 4^+}(4-\theta)^{-1}\Big|_a^6 = \lim_{a\to 4^+}\left(\frac{1}{-2} - \frac{1}{4-a}\right).$$

Since $1/(4-a)\to -\infty$ as $a\to 4$ from the right, the integral does not converge. It is not necessary to check the convergence of $\int_3^4 \frac{d\theta}{(4-\theta)^2}$. However, we could have started with $\int_3^4 \frac{d\theta}{(4-\theta)^2}$, instead of $\int_4^6 \frac{d\theta}{(4-\theta)^2}$, and arrived at the same conclusion.

Problems

33. Since the graph is above the x-axis for $x\ge 0$, we have

$$\text{Area} = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$

$$= \lim_{b\to\infty} \left(-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx \right)$$
$$= \lim_{b\to\infty} \left(-be^{-b} - e^{-x}\Big|_0^b \right)$$
$$= \lim_{b\to\infty} (-be^{-b} - e^{-b} + e^0) = 1.$$

34. The curve has an asymptote at $t = \frac{\pi}{2}$, and so the area integral is improper there.

$$\text{Area } = \int_0^{\frac{\pi}{2}} \frac{dt}{\cos^2 t} = \lim_{b\to\frac{\pi}{2}} \int_0^b \frac{dt}{\cos^2 t} = \lim_{b\to\frac{\pi}{2}} \tan t \Big|_0^b,$$

which diverges. Therefore the area is infinite.

35. We let $t = (x-a)/\sqrt{b}$. This means that $dt = dx/\sqrt{b}$, and that $t = \pm\infty$ when $x = \pm\infty$. We have

$$\int_{-\infty}^{\infty} e^{-(x-a)^2/b}\,dx = \int_{-\infty}^{\infty} e^{-t^2}\,(\sqrt{b}\,dt) = \sqrt{b}\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{b}\sqrt{\pi} = \sqrt{b\pi}.$$

36. The factor $\ln x$ grows slowly enough not to change the convergence or divergence of the integral, although it will change what it converges or diverges to.

Integrating by parts or using the table of integrals, we get

$$\int_e^{\infty} x^p \ln x\,dx = \lim_{b\to\infty} \int_e^b x^p \ln x\,dx$$
$$= \lim_{b\to\infty} \left[\frac{1}{p+1} x^{p+1} \ln x - \frac{1}{(p+1)^2} x^{p+1} \right] \Bigg|_e^b$$
$$= \lim_{b\to\infty} \left[\left(\frac{1}{p+1} b^{p+1} \ln b - \frac{1}{(p+1)^2} b^{p+1} \right) - \left(\frac{1}{p+1} e^{p+1} - \frac{1}{(p+1)^2} e^{p+1} \right) \right].$$

If $p > -1$, then $(p+1)$ is positive and the limit does not exist since b^{p+1} and $\ln b$ both approach ∞ as b does.

If $p < -1$, then $(p+1)$ is negative and both b^{p+1} and $b^{p+1}\ln b$ approach 0 as $b \to \infty$. (This follows by looking at graphs of $x^{p+1}\ln x$ (for different values of p), or by noting that $\ln x$ grows more slowly than x^{p+1} tends to 0.) So the value of the integral is $-pe^{p+1}/(p+1)^2$.

The case $p = -1$ has to be handled separately. For $p = -1$,

$$\int_e^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_e^b \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \frac{(\ln x)^2}{2}\Bigg|_e^b = \lim_{b\to\infty} \left(\frac{(\ln b)^2 - 1}{2} \right).$$

As $b \to \infty$, this limit does not exist, so the integral diverges if $p = -1$.

To summarize, $\int_e^{\infty} x^p \ln x\,dx$ converges for $p < -1$ to the value $-pe^{p+1}/(p+1)^2$.

37. The factor $\ln x$ grows slowly enough (as $x \to 0^+$) not to change the convergence or divergence of the integral, although it will change what it converges or diverges to.

The integral is always improper, because $\ln x$ is not defined for $x = 0$. Integrating by parts (or, alternatively, the integral table) yields

$$\int_0^e x^p \ln x\,dx = \lim_{a\to 0^+} \int_a^e x^p \ln x\,dx$$
$$= \lim_{a\to 0^+} \left(\frac{1}{p+1} x^{p+1} \ln x - \frac{1}{(p+1)^2} x^{p+1} \right) \Bigg|_a^e$$
$$= \lim_{a\to 0^+} \left[\left(\frac{1}{p+1} e^{p+1} - \frac{1}{(p+1)^2} e^{p+1} \right) - \left(\frac{1}{p+1} a^{p+1} \ln a - \frac{1}{(p+1)^2} a^{p+1} \right) \right].$$

If $p < -1$, then $(p+1)$ is negative, so as $a \to 0^+$, $a^{p+1} \to \infty$ and $\ln a \to -\infty$, and therefore the limit does not exist.

If $p > -1$, then $(p+1)$ is positive and it's easy to see that $a^{p+1} \to 0$ as $a \to 0$. Looking at graphs of $x^{p+1} \ln x$ (for different values of p) shows that $a^{p+1} \ln a \to 0$ as $a \to 0$. This isn't so easy to see analytically. It's true because if we let $t = \frac{1}{a}$ then

$$\lim_{a\to 0^+} a^{p+1} \ln a = \lim_{t\to\infty} \left(\frac{1}{t}\right)^{p+1} \ln\left(\frac{1}{t}\right) = \lim_{t\to\infty} -\frac{\ln t}{t^{p+1}}.$$

This last limit is zero because $\ln t$ grows very slowly, much more slowly than t^{p+1}. So if $p > -1$, the integral converges and equals $e^{p+1}[1/(p+1) - 1/(p+1)^2] = pe^{p+1}/(p+1)^2$.

What happens if $p = -1$? Then we get

$$\begin{aligned}\int_0^e \frac{\ln x}{x}\,dx &= \lim_{a\to 0^+} \int_a^e \frac{\ln x}{x}\,dx \\ &= \lim_{a\to 0^+} \frac{(\ln x)^2}{2}\bigg|_a^e \\ &= \lim_{a\to 0^+} \left(\frac{1-(\ln a)^2}{2}\right).\end{aligned}$$

Since $\ln a \to -\infty$ as $a \to 0^+$, this limit does not exist.

To summarize, $\int_0^e x^p \ln x$ converges for $p > -1$ to the value $pe^{p+1}/(p+1)^2$.

38. **(a)**

$$\begin{aligned}\Gamma(1) &= \int_0^\infty e^{-t}\,dt \\ &= \lim_{b\to\infty} \int_0^b e^{-t}\,dt \\ &= \lim_{b\to\infty} -e^{-t}\bigg|_0^b \\ &= \lim_{b\to\infty} [1 - e^{-b}] = 1.\end{aligned}$$

Using Problem 9,

$$\Gamma(2) = \int_0^\infty te^{-t}\,dt = 1.$$

(b) We integrate by parts. Let $u = t^n$, $v' = e^{-t}$. Then $u' = nt^{n-1}$ and $v = -e^{-t}$, so

$$\int t^n e^{-t}\,dt = -t^n e^{-t} + n \int t^{n-1} e^{-t}\,dt.$$

So

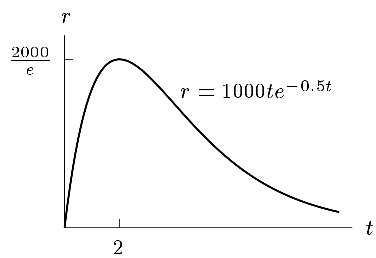
$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty t^n e^{-t}\,dt \\ &= \lim_{b\to\infty} \int_0^b t^n e^{-t}\,dt \\ &= \lim_{b\to\infty} \left[-t^n e^{-t}\bigg|_0^b + n\int_0^b t^{n-1}e^{-t}\,dt\right] \\ &= \lim_{b\to\infty} -b^n e^{-b} + \lim_{b\to\infty} n \int_0^b t^{n-1} e^{-t}\,dt \\ &= 0 + n\int_0^\infty t^{n-1}e^{-t}\,dt \\ &= n\Gamma(n).\end{aligned}$$

(c) We already have $\Gamma(1) = 1$ and $\Gamma(2) = 1$. Using $\Gamma(n+1) = n\Gamma(n)$ we can get

$$\Gamma(3) = 2\Gamma(2) = 2$$
$$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2$$
$$\Gamma(5) = 4\Gamma(4) = 4 \cdot 3 \cdot 2.$$

So it appears that $\Gamma(n)$ is just the first $n-1$ numbers multiplied together, so $\Gamma(n) = (n-1)!$.

39. (a) Using a calculator or a computer, the graph is:



(b) People are getting sick fastest when the rate of infection is highest, i.e. when r is at its maximum. Since

$$\begin{aligned} r' &= 1000e^{-0.5t} - 1000(0.5)te^{-0.5t} \\ &= 500e^{-0.5t}(2-t) \end{aligned}$$

this must occur at $t = 2$.

(c) The total number of sick people $= \displaystyle\int_0^\infty 1000te^{-0.5t}\,dt$.

Using integration by parts, with $u = t$, $v' = e^{-0.5t}$:

$$\begin{aligned} \text{Total} &= \lim_{b\to\infty} 1000\left(\frac{-t}{0.5}e^{-0.5t}\bigg|_0^b - \int_0^b \frac{-1}{0.5}e^{-0.5t}\,dt\right) \\ &= \lim_{b\to\infty} 1000\left(-2be^{-0.5b} - \frac{2}{0.5}e^{-0.5b}\right)\bigg|_0^b \\ &= \lim_{b\to\infty} 1000\left(-2be^{-0.5b} - 4e^{-0.5b} + 4\right) \\ &= 4000 \text{ people.} \end{aligned}$$

40. The energy required is

$$\begin{aligned} E &= \int_1^\infty \frac{kq_1q_2}{r^2}\,dr = kq_1q_2 \lim_{b\to\infty} -\frac{1}{r}\bigg|_1^b \\ &= (9\times10^9)(1)(1)(1) = 9\times10^9 \text{ joules} \end{aligned}$$

Solutions for Section 7.8

Exercises

1. For large x, the integrand behaves like $1/x^2$ because

$$\frac{x^2}{x^4+1} \approx \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Since $\int_1^\infty \frac{dx}{x^2}$ converges, we expect our integral to converge. More precisely, since $x^4+1 > x^4$, we have

$$\frac{x^2}{x^4+1} < \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Since $\int_1^\infty \frac{dx}{x^2}$ is convergent, the comparison test tells us that $\int_1^\infty \frac{x^2}{x^4+1}\,dx$ converges also.

2. For large x, the integrand behaves like $1/x$ because

$$\frac{x^3}{x^4-1} \approx \frac{x^3}{x^4} = \frac{1}{x}.$$

Since $\int_2^\infty \frac{1}{x}\,dx$ does not converge, we expect our integral not to converge. More precisely, since $x^4-1 < x^4$, we have

$$\frac{x^3}{x^4-1} > \frac{x^3}{x^4} = \frac{1}{x}.$$

Since $\int_2^\infty \frac{1}{x}\,dx$ does not converge, the comparison test tells us that $\int_2^\infty \frac{x^3}{x^4-1}\,dx$ does not converge either.

3. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x^2+1}{x^3+3x+2}$ behaves like $\frac{x^2}{x^3}$ or $\frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, we predict that the given integral will diverge.

4. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{1}{x^2+5x+1}$ behaves like $\frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, we predict that the given integral will converge.

5. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x}{x^2+2x+4}$ behaves like $\frac{x}{x^2}$ or $\frac{1}{x}$. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, we predict that the given integral will diverge.

6. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x^2-6x+1}{x^2+4}$ behaves like $\frac{x^2}{x^2}$ or 1. Since $\int_1^\infty 1\,dx$ diverges, we predict that the given integral will diverge.

7. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{5x+2}{x^4+8x^2+4}$ behaves like $\frac{5x}{x^4}$ or $\frac{5}{x^3}$. Since $\int_1^\infty \frac{5}{x^3}\,dx$ converges, we predict that the given integral will converge.

8. For large t, the 2 is negligible in comparison to e^{5t}, so the integrand behaves like e^{-5t}. Thus

$$\frac{1}{e^{5t}+2} \approx \frac{1}{e^{5t}} = e^{-5t}.$$

More precisely, since $e^{5t}+2 > e^{5t}$, we have

$$\frac{1}{e^{5t}+2} < \frac{1}{e^{5t}} = e^{-5t}.$$

Since $\int_1^\infty e^{-5t}\,dt$ converges, by the Comparison Theorem $\int_1^\infty \frac{1}{e^{5t}+2}\,dt$ converges also.

9. The integrand is continuous for all $x \geq 1$, so whether the integral converges or diverges depends only on the behavior of the function as $x \to \infty$. As $x \to \infty$, polynomials behave like the highest powered term. Thus, as $x \to \infty$, the integrand $\frac{x^2+4}{x^4+3x^2+11}$ behaves like $\frac{x^2}{x^4}$ or $\frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges, we predict that the given integral will converge.

10. It converges:

$$\int_{50}^{\infty} \frac{dz}{z^3} = \lim_{b\to\infty} \int_{50}^{b} \frac{dz}{z^3} = \lim_{b\to\infty} \left(-\frac{1}{2} z^{-2} \Big|_{50}^{b} \right) = \frac{1}{2} \lim_{b\to\infty} \left(\frac{1}{50^2} - \frac{1}{b^2} \right) = \frac{1}{5000}$$

11. Since $\dfrac{1}{1+x} \geq \dfrac{1}{2x}$ and $\dfrac{1}{2}\displaystyle\int_0^{\infty} \frac{1}{x}\,dx$ diverges, we have that $\displaystyle\int_1^{\infty} \frac{dx}{1+x}$ diverges.

12. If $x \geq 1$, we know that $\dfrac{1}{x^3+1} \leq \dfrac{1}{x^3}$, and since $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$ converges, the improper integral $\displaystyle\int_1^{\infty} \frac{dx}{x^3+1}$ converges.

13. The integrand is unbounded as $t \to 5$. We substitute $w = t - 5$, so $dw = dt$. When $t = 5$, $w = 0$ and when $t = 8$, $w = 3$.

$$\int_5^8 \frac{6}{\sqrt{t-5}}\,dt = \int_0^3 \frac{6}{\sqrt{w}}\,dw.$$

Since

$$\int_0^3 \frac{6}{\sqrt{w}}\,dw = \lim_{a\to 0^+} 6\int_a^3 \frac{1}{\sqrt{w}}\,dw = 6 \lim_{a\to 0^+} 2w^{1/2}\Big|_a^3 = 12 \lim_{a\to 0^+} (\sqrt{3} - \sqrt{a}) = 12\sqrt{3},$$

our integral converges.

14. The integral converges.

$$\int_0^1 \frac{1}{x^{19/20}}\,dx = \lim_{a\to 0} \int_a^1 \frac{1}{x^{19/20}}\,dx = \lim_{a\to 0} 20x^{1/20}\Big|_a^1 = \lim_{a\to 0} 20\left(1 - a^{1/20}\right) = 20.$$

15. This integral diverges. To see this, substitute $t + 1 = w$, $dt = dw$. So,

$$\int_{t=-1}^{t=5} \frac{dt}{(t+1)^2} = \int_{w=0}^{w=6} \frac{dw}{w^2},$$

which diverges.

16. Since we know the antiderivative of $\dfrac{1}{1+u^2}$, we can use the Fundamental Theorem of Calculus to evaluate the integral. Since the integrand is even, we write

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{du}{1+u^2} &= 2\int_0^{\infty} \frac{du}{1+u^2} = 2\lim_{b\to\infty} \int_0^b \frac{du}{1+u^2} \\ &= 2\lim_{b\to\infty} \arctan b = 2\left(\frac{\pi}{2}\right) = \pi. \end{aligned}$$

Thus, the integral converges to π.

17. Since $\dfrac{1}{u+u^2} < \dfrac{1}{u^2}$ for $u \geq 1$, and since $\displaystyle\int_1^{\infty} \frac{du}{u^2}$ converges, $\displaystyle\int_1^{\infty} \frac{du}{u+u^2}$ converges.

18. This improper integral diverges. We expect this because, for large θ, $\dfrac{1}{\sqrt{\theta^2+1}} \approx \dfrac{1}{\sqrt{\theta^2}} = \dfrac{1}{\theta}$ and $\displaystyle\int_1^{\infty} \frac{d\theta}{\theta}$ diverges. More precisely, for $\theta \geq 1$

$$\frac{1}{\sqrt{\theta^2+1}} \geq \frac{1}{\sqrt{\theta^2+\theta^2}} = \frac{1}{\sqrt{2}\sqrt{\theta^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\theta}$$

and $\displaystyle\int_1^{\infty} \frac{d\theta}{\theta}$ diverges. (The factor $\frac{1}{\sqrt{2}}$ doesn't affect the divergence.)

19. For $\theta \geq 2$, we have $\dfrac{1}{\sqrt{\theta^3+1}} \leq \dfrac{1}{\sqrt{\theta^3}} = \dfrac{1}{\theta^{\frac{3}{2}}}$, and $\displaystyle\int_2^{\infty} \frac{d\theta}{\theta^{3/2}}$ converges (check by integration), so $\displaystyle\int_2^{\infty} \frac{d\theta}{\sqrt{\theta^3+1}}$ converges.

20. This integral is improper at $\theta = 0$. For $0 \leq \theta \leq 1$, we have $\dfrac{1}{\sqrt{\theta^3+\theta}} \leq \dfrac{1}{\sqrt{\theta}}$, and since $\displaystyle\int_0^1 \frac{1}{\sqrt{\theta}}\,d\theta$ converges, $\displaystyle\int_0^1 \frac{d\theta}{\sqrt{\theta^3+\theta}}$ converges.

21. Since $\dfrac{1}{1+e^y} \leq \dfrac{1}{e^y} = e^{-y}$ and $\displaystyle\int_0^{\infty} e^{-y}\,dy$ converges, the integral $\displaystyle\int_0^{\infty} \frac{dy}{1+e^y}$ converges.

22. This integral is convergent because, for $\phi \geq 1$,

$$\frac{2+\cos\phi}{\phi^2} \leq \frac{3}{\phi^2},$$

and $\int_1^\infty \frac{3}{\phi^2} d\phi = 3\int_1^\infty \frac{1}{\phi^2} d\phi$ converges.

23. Since $\frac{1}{e^z+2^z} < \frac{1}{e^z} = e^{-z}$ for $z \geq 0$, and $\int_0^\infty e^{-z} dz$ converges, $\int_0^\infty \frac{dz}{e^z+2^z}$ converges.

24. Since $\frac{1}{\phi^2} \leq \frac{2-\sin\phi}{\phi^2}$ for $0 < \phi \leq \pi$, and since $\int_0^\pi \frac{1}{\phi^2} d\phi$ diverges, $\int_0^\pi \frac{2-\sin\phi}{\phi^2} d\phi$ must diverge.

25. Since $\frac{3+\sin\alpha}{\alpha} \geq \frac{2}{\alpha}$ for $\alpha \geq 4$, and since $\int_4^\infty \frac{2}{\alpha} d\alpha$ diverges, then $\int_4^\infty \frac{3+\sin\alpha}{\alpha} d\alpha$ diverges.

26. If we integrate e^{-x^2} from 1 to 10, we get 0.139. This answer doesn't change noticeably if you extend the region of integration to from 1 to 11, say, or even up to 1000. There's a reason for this; and the reason is that the tail, $\int_{10}^\infty e^{-x^2}\,dx$, is very small indeed. In fact

$$\int_{10}^\infty e^{-x^2}\,dx \leq \int_{10}^\infty e^{-x}\,dx = e^{-10},$$

which is very small. (In fact, the tail integral is less than $e^{-100}/10$. Can you prove that? [Hint: $e^{-x^2} \leq e^{-10x}$ for $x \geq 10$.])

27. Approximating the integral by $\int_0^{10} e^{-x^2}\cos^2 x\,dx$ yields 0.606 to two decimal places. This is a good approximation to the improper integral because the "tail" is small:

$$\int_{10}^\infty e^{-x^2}\cos^2 x\,dx \leq \int_{10}^\infty e^{-x}\,dx = e^{-10},$$

which is very small.

Problems

28. **(a)** The area is infinite. The area under $1/x$ is infinite and the area under $1/x^2$ is 1. So the area between the two has to be infinite also.

(b) Since $f(x)$ is bounded between 0 and $1/x^2$, and the area under $1/x^2$ is finite, $f(x)$ will have finite area by the comparison test. Similarly, $h(x)$ lies above $1/x$, whose area is infinite, so $h(x)$ must have infinite area as well. We can tell nothing about the area of $g(x)$, because the comparison test tells us nothing about a function larger than a function with finite area but smaller than one with infinite area. Finally, $k(x)$ will certainly have infinite area, because it has a lower bound m, for some $m > 0$. Thus, $\int_0^a k(x)\,dx \geq ma$, and since the latter does not converge as $a \to \infty$, neither can the former.

29. First let's calculate the indefinite integral $\int \frac{dx}{x(\ln x)^p}$. Let $\ln x = w$, then $\frac{dx}{x} = dw$. So

$$\begin{aligned}\int \frac{dx}{x(\ln x)^p} &= \int \frac{dw}{w^p} \\ &= \begin{cases} \ln|w| + C, & \text{if } p = 1 \\ \frac{1}{1-p}w^{1-p} + C, & \text{if } p \neq 1 \end{cases} \\ &= \begin{cases} \ln|\ln x| + C, & \text{if } p = 1 \\ \frac{1}{1-p}(\ln x)^{1-p} + C, & \text{if } p \neq 1. \end{cases}\end{aligned}$$

Notice that $\lim\limits_{x\to\infty} \ln x = +\infty$.

(a) $p = 1$:

$$\int_2^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\Big(\ln|\ln b| - \ln|\ln 2|\Big) = +\infty.$$

(b) $p < 1$:

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \frac{1}{1-p}\left(\lim_{b\to\infty}(\ln b)^{1-p} - (\ln 2)^{1-p}\right) = +\infty.$$

(c) $p > 1$:

$$\begin{aligned}\int_2^\infty \frac{dx}{x(\ln x)^p} &= \frac{1}{1-p}\left(\lim_{b\to\infty}(\ln b)^{1-p} - (\ln 2)^{1-p}\right)\\ &= \frac{1}{1-p}\left(\lim_{b\to\infty}\frac{1}{(\ln b)^{p-1}} - (\ln 2)^{1-p}\right)\\ &= -\frac{1}{1-p}(\ln 2)^{1-p}.\end{aligned}$$

Thus, $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$ is convergent for $p > 1$, divergent for $p \le 1$.

30. The indefinite integral $\displaystyle\int \frac{dx}{x(\ln x)^p}$ is computed in Problem 29. Let $\ln x = w$, then $\dfrac{dx}{x} = dw$. Notice that $\lim_{x\to 1}\ln x = 0$, and $\lim_{x\to 0^+}\ln x = -\infty$.

For this integral notice that $\ln 1 = 0$, so the integrand blows up at $x = 1$.

(a) $p = 1$:

$$\int_1^2 \frac{dx}{x\ln x} = \lim_{a\to 1^+}(\ln|\ln 2| - \ln|\ln a|)$$

Since $\ln a \to 0$ as $a \to 1$, $\ln|\ln a| \to -\infty$ as $b \to 1$. So the integral is divergent.

(b) $p < 1$:

$$\begin{aligned}\int_1^2 \frac{dx}{x(\ln x)^p} &= \frac{1}{1-p}\lim_{a\to 1^+}\left((\ln 2)^{1-p} - (\ln a)^{1-p}\right)\\ &= \frac{1}{1-p}(\ln 2)^{1-p}.\end{aligned}$$

(c) $p > 1$:

$$\int_1^2 \frac{dx}{x(\ln x)^p} = \frac{1}{1-p}\lim_{a\to 1^+}\left((\ln 2)^{1-p} - (\ln a)^{1-p}\right)$$

As $\displaystyle\lim_{a\to 1^+}(\ln a)^{1-p} = \lim_{a\to 1^+}\frac{1}{(\ln a)^{p-1}} = +\infty$, the integral diverges.

Thus, $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ is convergent for $p < 1$, divergent for $p \ge 1$.

31. To find a, we first calculate $\int_0^{10} e^{-\frac{x^2}{2}}\,dx$. Since $\frac{x^2}{2} \ge x$ for $x \ge 10$, this will differ from $\int_0^\infty e^{-\frac{x^2}{2}}\,dx$ by at most

$$\int_{10}^\infty e^{-\frac{x^2}{2}}\,dx \le \int_{10}^\infty e^{-x}\,dx = e^{-10},$$

which is very small. Using Simpson's rule with 100 intervals (well more than necessary), we find $\int_0^{10} e^{-\frac{x^2}{2}}\,dx \approx 1.253314137$. Thus, since $e^{-\frac{x^2}{2}}$ is even, $\int_{-10}^{10} e^{-\frac{x^2}{2}}\,dx \approx 2.506628274$, and this is extremely close to $\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\,dx$. To find a, we need $\int_{-\infty}^\infty ae^{-\frac{x^2}{2}}\,dx = 1$.

$$a = \frac{1}{\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\,dx} \approx 0.399 \text{ to three decimal places.}$$

32. **(a)** If we substitute $w = x - k$ and $dw = dx$, we find

$$\int_{-\infty}^{\infty} ae^{-\frac{(x-k)^2}{2}}\,dx = \int_{-\infty}^{\infty} ae^{-\frac{w^2}{2}}\,dw.$$

This integral is the same as the integral in Problem 31, so the value of a will be the same, namely 0.399.

(b) The answer is the same because $g(x)$ is the same as $f(x)$ in Problem 31 except that it is shifted by k to the right. Since we are integrating from $-\infty$ to ∞, however, this shift doesn't mean anything for the integral.

33. **(a)** Since $e^{-x^2} \leq e^{-3x}$ for $x \geq 3$,

$$\int_3^{\infty} e^{-x^2}\,dx \leq \int_3^{\infty} e^{-3x}\,dx$$

Now

$$\int_3^{\infty} e^{-3x}\,dx = \lim_{b\to\infty}\int_3^{b} e^{-3x}\,dx = \lim_{b\to\infty} -\frac{1}{3}e^{-3x}\Big|_3^b$$
$$= \lim_{b\to\infty}\frac{e^{-9}}{3} - \frac{e^{-3b}}{3} = \frac{e^{-9}}{3}.$$

Thus

$$\int_3^{\infty} e^{-x^2}\,dx \leq \frac{e^{-9}}{3}.$$

(b) By reasoning similar to part (a),

$$\int_n^{\infty} e^{-x^2}\,dx \leq \int_n^{\infty} e^{-nx}\,dx,$$

and

$$\int_n^{\infty} e^{-nx}\,dx = \frac{1}{n}e^{-n^2},$$

so

$$\int_n^{\infty} e^{-x^2}\,dx \leq \frac{1}{n}e^{-n^2}.$$

34. **(a)** The tangent line to e^t has slope $(e^t)' = e^t$. Thus at $t = 0$, the slope is $e^0 = 1$. The line passes through $(0, e^0) = (0, 1)$. Thus the equation of the tangent line is $y = 1 + t$. Since e^t is everywhere concave up, its graph is always above the graph of any of its tangent lines; in particular, e^t is always above the line $y = 1 + t$. This is tantamount to saying

$$1 + t \leq e^t,$$

with equality holding only at the point of tangency, $t = 0$.

(b) If $t = \frac{1}{x}$, then the above inequality becomes

$$1 + \frac{1}{x} \leq e^{1/x}, \text{ or } e^{1/x} - 1 \geq \frac{1}{x}.$$

Since $t = \frac{1}{x}$, t is never zero. Therefore, the inequality is strict, and we write

$$e^{1/x} - 1 > \frac{1}{x}.$$

(c) Since $e^{1/x} - 1 > \frac{1}{x}$,

$$\frac{1}{x^5\left(e^{1/x} - 1\right)} < \frac{1}{x^5\left(\frac{1}{x}\right)} = \frac{1}{x^4}.$$

Since $\int_1^{\infty} \frac{dx}{x^4}$ converges, $\int_1^{\infty} \frac{dx}{x^5\left(e^{1/x} - 1\right)}$ converges.

Solutions for Chapter 7 Review

Exercises

1. Since $\frac{d}{dt}\cos t = -\sin t$, we have

$$\int \sin t\,dt = -\cos t + C, \quad \text{where } C \text{ is a constant.}$$

2. Let $2t = w$, then $2dt = dw$, so $dt = \frac{1}{2}dw$, so

$$\int \cos 2t\,dt = \int \frac{1}{2}\cos w\,dw = \frac{1}{2}\sin w + C = \frac{1}{2}\sin 2t + C,$$

where C is a constant.

3. Let $5z = w$, then $5dz = dw$, which means $dz = \frac{1}{5}dw$, so

$$\int e^{5z}\,dz = \int e^w \cdot \frac{1}{5}dw = \frac{1}{5}\int e^w\,dw = \frac{1}{5}e^w + C = \frac{1}{5}e^{5z} + C,$$

where C is a constant.

4. Using the power rule gives $\frac{3}{2}w^2 + 7w + C$.

5. Since $\int \sin w\,d\theta = -\cos w + C$, the substitution $w = 2\theta$, $dw = 2\,d\theta$ gives $\int \sin 2\theta\,d\theta = -\frac{1}{2}\cos 2\theta + C$.

6. Let $w = x^3 - 1$, then $dw = 3x^2dx$ so that

$$\int (x^3-1)^4 x^2 dx = \frac{1}{3}\int w^4 dw = \frac{1}{15}w^5 + C = \frac{1}{15}(x^3-1)^5 + C.$$

7. The power rule gives $\frac{2}{5}x^{5/2} + \frac{3}{5}x^{5/3} + C$

8. From the rule for antidifferentiation of exponentials, we get

$$\int (e^x + 3^x)\,dx = e^x + \frac{1}{\ln 3}\cdot 3^x + C.$$

9. Either expand $(r+1)^3$ or use the substitution $w = r+1$. If $w = r+1$, then $dw = dr$ and

$$\int (r+1)^3\,dr = \int w^3\,dw = \frac{1}{4}w^4 + C = \frac{1}{4}(r+1)^4 + C.$$

10. Rewrite the integrand as

$$\int \left(\frac{4}{x^2} - \frac{3}{x^3}\right)\,dx = 4\int x^{-2}\,dx - 3\int x^{-3}\,dx = -4x^{-1} + \frac{3}{2}x^{-2} + C.$$

11. Dividing by x^2 gives

$$\int \left(\frac{x^3+x+1}{x^2}\right)dx = \int \left(x + \frac{1}{x} + \frac{1}{x^2}\right)dx = \frac{1}{2}x^2 + \ln|x| - \frac{1}{x} + C.$$

12. Let $w = 1 + \ln x$, then $dw = dx/x$ so that

$$\int \frac{(1+\ln x)^2}{x}\,dx = \int w^2 dw = \frac{1}{3}w^3 + C = \frac{1}{3}(1+\ln x)^3 + C.$$

13. Substitute $w = t^2$, so $dw = 2t\,dt$.

$$\int te^{t^2}\,dt = \frac{1}{2}\int e^{t^2}2t\,dt = \frac{1}{2}\int e^w\,dw = \frac{1}{2}e^w + C = \frac{1}{2}e^{t^2} + C.$$

Check:

$$\frac{d}{dt}\left(\frac{1}{2}e^{t^2} + C\right) = 2t\left(\frac{1}{2}e^{t^2}\right) = te^{t^2}.$$

14. Integration by parts with $u = x$, $v' = \cos x$ gives

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx + C = x\sin x + \cos x + C.$$

Or use III-16 with $p(x) = x$ and $a = 1$ in the integral table.

15. Integration by parts twice gives

$$\int x^2e^{2x}\,dx = \frac{x^2e^{2x}}{2} - \int 2xe^{2x}\,dx = \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + C$$
$$= (\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4})e^{2x} + C.$$

Or use the integral table, III-14 with $p(x) = x^2$ and $a = 1$.

16. Using substitution with $w = 1 - x$ and $dw = -dx$, we get

$$\int x\sqrt{1-x}\,dx = -\int (1-w)\sqrt{w}dw = \frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2} + C = \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C.$$

17. Integration by parts with $u = \ln x$, $v' = x$ gives

$$\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

Or use the integral table, III-13, with $n = 1$.

18. We integrate by parts, with $u = y$, $v' = \sin y$. We have $u' = 1$, $v = -\cos y$, and

$$\int y\sin y\,dy = -y\cos y - \int (-\cos y)\,dy = -y\cos y + \sin y + C.$$

Check:

$$\frac{d}{dy}(-y\cos y + \sin y + C) = -\cos y + y\sin y + \cos y = y\sin y.$$

19. We integrate by parts, using $u = (\ln x)^2$ and $v' = 1$. Then $u' = 2\frac{\ln x}{x}$ and $v = x$, so

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx.$$

But, integrating by parts or using the integral table, $\int \ln x\,dx = x\ln x - x + C$. Therefore,

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C.$$

Check:

$$\frac{d}{dx}\left[x(\ln x)^2 - 2x\ln x + 2x + C\right] = (\ln x)^2 + x\frac{2\ln x}{x} - 2\ln x - 2x\frac{1}{x} + 2 = (\ln x)^2.$$

20. Remember that $\ln(x^2) = 2\ln x$. Therefore,

$$\int \ln(x^2)\,dx = 2\int \ln x\,dx = 2x\ln x - 2x + C.$$

Check:

$$\frac{d}{dx}(2x\ln x - 2x + C) = 2\ln x + \frac{2x}{x} - 2 = 2\ln x = \ln(x^2).$$

21. Using the exponent rules and the chain rule, we have

$$\int e^{0.5-0.3t}\,dt = e^{0.5}\int e^{-0.3t}\,dt = -\frac{e^{0.5}}{0.3}e^{-0.3t} + C = -\frac{e^{0.5-0.3t}}{0.3} + C.$$

22. Let $\sin\theta = w$, then $\cos\theta\,d\theta = dw$, so

$$\int \sin^2\theta\cos\theta\,d\theta = \int w^2\,dw = \frac{1}{3}w^3 + C = \frac{1}{3}\sin^3\theta + C,$$

where C is a constant.

23. Substitute $w = 4 - x^2$, $dw = -2x\,dx$:

$$\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int \sqrt{w}\,dw = -\frac{1}{3}w^{3/2} + C = -\frac{1}{3}(4-x^2)^{3/2} + C.$$

Check

$$\frac{d}{dx}\left[-\frac{1}{3}(4-x^2)^{3/2} + C\right] = -\frac{1}{3}\left[\frac{3}{2}(4-x^2)^{1/2}(-2x)\right] = x\sqrt{4-x^2}.$$

24. Expanding the numerator and dividing, we have

$$\begin{aligned}\int \frac{(u+1)^3}{u^2}\,du &= \int \frac{(u^3+3u^2+3u+1)}{u^2}\,du = \int \left(u + 3 + \frac{3}{u} + \frac{1}{u^2}\right)\,du \\ &= \frac{u^2}{2} + 3u + 3\ln|u| - \frac{1}{u} + C.\end{aligned}$$

Check:

$$\frac{d}{du}\left(\frac{u^2}{2} + 3u + 3\ln|u| - \frac{1}{u} + C\right) = u + 3 + 3/u + 1/u^2 = \frac{(u+1)^3}{u^2}.$$

25. Substitute $w = \sqrt{y}$, $dw = 1/(2\sqrt{y})\,dy$. Then

$$\int \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy = 2\int \cos w\,dw = 2\sin w + C = 2\sin\sqrt{y} + C.$$

Check:

$$\frac{d}{dy}2\sin\sqrt{y} + C = \frac{2\cos\sqrt{y}}{2\sqrt{y}} = \frac{\cos\sqrt{y}}{\sqrt{y}}.$$

26. Since $\dfrac{d}{dz}(\tan z) = \dfrac{1}{\cos^2 z}$, we have

$$\int \frac{1}{\cos^2 z}\,dz = \tan z + C.$$

Check:

$$\frac{d}{dz}(\tan z + C) = \frac{d}{dz}\frac{\sin z}{\cos z} = \frac{(\cos z)(\cos z) - (\sin z)(-\sin z)}{\cos^2 z} = \frac{1}{\cos^2 z}.$$

27. Denote $\int \cos^2\theta\,d\theta$ by A. Let $u = \cos\theta$, $v' = \cos\theta$. Then, $v = \sin\theta$ and $u' = -\sin\theta$. Integrating by parts, we get:

$$A = \cos\theta\sin\theta - \int(-\sin\theta)\sin\theta\,d\theta.$$

Employing the identity $\sin^2\theta = 1 - \cos^2\theta$, the equation above becomes:

$$\begin{aligned}A &= \cos\theta\sin\theta + \int d\theta - \int \cos^2\theta\,d\theta \\ &= \cos\theta\sin\theta + \theta - A + C.\end{aligned}$$

Solving this equation for A, and using the identity $\sin 2\theta = 2\cos\theta\sin\theta$ we get:

$$A = \int \cos^2\theta\, d\theta = \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C.$$

[Note: An alternate solution would have been to use the identity $\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$.]

28. Multiplying out and integrating term by term:

$$\int t^{10}(t-10)\,dt = \int (t^{11} - 10t^{10})\,dt = \int t^{11}dt - 10\int t^{10}\,dt = \frac{1}{12}t^{12} - \frac{10}{11}t^{11} + C.$$

29. Substitute $w = 2x - 6$. Then $dw = 2\,dx$ and

$$\int \tan(2x-6)\,dx = \frac{1}{2}\int \tan w\,dw = \frac{1}{2}\int \frac{\sin w}{\cos w}\,dw$$

$$= -\frac{1}{2}\ln|\cos w| + C \text{ by substitution or by I-7 of the integral table.}$$

$$= -\frac{1}{2}\ln|\cos(2x-6)| + C.$$

30. Using integration by parts, we have

$$\int_1^3 \ln(x^3)\,dx = 3\int_1^3 \ln x\,dx = 3(x\ln x - x)\Big|_1^3 = 9\ln 3 - 6 \approx 3.8875.$$

This matches the approximation given by Simpson's rule with 10 intervals.

31. In Problem 19, we found that

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C.$$

Thus

$$\int_1^e (\ln x)^2\,dx = [x(\ln x)^2 - 2x\ln x + 2x]\Big|_1^e = e - 2 \approx 0.71828.$$

This matches the approximation given by Simpson's rule with 10 intervals.

32. Integrating by parts, we take $u = e^{2x}$, $u' = 2e^{2x}$, $v' = \sin 2x$, and $v = -\frac{1}{2}\cos 2x$, so

$$\int e^{2x}\sin 2x\,dx = -\frac{e^{2x}}{2}\cos 2x + \int e^{2x}\cos 2x\,dx.$$

Integrating by parts again, with $u = e^{2x}$, $u' = 2e^{2x}$, $v' = \cos 2x$, and $v = \frac{1}{2}\sin 2x$, we get

$$\int e^{2x}\cos 2x\,dx = \frac{e^{2x}}{2}\sin 2x - \int e^{2x}\sin 2x\,dx.$$

Substituting into the previous equation, we obtain

$$\int e^{2x}\sin 2x\,dx = -\frac{e^{2x}}{2}\cos 2x + \frac{e^{2x}}{2}\sin 2x - \int e^{2x}\sin 2x\,dx.$$

Solving for $\int e^{2x}\sin 2x\,dx$ gives

$$\int e^{2x}\sin 2x\,dx = \frac{1}{4}e^{2x}(\sin 2x - \cos 2x) + C.$$

This result can also be obtained using II-8 in the integral table. Thus

$$\int_{-\pi}^{\pi} e^{2x}\sin 2x = [\frac{1}{4}e^{2x}(\sin 2x - \cos 2x)]\Big|_{-\pi}^{\pi} = \frac{1}{4}(e^{-2\pi} - e^{2\pi}) \approx -133.8724.$$

We get -133.37 using Simpson's rule with 10 intervals. With 100 intervals, we get -133.8724. Thus our answer matches the approximation of Simpson's rule.

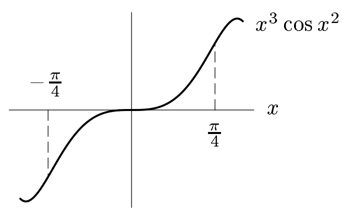
33.

$$\begin{aligned}\int_0^{10} ze^{-z}\,dz &= [-ze^{-z}]\Big|_0^{10} - \int_0^{10} -e^{-z}\,dz \qquad (\text{let } z=u, e^{-z}=v', -e^{-z}=v)\\ &= -10e^{-10} - [e^{-z}]\Big|_0^{10}\\ &= -10e^{-10} - e^{-10} + 1\\ &= -11e^{-10} + 1.\end{aligned}$$

34. Let $\sin\theta = w$, $\cos\theta\,d\theta = dw$. So, if $\theta = -\dfrac{\pi}{3}$, then $w = -\dfrac{\sqrt{3}}{2}$, and if $\theta = \dfrac{\pi}{4}$, then $w = \dfrac{\sqrt{2}}{2}$. So we have

$$\int_{-\pi/3}^{\pi/4} \sin^3\theta\cos\theta\,d\theta = \int_{-\sqrt{3}/2}^{\sqrt{2}/2} w^3\,dw = \frac{1}{4}w^4\Big|_{-\sqrt{3}/2}^{\sqrt{2}/2} = \frac{1}{4}\left[\left(\frac{\sqrt{2}}{2}\right)^4 - \left(\frac{-\sqrt{3}}{2}\right)^4\right] = -\frac{5}{64}.$$

35. This integral is 0 because the function $x^3\cos(x^2)$ is odd (meaning $f(-x) = -f(x)$), and so the negative contribution to the integral from $-\frac{\pi}{4} < x < 0$ exactly cancels the positive contribution from $0 < x < \frac{\pi}{4}$. See figure below.



36. Let $\sqrt{x} = w$, $\frac{1}{2}x^{-\frac{1}{2}}dx = dw$, $\frac{dx}{\sqrt{x}} = 2\,dw$. If $x = 1$ then $w = 1$, and if $x = 4$ so $w = 2$. So we have

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^2 e^w \cdot 2\,dw = 2e^w\Big|_1^2 = 2(e^2 - e) \approx 9.34.$$

37.

$$\int_0^1 \frac{dx}{x^2+1} = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

38. Let $\ln x = w$, then $\frac{1}{x}\,dx = dw$, so

$$\int \frac{(\ln x)^2}{x}\,dx = \int w^2\,dw = \frac{1}{3}w^3 + C = \frac{1}{3}(\ln x)^3 + C, \text{ where } C \text{ is a constant.}$$

39. Multiplying out, dividing, and then integrating yields

$$\int \frac{(t+2)^2}{t^3}\,dt = \int \frac{t^2+4t+4}{t^3}\,dt = \int \frac{1}{t}\,dt + \int \frac{4}{t^2}\,dt + \int \frac{4}{t^3}\,dt = \ln|t| - \frac{4}{t} - \frac{2}{t^2} + C,$$

where C is a constant.

40. Integrating term by term:

$$\int \left(x^2 + 2x + \frac{1}{x}\right)dx = \frac{1}{3}x^3 + x^2 + \ln|x| + C,$$

where C is a constant.

41. Dividing and then integrating, we obtain

$$\int \frac{t+1}{t^2}\,dt = \int \frac{1}{t}\,dt + \int \frac{1}{t^2}\,dt = \ln|t| - \frac{1}{t} + C, \text{ where } C \text{ is a constant.}$$

42. Let $t^2+1=w$, then $2t\,dt=dw,\ tdt=\frac{1}{2}\,dw$, so

$$\int te^{t^2+1}\,dt=\int e^w\cdot\frac{1}{2}\,dw=\frac{1}{2}\int e^w\,dw=\frac{1}{2}e^w+C=\frac{1}{2}e^{t^2+1}+C,$$

where C is a constant.

43. Let $\cos\theta=w$, then $-\sin\theta\,d\theta=dw$, so

$$\int\tan\theta\,d\theta=\int\frac{\sin\theta}{\cos\theta}\,d\theta=\int\frac{-1}{w}\,dw$$
$$=-\ln|w|+C=-\ln|\cos\theta|+C,$$

where C is a constant.

44. If $u=\sin(5\theta)$, $du=\cos(5\theta)\cdot 5\,d\theta$, so

$$\int\sin(5\theta)\cos(5\theta)d\theta=\frac{1}{5}\int\sin(5\theta)\cdot 5\cos(5\theta)d\theta=\frac{1}{5}\int u\,du$$
$$=\frac{1}{5}\left(\frac{u^2}{2}\right)+C=\frac{1}{10}\sin^2(5\theta)+C$$

or

$$\int\sin(5\theta)\cos(5\theta)d\theta=\frac{1}{2}\int 2\sin(5\theta)\cos(5\theta)d\theta=\frac{1}{2}\int\sin(10\theta)d\theta\quad\text{(using }\sin(2x)=2\sin x\cos x\text{)}$$
$$=\frac{-1}{20}\cos(10\theta)+C.$$

45. Using substitution,

$$\int\frac{x}{x^2+1}\,dx=\int\frac{1/2}{w}\,dw\qquad(x^2+1=w,2x\,dx=dw,xdx=\frac{1}{2}\,dw)$$
$$=\frac{1}{2}\int\frac{1}{w}\,dw=\frac{1}{2}\ln|w|+C=\frac{1}{2}\ln|x^2+1|+C,$$

where C is a constant.

46. Since $\dfrac{d}{dz}(\arctan z)=\dfrac{1}{1+z^2}$, we have

$$\int\frac{dz}{1+z^2}=\arctan z+C,\ \text{ where } C \text{ is a constant.}$$

47. Let $w=2z$, so $dw=2dz$. Then, since $\dfrac{d}{dw}\arctan w=\dfrac{1}{1+w^2}$, we have

$$\int\frac{dz}{1+4z^2}=\int\frac{\frac{1}{2}dw}{1+w^2}=\frac{1}{2}\arctan w+C=\frac{1}{2}\arctan 2z+C.$$

48. Let $w=\cos 2\theta$. Then $dw=-2\sin 2\theta\,d\theta$, hence

$$\int\cos^3 2\theta\sin 2\theta\,d\theta=-\frac{1}{2}\int w^3\,dw=-\frac{w^4}{8}+C=-\frac{\cos^4 2\theta}{8}+C.$$

Check:

$$\frac{d}{d\theta}\left(-\frac{\cos^4 2\theta}{8}\right)=-\frac{(4\cos^3 2\theta)(-\sin 2\theta)(2)}{8}=\cos^3 2\theta\sin 2\theta.$$

49. Let $\cos 5\theta = w$, then $-5\sin 5\theta\, d\theta = dw,\ \sin 5\theta\, d\theta = -\frac{1}{5}dw$. So

$$\int \sin 5\theta \cos^3 5\theta\, d\theta = \int w^3 \cdot (-\frac{1}{5})\, dw = -\frac{1}{5}\int w^3\, dw = -\frac{1}{20}w^4 + C$$
$$= -\frac{1}{20}\cos^4 5\theta + C,$$

where C is a constant.

50.

$$\begin{aligned}\int \sin^3 z \cos^3 z\, dz &= \int \sin z(1-\cos^2 z)\cos^3 z\, dz \\ &= \int \sin z \cos^3 z\, dz - \int \sin z \cos^5 z\, dz \\ &= \int w^3\,(-dw) - \int w^5\,(-dw) \quad (\text{let } \cos z = w, \text{ so } -\sin z\, dz = dw) \\ &= -\int w^3\, dw + \int w^5\, dw \\ &= -\frac{1}{4}w^4 + \frac{1}{6}w^6 + C \\ &= -\frac{1}{4}\cos^4 z + \frac{1}{6}\cos^6 z + C,\end{aligned}$$

where C is a constant.

51. If $u = t - 10$, $t = u + 10$ and $dt = 1\, du$, so substituting we get

$$\int (u+10)u^{10}du = \int (u^{11} + 10u^{10})\, du = \frac{1}{12}u^{12} + \frac{10}{11}u^{11} + C$$
$$= \frac{1}{12}(t-10)^{12} + \frac{10}{11}(t-10)^{11} + C.$$

52. Let $\sin\theta = w$, then $\cos\theta\, d\theta = dw$, so

$$\int \cos\theta\sqrt{1+\sin\theta}\, d\theta = \int \sqrt{1+w}\, dw$$
$$= \frac{(1+w)^{3/2}}{3/2} + C = \frac{2}{3}(1+\sin\theta)^{3/2} + C,$$

where C is a constant.

53.

$$\int xe^x\, dx = xe^x - \int e^x\, dx \qquad (\text{let } x = u, e^x = v', e^x = v)$$
$$= xe^x - e^x + C,$$

where C is a constant.

54.

$$\begin{aligned}\int t^3e^t\, dt &= t^3e^t - \int 3t^2e^t\, dt \qquad (\text{let } t^3 = u, e^t = v', 3t^2 = u', e^t = v) \\ &= t^3e^t - 3\int t^2e^t\, dt \qquad (\text{let } t^2 = u, e^t = v') \\ &= t^3e^t - 3(t^2e^t - \int 2te^t\, dt) \\ &= t^3e^t - 3t^2e^t + 6\int te^t\, dt \qquad (\text{let } t = u, e^t = v')\end{aligned}$$

$$= t^3e^t - 3t^2e^t + 6(te^t - \int e^t\,dt)$$
$$= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C,$$

where C is a constant.

55. Let $x^2 = w$, then $2xdx = dw$, $x = 1 \Rightarrow w = 1$, $x = 3 \Rightarrow w = 9$. Thus,

$$\begin{aligned}\int_1^3 x(x^2+1)^{70}\,dx &= \int_1^9 (w+1)^{70}\frac{1}{2}\,dw \\ &= \frac{1}{2}\cdot\frac{1}{71}(w+1)^{71}\Big|_1^9 \\ &= \frac{1}{142}(10^{71} - 2^{71}).\end{aligned}$$

56. Let $w = 3z+5$ and $dw = 3\,dz$. Then

$$\int (3z+5)^3\,dz = \frac{1}{3}\int w^3 dw = \frac{1}{12}w^4 + C = \frac{1}{12}(3z+5)^4 + C.$$

57. Rewrite $9+u^2$ as $9[1+(u/3)^2]$ and let $w = u/3$, then $dw = du/3$ so that

$$\int \frac{du}{9+u^2} = \frac{1}{3}\int \frac{dw}{1+w^2} = \frac{1}{3}\arctan w + C = \frac{1}{3}\arctan\left(\frac{u}{3}\right) + C.$$

58. Let $u = \sin w$, then $du = \cos w\,dw$ so that

$$\int \frac{\cos w}{1+\sin^2 w}\,dw = \int \frac{du}{1+u^2} = \arctan u + C = \arctan(\sin w) + C.$$

59. Let $w = \ln x$, then $dw = (1/x)dx$ which gives

$$\int \frac{1}{x}\tan(\ln x)\,dx = \int \tan w\,dw = \int \frac{\sin w}{\cos w}\,dw = -\ln(|\cos w|) + C = -\ln(|\cos(\ln x)|) + C.$$

60. Let $w = \ln x$, then $dw = (1/x)dx$ so that

$$\int \frac{1}{x}\sin(\ln x)\,dx = \int \sin w\,dw = -\cos w + C = -\cos(\ln x) + C.$$

61. Let $u = 2x$, then $du = 2\,dx$ so that

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(2x) + C.$$

62. Let $u = 16 - w^2$, then $du = -2w\,dw$ so that

$$\int \frac{wdw}{\sqrt{16-w^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{16-w^2} + C.$$

63. Dividing and then integrating term by term, we get

$$\begin{aligned}\int \frac{e^{2y}+1}{e^{2y}}\,dy &= \int \left(\frac{e^{2y}}{e^{2y}} + \frac{1}{e^{2y}}\right)\,dy = \int (1+e^{-2y})\,dy = \int dy + \left(-\frac{1}{2}\right)\int e^{-2y}(-2)\,dy \\ &= y - \frac{1}{2}e^{-2y} + C.\end{aligned}$$

64. Let $u = 1 - \cos w$, then $du = \sin w\,dw$ which gives

$$\int \frac{\sin w\,dw}{\sqrt{1-\cos w}} = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1-\cos w} + C.$$

65. Let $w = \ln x$. Then $dw = (1/x)dx$ which gives

$$\int \frac{dx}{x \ln x} = \int \frac{dw}{w} = \ln|w| + C = \ln|\ln x| + C.$$

66. Let $w = 3u + 8$, then $dw = 3du$ and

$$\int \frac{du}{3u+8} = \int \frac{dw}{3w} = \frac{1}{3}\ln|3u+8| + C.$$

67. Let $w = \sqrt{x^2+1}$, then $dw = \dfrac{xdx}{\sqrt{x^2+1}}$ so that

$$\int \frac{x}{\sqrt{x^2+1}} \cos\sqrt{x^2+1}\,dx = \int \cos w\,dw = \sin w + C = \sin\sqrt{x^2+1} + C.$$

68. Integrating by parts using $u = t^2$ and $dv = \frac{tdt}{\sqrt{1+t^2}}$ gives $du = 2t\,dt$ and $v = \sqrt{1+t^2}$. Now

$$\begin{aligned}\int \frac{t^3}{\sqrt{1+t^2}}\,dt &= t^2\sqrt{1+t^2} - \int 2t\sqrt{1+t^2}\,dt \\ &= t^2\sqrt{1+t^2} - \frac{2}{3}(1+t^2)^{3/2} + C \\ &= \sqrt{1+t^2}(t^2 - \frac{2}{3}(1+t^2)) + C \\ &= \sqrt{1+t^2}\frac{(t^2-2)}{3} + C.\end{aligned}$$

69. Using integration by parts, let $r = u$ and $dt = e^{ku}du$, so $dr = du$ and $t = (1/k)e^{ku}$. Thus

$$\int ue^{ku}\,du = \frac{u}{k}e^{ku} - \frac{1}{k}\int e^{ku}\,du = \frac{u}{k}e^{ku} - \frac{1}{k^2}e^{ku} + C.$$

70. Let $u = w + 5$, then $du = dw$ and noting that $w = u - 5$ we obtain

$$\begin{aligned}\int (w+5)^4 w\,dw &= \int u^4(u-5)\,du \\ &= \int \left(u^5 - 5u^4\right)\,du \\ &= \frac{1}{6}u^6 - u^5 + C \\ &= \frac{1}{6}(w+5)^6 - (w+5)^5 + C.\end{aligned}$$

71. $\int e^{\sqrt{2}x+3}dx = \dfrac{1}{\sqrt{2}}\int e^{\sqrt{2}x+3}\sqrt{2}dx$. If $u = \sqrt{2}x + 3$, $du = \sqrt{2}dx$, so

$$\frac{1}{\sqrt{2}}\int e^u du = \frac{1}{\sqrt{2}}e^u + C = \frac{1}{\sqrt{2}}e^{\sqrt{2}x+3} + C.$$

72. Integrate by parts letting $u = (\ln r)^2$ and $dv = rdr$, then $du = (2/r)\ln r dr$ and $v = r^2/2$. We get

$$\int r(\ln r)^2\,dr = \frac{1}{2}r^2(\ln r)^2 - \int r\ln r\,dr.$$

Then using integration by parts again with $u = \ln r$ and $dv = rdr$, so $du = dr/r$ and $v = r^2/2$, we get

$$\int r\ln^2 r\,dr = \frac{1}{2}r^2(\ln r)^2 - \left[\frac{1}{2}r^2\ln r - \frac{1}{2}\int r\,dr\right] = \frac{1}{2}r^2(\ln r)^2 - \frac{1}{2}r^2\ln r + \frac{1}{4}r^2 + C.$$

73. $\int (e^x + x)^2 dx = \int (e^{2x} + 2xe^x + x^2) dy$. Separating into three integrals, we have

$$\int e^{2x} dx = \frac{1}{2}\int e^{2x} 2\, dx = \frac{1}{2}e^{2x} + C_1,$$

$$\int 2xe^x dx = 2\int xe^x dx = 2xe^x - 2e^x + C_2$$

from Formula II-13 of the integral table or integration by parts, and

$$\int x^2\, dx = \frac{x^3}{3} + C_3.$$

Combining the results and writing $C = C_1 + C_2 + C_3$, we get

$$\frac{1}{2}e^{2x} + 2xe^x - 2e^x + \frac{x^3}{3} + C.$$

74. Integrate by parts, $r = \ln u$ and $dt = u^2\, du$, so $dr = (1/u)\, du$ and $t = (1/3)u^3$. We have

$$\int u^2 \ln u\, du = \frac{1}{3}u^3 \ln u - \frac{1}{3}\int u^2\, du = \frac{1}{3}u^3 \ln u - \frac{1}{9}u^3 + C.$$

75. The integral table yields

$$\begin{aligned}\int \frac{5x+6}{x^2+4}\, dx &= \frac{5}{2}\ln|x^2+4| + \frac{6}{2}\arctan\frac{x}{2} + C \\ &= \frac{5}{2}\ln|x^2+4| + 3\arctan\frac{x}{2} + C.\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{dx}\left(\frac{5}{2}\ln|x^2+4| + \frac{6}{2}\arctan\frac{x}{2} + C\right) &= \frac{5}{2}\left(\frac{1}{x^2+4}(2x) + 3\frac{1}{1+(x/2)^2}\frac{1}{2}\right) \\ &= \frac{5x}{x^2+4} + \frac{6}{x^2+4} = \frac{5x+6}{x^2+4}.\end{aligned}$$

76. Using Table IV-19, let $m = 3$, $w = 2x$, and $dw = 2dx$. Then

$$\begin{aligned}\int \frac{1}{\sin^3(2x)}\, dx &= \frac{1}{2}\int \frac{1}{\sin^3 w}\, dw \\ &= \frac{1}{2}\left[\frac{-1}{(3-1)}\frac{\cos w}{\sin^2 w}\right] + \frac{1}{4}\int \frac{1}{\sin w}\, dw,\end{aligned}$$

and using Table IV-20, we have

$$\int \frac{1}{\sin w}\, dw = \frac{1}{2}\ln\left|\frac{\cos w - 1}{\cos w + 1}\right| + C.$$

Thus,

$$\int \frac{1}{\sin^3(2x)}\, dx = -\frac{\cos 2x}{4\sin^2 2x} + \frac{1}{8}\ln\left|\frac{\cos 2x - 1}{\cos 2x + 1}\right| + C.$$

77. We can factor $r^2 - 100 = (r-10)(r+10)$ so we can use Table V-26 (with $a = 10$ and $b = -10$) to get

$$\int \frac{dr}{r^2 - 100} = \frac{1}{20}\left[\ln|r-10| + \ln|r+10|\right] + C.$$

78. Integration by parts will be used twice here. First let $u = y^2$ and $dv = \sin(cy)\,dy$, then $du = 2y\,dy$ and $v = -(1/c)\cos(cy)$. Thus

$$\int y^2 \sin(cy)\,dy = -\frac{y^2}{c}\cos(cy) + \frac{2}{c}\int y\cos(cy)\,dy.$$

Now use integration by parts to evaluate the integral in the right hand expression. Here let $u = y$ and $dv = \cos(cy)dy$ which gives $du = dy$ and $v = (1/c)\sin(cy)$. Then we have

$$\begin{aligned}\int y^2 \sin(cy)\,dy &= -\frac{y^2}{c}\cos(cy) + \frac{2}{c}\left(\frac{y}{c}\sin(cy) - \frac{1}{c}\int \sin(cy)\,dy\right)\\ &= -\frac{y^2}{c}\cos(cy) + \frac{2y}{c^2}\sin(cy) + \frac{2}{c^3}\cos(cy) + C.\end{aligned}$$

79. Integration by parts will be used twice. First let $u = e^{-ct}$ and $dv = \sin(kt)dt$, then $du = -ce^{-ct}dt$ and $v = (-1/k)\cos kt$. Then

$$\begin{aligned}\int e^{-ct}\sin kt\,dt &= -\frac{1}{k}e^{-ct}\cos kt - \frac{c}{k}\int e^{-ct}\cos kt\,dt\\ &= -\frac{1}{k}e^{-ct}\cos kt - \frac{c}{k}\left(\frac{1}{k}e^{-ct}\sin kt + \frac{c}{k}\int e^{-ct}\sin kt\,dt\right)\\ &= -\frac{1}{k}e^{-ct}\cos kt - \frac{c}{k^2}e^{-ct}\sin kt - \frac{c^2}{k^2}\int e^{-ct}\sin kt\,dt\end{aligned}$$

Solving for $\int e^{-ct}\sin kt\,dt$ gives

$$\frac{k^2+c^2}{k^2}\int e^{-ct}\sin kt\,dt = -\frac{e^{-ct}}{k^2}\left(k\cos kt + c\sin kt\right),$$

so

$$\int e^{-ct}\sin kt\,dt = -\frac{e^{-ct}}{k^2+c^2}\left(k\cos kt + c\sin kt\right) + C.$$

80. Using II-9 from the integral table, with $a = 5$ and $b = 3$, we have

$$\begin{aligned}\int e^{5x}\cos(3x)\,dx &= \frac{1}{25+9}e^{5x}\left[5\cos(3x) + 3\sin(3x)\right] + C\\ &= \frac{1}{34}e^{5x}\left[5\cos(3x) + 3\sin(3x)\right] + C.\end{aligned}$$

81. Since $\displaystyle\int (x^{\sqrt{k}} + (\sqrt{k})^x)dx = \int x^{\sqrt{k}}dx + \int (\sqrt{k})^x\,dx$, for the first integral, use Formula I-1 with $n = \sqrt{k}$. For the second integral, use Formula I-3 with $a = \sqrt{k}$. The result is

$$\int (x^{\sqrt{x}} + (\sqrt{k})^x)\,dx = \frac{x^{(\sqrt{k})+1}}{(\sqrt{k})+1} + \frac{(\sqrt{k})^x}{\ln\sqrt{k}} + C.$$

82. Factor $\sqrt{3}$ out of the integrand and use VI-30 of the integral table with $u = 2x$ and $du = 2dx$ to get

$$\begin{aligned}\int \sqrt{3+12x^2}\,dx &= \int \sqrt{3}\sqrt{1+4x^2}\,dx\\ &= \frac{\sqrt{3}}{2}\int \sqrt{1+u^2}\,du\\ &= \frac{\sqrt{3}}{4}\left(u\sqrt{1+u^2} + \int \frac{1}{\sqrt{1+u^2}}\,du\right).\end{aligned}$$

Then from VI-29, simplify the integral on the right to get

$$\int \sqrt{3+12x^2}\,dx = \frac{\sqrt{3}}{4}\left(u\sqrt{1+u^2}+\ln|u+\sqrt{1+u^2}|\right)+C$$
$$= \frac{\sqrt{3}}{4}\left(2x\sqrt{1+(2x)^2}+\ln|2x+\sqrt{1+(2x)^2}|\right)+C.$$

83. We know $x^2+5x+4=(x+1)(x+4)$, so we can use V-26 of the integral table with $a=-1$ and $b=-4$ to write

$$\int \frac{dx}{x^2+5x+4} = \frac{1}{3}(\ln|x+1|-\ln|x+4|)+C.$$

84. By completing the square, we get

$$x^2-3x+2=(x^2-3x+(-\frac{3}{2})^2)+2-\frac{9}{4}=(x-\frac{3}{2})^2-\frac{1}{4}.$$

Then

$$\int \frac{1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{1}{\sqrt{(x-\frac{3}{2})^2-\frac{1}{4}}}\,dx.$$

Let $w=(x-(3/2))$, then $dw=dx$ and $a^2=1/4$. Then we have

$$\int \frac{1}{\sqrt{x^2-3x+2}}\,dx = \int \frac{1}{\sqrt{w^2-a^2}}\,dw$$

and from VI-29 of the integral table we have

$$\int \frac{1}{\sqrt{w^2-a^2}}\,dw = \ln\left|w+\sqrt{w^2-a^2}\right|+C$$
$$= \ln\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\frac{1}{4}}\right|+C$$
$$= \ln\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C.$$

85. First divide x^2+3x+2 into x^3 to obtain

$$\frac{x^3}{x^2+3x+2} = x-3+\frac{7x+6}{x^2+3x+2}.$$

Since $x^2+3x+2=(x+1)(x+2)$, we can use V-27 of the integral table (with $c=7$, $d=6$, $a=-1$, and $b=-2$) to get

$$\int \frac{7x+6}{x^2+3x+2}\,dx = -\ln|x+1|+8\ln|x+2|+C.$$

Including the terms $x-3$ from the long division and integrating them gives

$$\int \frac{x^3}{x^2+3x+2}\,dx = \int \left(x-3+\frac{7x+6}{x^2+3x+6}\right)\,dx = \frac{1}{2}x^2-3x-\ln|x+1|+8\ln|x+2|+C.$$

86. First divide x^2+1 by x^2-3x+2 to obtain

$$\frac{x^2+1}{x^2-3x+2} = 1+\frac{3x-1}{x^2-3x+2}.$$

Factoring $x^2-3x+2=(x-2)(x-1)$ we can use V-27 (with $c=3$, $d=-1$, $a=2$ and $b=1$) to write

$$\int \frac{3x-1}{x^2-3x+2}\,dx = 5\ln|x-2|-2\ln|x-1|+C.$$

Remembering to include the extra term of $+1$ we got when dividing, we get

$$\int \frac{x^2+1}{x^2-3x+2}\,dx = \int \left(1+\frac{3x-1}{x^2-3x+2}\right)\,dx = x+5\ln|x-2|-2\ln|x-1|+C.$$

87. We can factor the denominator into $ax(x+\frac{b}{a})$, so

$$\int \frac{dx}{ax^2+bx} = \frac{1}{a}\int \frac{1}{x(x+\frac{b}{a})}$$

Now we can use V-26 (with $A=0$ and $B=-\frac{b}{a}$ to give

$$\frac{1}{a}\int \frac{1}{x(x+\frac{b}{a})} = \frac{1}{a}\cdot\frac{a}{b}\left(\ln|x| - \ln\left|x+\frac{b}{a}\right|\right) + C = \frac{1}{b}\left(\ln|x| - \ln\left|x+\frac{b}{a}\right|\right) + C.$$

88. Let $w = ax^2+2bx+c$, then $dw = (2ax+2b)dx$ so that

$$\int \frac{ax+b}{ax^2+2bx+c}\,dx = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|ax^2+2bx+c| + C.$$

89. This can be done by formula V–26 in the integral table or by partial fractions

$$\int \frac{dz}{z^2+z} = \int \frac{dz}{z(z+1)} = \int \left(\frac{1}{z} - \frac{1}{z+1}\right) dz = \ln|z| - \ln|z+1| + C.$$

Check:

$$\frac{d}{dz}\left(\ln|z| - \ln|z+1| + C\right) = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z^2+z}.$$

90. Multiplying out and integrating term by term,

$$\int \left(\frac{x}{3}+\frac{3}{x}\right)^2 dx = \int \left(\frac{x^2}{9}+2+\frac{9}{x^2}\right) dx = \frac{1}{9}\left(\frac{x^3}{3}\right) + 2x + 9\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^3}{27} + 2x - \frac{9}{x} + C.$$

91. If $u = 2^t+1$, $du = 2^t(\ln 2)\,dt$, so

$$\int \frac{2^t}{2^t+1}\,dt = \frac{1}{\ln 2}\int \frac{2^t\ln 2}{2^t+1}\,dt = \frac{1}{\ln 2}\int \frac{1}{u} = \frac{1}{\ln 2}\ln|u| + C = \frac{1}{\ln 2}\ln|2^t+1| + C.$$

92. If $u = 1-x$, $du = -1\,dx$, so

$$\int 10^{1-x}dx = -1\int 10^{1-x}(-1\,dx) = -1\int 10^u\,du = -1\frac{10^u}{\ln 10} + C = -\frac{1}{\ln 10}10^{1-x} + C.$$

93. Multiplying out and integrating term by term gives

$$\begin{aligned}\int (x^2+5)^3dx &= \int (x^6+15x^4+75x^2+125)dx = \frac{1}{7}x^7 + 15\frac{x^5}{5} + 75\frac{x^3}{3} + 125x + C\\ &= \frac{1}{7}x^7 + 3x^5 + 25x^3 + 125x + C.\end{aligned}$$

94. Integrate by parts letting $r = v$ and $dt = \arcsin v\,dv$ then $dr = dv$ and to find t we integrate $\arcsin v\,dv$ by parts letting $x = \arcsin v$ and $dy = dv$. This gives

$$t = v\arcsin v - \int (1/\sqrt{1-v^2})v\,dv = v\arcsin v + \sqrt{1-v^2}.$$

Now, back to the original integration by parts, and we have

$$\int v\arcsin v\,dv = v^2\arcsin v + v\sqrt{1-v^2} - \int \left[v\arcsin v + \sqrt{1-v^2}\right]\,dv.$$

Adding $\int v \arcsin v\, dv$ to both sides of the above line we obtain

$$2\int v \arcsin v\, dv = v^2 \arcsin v + v\sqrt{1-v^2} - \int \sqrt{1-v^2}\, dv$$
$$= v^2 \arcsin v + v\sqrt{1-v^2} - \frac{1}{2}v\sqrt{1-v^2} - \frac{1}{2}\arcsin v + C.$$

Dividing by 2 gives

$$\int v \arcsin v\, dv = \left(\frac{v^2}{2} - \frac{1}{4}\right)\arcsin v + \frac{1}{4}v\sqrt{1-v^2} + K,$$

where $K = C/2$.

95. By VI-30 in the table of integrals, we have

$$\int \sqrt{4-x^2}\, dx = \frac{x\sqrt{4-x^2}}{2} + 2\int \frac{1}{\sqrt{4-x^2}}\, dx.$$

The same table informs us in formula VI-28 that

$$\int \frac{1}{\sqrt{4-x^2}}\, dx = \arcsin\frac{x}{2} + C.$$

Thus

$$\int \sqrt{4-x^2}\, dx = \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\frac{x}{2} + C.$$

96. By long division, $\dfrac{z^3}{z-5} = z^2 + 5z + 25 + \dfrac{125}{z-5}$, so

$$\int \frac{z^3}{z-5} dz = \int \left(z^2 + 5z + 25 + \frac{125}{z-5}\right) dz = \frac{z^3}{3} + \frac{5z^2}{2} + 25z + 125\int \frac{1}{z-5} dz$$
$$= \frac{z^3}{3} + \frac{5}{2}z^2 + 25z + 125\ln|z-5| + C.$$

97. If $u = 1 + \cos^2 w$, $du = 2(\cos w)^1(-\sin w)\, dw$, so

$$\int \frac{\sin w \cos w}{1+\cos^2 w}\, dw = -\frac{1}{2}\int \frac{-2\sin w\cos w}{1+\cos^2 w}\, dw = -\frac{1}{2}\int \frac{1}{u}\, du = -\frac{1}{2}\ln|u| + C$$
$$= -\frac{1}{2}\ln|1+\cos^2 w| + C.$$

98. $\displaystyle\int \frac{1}{\tan(3\theta)} d\theta = \int \frac{1}{\left(\frac{\sin(3\theta)}{\cos(3\theta)}\right)}\, d\theta = \int \frac{\cos(3\theta)}{\sin(3\theta)} d\theta$. If $u = \sin(3\theta)$, $du = \cos(3\theta)\cdot 3d\theta$, so

$$\int \frac{\cos(3\theta)}{\sin(3\theta)}\, d\theta = \frac{1}{3}\int \frac{3\cos(3\theta)}{\sin(3\theta)}\, d\theta = \frac{1}{3}\int \frac{1}{u} du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|\sin(3\theta)| + C.$$

99. $\displaystyle\int \frac{x}{\cos^2 x} dx = \int x\frac{1}{\cos^2 x} dx$. Using integration by parts with $u = x$, $du = dx$ and $dv = \dfrac{1}{\cos^2 x} dx$, $v = \tan x$, we have

$$\int x\left(\frac{1}{\cos^2 x} dx\right) = x\tan x - \int \tan x dx.$$

Formula I-7 gives the final result of $x\tan x - (-\ln|\cos x|) + C = x\tan x + \ln|\cos x| + C$.

100. Dividing and integrating term by term gives

$$\int \frac{x+1}{\sqrt{x}}\, dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)\, dx = \int (x^{1/2} + x^{-1/2})\, dx = \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}x^{3/2} + 2\sqrt{x} + C.$$

101. If $u = \sqrt{x+1}$, $u^2 = x+1$ with $x = u^2 - 1$ and $dx = 2u\,du$. Substituting, we get

$$\int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{(u^2-1)2u\,du}{u} = \int (u^2-1)2\,du = 2\int (u^2-1)\,du$$
$$= \frac{2u^3}{3} - 2u + C = \frac{2(\sqrt{x+1})^3}{3} - 2\sqrt{x+1} + C.$$

102. $\int \frac{\sqrt{\sqrt{x}+1}}{\sqrt{x}} = \int (\sqrt{x}+1)^{1/2}\frac{1}{\sqrt{x}}\,dx$; if $u = \sqrt{x}+1$, $du = \frac{1}{2\sqrt{x}}\,dx$, so we have

$$2\int (\sqrt{x}+1)^{1/2}\frac{1}{2\sqrt{x}}\,dx = 2\int u^{1/2}\,du = 2\left(\frac{u^{3/2}}{\frac{3}{2}}\right) + C = \frac{4}{3}u^{3/2} + C = \frac{4}{3}(\sqrt{x}+1)^{3/2} + C.$$

103. If $u = e^{2y}+1$, then $du = e^{2y}2\,dy$, so

$$\int \frac{e^{2y}}{e^{2y}+1}\,dy = \frac{1}{2}\int \frac{2e^{2y}}{e^{2y}+1}\,dy = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|e^{2y}+1| + C.$$

104. If $u = z^2 - 5$, $du = 2z\,dz$, then

$$\int \frac{z}{(z^2-5)^3}\,dz = \int (z^2-5)^{-3}z\,dz = \frac{1}{2}\int (z^2-5)^{-3}2z\,dz = \frac{1}{2}\int u^{-3}\,du = \frac{1}{2}\left(\frac{u^{-2}}{-2}\right) + C$$
$$= \frac{1}{-4(z^2-5)^2} + C.$$

105. Letting $u = z-5$, $z = u+5$, $dz = du$, and substituting, we have

$$\int \frac{z}{(z-5)^3}dz = \int \frac{u+5}{u^3}du = \int (u^{-2}+5u^{-3})du = \frac{u^{-1}}{-1} + 5\left(\frac{u^{-2}}{-2}\right) + C$$
$$= \frac{-1}{(z-5)} + \frac{-5}{2(z-5)^2} + C.$$

106. If $u = 1+\tan x$ then $du = \frac{1}{\cos^2 x}\,dx$, and so

$$\int \frac{(1+\tan x)^3}{\cos^2 x}\,dx = \int (1+\tan x)^3\frac{1}{\cos^2 x}\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(1+\tan x)^4}{4} + C.$$

107. $\int \frac{(2x-1)e^{x^2}}{e^x}dx = \int e^{x^2-x}(2x-1)dx$. If $u = x^2 - x$, $du = (2x-1)dx$, so

$$\int e^{x^2-x}(2x-1)dx = \int e^u\,du$$
$$= e^u + c$$
$$= e^{x^2-x} + C.$$

108. We use the substitution $w = x^2 + x$, $dw = (2x+1)\,dx$.

$$\int (2x+1)e^{x^2}e^x\,dx = \int (2x+1)e^{x^2+x}\,dx = \int e^w\,dw$$
$$= e^w + C = e^{x^2+x} + C.$$

Check: $\frac{d}{dx}(e^{x^2+x} + C) = e^{x^2+x}\cdot(2x+1) = (2x+1)e^{x^2}e^x$.

109. Let $w = 2 + 3\cos x$, so $dw = -3\sin x\,dx$, giving $-\frac{1}{3}\,dw = \sin x\,dx$. Then

$$\begin{aligned}\int \sin x\left(\sqrt{2+3\cos x}\right)\,dx &= \int \sqrt{w}\left(-\frac{1}{3}\right)\,dw = -\frac{1}{3}\int \sqrt{w}\,dw\\ &= \left(-\frac{1}{3}\right)\frac{w^{\frac{3}{2}}}{\frac{3}{2}} + C = -\frac{2}{9}(2+3\cos x)^{\frac{3}{2}} + C.\end{aligned}$$

110. Using Table III-14, with $a = -4$ we have

$$\begin{aligned}\int (x^2 - 3x + 2)e^{-4x}\,dx &= -\frac{1}{4}(x^2 - 3x + 2)e^{-4x}\\ &\quad -\frac{1}{16}(2x-3)e^{-4x} - \frac{1}{64}(2)e^{-4x} + C.\\ &= \frac{1}{32}e^{-4x}(-11 + 20x - 8x^2) + C.\end{aligned}$$

111. Let $x = 2\theta$, then $dx = 2d\theta$. Thus

$$\int \sin^2(2\theta)\cos^3(2\theta)\,d\theta = \frac{1}{2}\int \sin^2 x\cos^3 x\,dx.$$

We let $w = \sin x$ and $dw = \cos x\,dx$. Then

$$\begin{aligned}\frac{1}{2}\int \sin^2 x\cos^3 x\,dx &= \frac{1}{2}\int \sin^2 x\cos^2 x\cos x\,dx\\ &= \frac{1}{2}\int \sin^2 x(1-\sin^2 x)\cos x\,dx\\ &= \frac{1}{2}\int w^2(1-w^2)\,dw = \frac{1}{2}\int (w^2 - w^4)\,dw\\ &= \frac{1}{2}\left(\frac{w^3}{3} - \frac{w^5}{5}\right) + C = \frac{1}{6}\sin^3 x - \frac{1}{10}\sin^5 x + C\\ &= \frac{1}{6}\sin^3(2\theta) - \frac{1}{10}\sin^5(2\theta) + C.\end{aligned}$$

112. If $u = 2\sin x$, then $du = 2\cos x\,dx$, so

$$\begin{aligned}\int \cos(2\sin x)\cos x\,dx &= \frac{1}{2}\int \cos(2\sin x)2\cos x\,dx = \frac{1}{2}\int \cos u\,du\\ &= \frac{1}{2}\sin u + C = \frac{1}{2}\sin(2\sin x) + C.\end{aligned}$$

113. Let $w = x + \sin x$, then $dw = (1 + \cos x)\,dx$ which gives

$$\int (x+\sin x)^3(1+\cos x)\,dx = \int w^3\,dw = \frac{1}{4}w^4 + C = \frac{1}{4}(x+\sin x)^4 + C.$$

114. Using Table III-16,

$$\begin{aligned}\int \left(2x^3 + 3x + 4\right)\cos(2x)\,dx &= \frac{1}{2}(2x^3 + 3x + 4)\sin(2x)\\ &\quad +\frac{1}{4}(6x^2 + 3)\cos(2x)\\ &\quad -\frac{1}{8}(12x)\sin(2x) - \frac{3}{4}\cos(2x) + C.\\ &= 2\sin(2x) + x^3\sin(2x) + \frac{3x^2}{2}\cos(2x) + C.\end{aligned}$$

115. Splitting the integrand into partial fractions with denominators $(x-2)$ and $(x+2)$, we have

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Multiplying by $(x-2)(x+2)$ gives the identity

$$1 = A(x+2) + B(x-2)$$

so

$$1 = (A+B)x + 2A - 2B.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} 2A - 2B &= 1 \\ A + B &= 0. \end{aligned}$$

Solving these equations gives $A = 1/4$, $B = -1/4$ and the integral becomes

$$\int \frac{1}{(x-2)(x+2)}dx = \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{1}{x+2}dx = \frac{1}{4}\left(\ln|x-2| - \ln|x+2|\right) + C.$$

116. Let $x = 5\sin t$. Then $dx = 5\cos t\, dt$, so substitution gives

$$\int \frac{1}{\sqrt{25-x^2}} = \int \frac{5\cos t}{\sqrt{25-25\sin^2 t}}dt = \int dt = t + C = \arcsin\left(\frac{x}{5}\right) + C.$$

117. Splitting the integrand into partial fractions with denominators x and $(x+5)$, we have

$$\frac{1}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5}.$$

Multiplying by $x(x+5)$ gives the identity

$$1 = A(x+5) + Bx$$

so

$$1 = (A+B)x + 5A.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$\begin{aligned} 5A &= 1 \\ A + B &= 0. \end{aligned}$$

Solving these equations gives $A = 1/5$, $B = -1/5$ and the integral becomes

$$\int \frac{1}{x(x+5)}dx = \frac{1}{5}\int \frac{1}{x}dx - \frac{1}{5}\int \frac{1}{x+5}dx = \frac{1}{5}\left(\ln|x| - \ln|x+5|\right) + C.$$

118. We use partial fractions and write

$$\frac{1}{3P-3P^2} = \frac{A}{3P} + \frac{B}{1-P},$$

multiply through by $3P(1-P)$, and then solve for A and B, getting $A = 1$ and $B = 1/3$. So

$$\begin{aligned} \int \frac{dP}{3P-3P^2} &= \int \left(\frac{1}{3P} + \frac{1}{3(1-P)}\right) dP = \frac{1}{3}\int \frac{dP}{P} + \frac{1}{3}\int \frac{dP}{1-P} \\ &= \frac{1}{3}\ln|P| - \frac{1}{3}\ln|1-P| + C = \frac{1}{3}\ln\left|\frac{P}{1-P}\right| + C. \end{aligned}$$

119. We use the trigonometric substitution $3x = \sin\theta$. Then $dx = \frac{1}{3}\cos\theta\, d\theta$ and substitution gives

$$\int \frac{1}{\sqrt{1-9x^2}}\, dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{3}\cos\theta\, d\theta = \frac{1}{3}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\, d\theta$$
$$= \frac{1}{3}\int 1\, d\theta = \frac{1}{3}\theta + C = \frac{1}{3}\arcsin(3x) + C.$$

120. Splitting the integrand into partial fractions with denominators x, $(x+2)$ and $(x-1)$, we have

$$\frac{2x+3}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}.$$

Multiplying by $x(x+2)(x-1)$ gives the identity

$$2x+3 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

so

$$2x+3 = (A+B+C)x^2 + (A-B+2C)x - 2A.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x on both sides must be equal. So

$$-2A = 3$$
$$A - B + 2C = 2$$
$$A + B + C = 0.$$

Solving these equations gives $A = -3/2$, $B = -1/6$ and $C = 5/3$. The integral becomes

$$\int \frac{2x+3}{x(x+2)(x-1)} dx = -\frac{3}{2}\int \frac{1}{x} dx - \frac{1}{6}\int \frac{1}{x+2} + \frac{5}{3}\int \frac{1}{x-1} dx$$
$$= -\frac{3}{2}\ln|x| - \frac{1}{6}\ln|x+2| + \frac{5}{3}\ln|x-1| + C.$$

121. The denominator can be factored to give $x(x-1)(x+1)$. Splitting the integrand into partial fractions with denominators x, $x-1$, and $x+1$, we have

$$\frac{3x+1}{x(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x}.$$

Multiplying by $x(x-1)(x+1)$ gives the identity

$$3x+1 = Ax(x+1) + Bx(x-1) + C(x-1)(x+1)$$

so

$$3x+1 = (A+B+C)x^2 + (A-B)x - C.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x and x^2 on both sides must be equal. So

$$-C = 1$$
$$A - B = 3$$
$$A + B + C = 0.$$

Solving these equations gives $A = 2$, $B = -1$ and $C = -1$. The integral becomes

$$\int \frac{3x+1}{x(x+1)(x-1)} dx = \int \frac{2}{x-1} dx - \int \frac{1}{x+1} dx - \int \frac{1}{x} dx$$
$$= 2\ln|x-1| - \ln|x+1| - \ln|x| + C.$$

122. Splitting the integrand into partial fractions with denominators $(1+x)$, $(1+x)^2$ and x, we have

$$\frac{1+x^2}{x(1+x)^2} = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{x}.$$

Multiplying by $x(1+x)^2$ gives the identity

$$1+x^2 = Ax(1+x) + Bx + C(1+x)^2$$

so

$$1+x^2 = (A+C)x^2 + (A+B+2C)x + C.$$

Since this equation holds for all x, the constant terms on both sides must be equal. Similarly, the coefficient of x and x^2 on both sides must be equal. So

$$\begin{aligned} C &= 1 \\ A+B+2C &= 0 \\ A+C &= 1. \end{aligned}$$

Solving these equations gives $A=0$, $B=-2$ and $C=1$. The integral becomes

$$\int \frac{1+x^2}{(1+x)^2 x}dx = -2\int \frac{1}{(1+x)^2}dx + \int \frac{1}{x}dx = \frac{2}{1+x} + \ln|x| + C.$$

123. Completing the square, we get

$$x^2+2x+2 = (x+1)^2+1.$$

We use the substitution $x+1=\tan t$, so $dx = (1/\cos^2 t)dt$. Since $\tan^2 t + 1 = 1/\cos^2 t$, the integral becomes

$$\int \frac{1}{(x+1)^2+1}dx = \int \frac{1}{\tan^2 t+1}\cdot\frac{1}{\cos^2 t}dt = \int dt = t + C = \arctan(x+1) + C.$$

124. Completing the square in the denominator gives

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1}.$$

We make the substitution $\tan\theta = x+2$. Then $dx = \frac{1}{\cos^2\theta}\,d\theta$.

$$\begin{aligned} \int \frac{dx}{(x+2)^2+1} &= \int \frac{d\theta}{\cos^2\theta(\tan^2\theta+1)} \\ &= \int \frac{d\theta}{\cos^2\theta(\frac{\sin^2\theta}{\cos^2\theta}+1)} \\ &= \int \frac{d\theta}{\sin^2\theta+\cos^2\theta} \\ &= \int d\theta = \theta + C \end{aligned}$$

But since $\tan\theta = x+2$, $\theta = \arctan(x+2)$, and so $\theta + C = \arctan(x+2) + C$.

125. Using partial fractions, we have:

$$\frac{3x+1}{x^2-3x+2} = \frac{3x+1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}.$$

Multiplying by $(x-1)$ and $(x-2)$, this becomes

$$\begin{aligned} 3x+1 &= A(x-2) + B(x-1) \\ &= (A+B)x - 2A - B \end{aligned}$$

which produces the system of equations

$$\begin{cases} A+B=3 \\ -2A-B=1. \end{cases}$$

Solving this system yields $A=-4$ and $B=7$. So,

$$\begin{aligned}\int \frac{3x+1}{x^2-3x+2}dx &= \int \left(-\frac{4}{x-1}+\frac{7}{x-2}\right)dx \\ &= -4\int \frac{dx}{x-1}+7\int \frac{dx}{x-2} \\ &= -4\ln|x-1|+7\ln|x-2|+C.\end{aligned}$$

126. We use the trigonometric substitution $bx=a\sin\theta$. Then $dx=\frac{a}{b}\cos\theta\,d\theta$, and we have

$$\begin{aligned}\int \frac{1}{\sqrt{a^2-(bx)^2}}\,dx &= \int \frac{1}{\sqrt{a^2-(a\sin\theta)^2}}\cdot\frac{a}{b}\cos\theta\,d\theta = \int \frac{1}{a\sqrt{1-\sin^2\theta}}\cdot\frac{a}{b}\cos\theta\,d\theta \\ &= \frac{1}{b}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta = \frac{1}{b}\int 1\,d\theta = \frac{1}{b}\theta+C = \frac{1}{b}\arcsin\left(\frac{bx}{a}\right)+C.\end{aligned}$$

127. $\int_4^\infty \frac{dt}{t^{3/2}}$ should converge, since $\int_1^\infty \frac{dt}{t^n}$ converges for $n>1$.
We calculate its value.
$\int_4^\infty \frac{dt}{t^{3/2}} = \lim_{b\to\infty}\int_4^b t^{-3/2}\,dt = \lim_{b\to\infty} -2t^{-1/2}\Big|_4^b = \lim_{b\to\infty}\left(1-\frac{2}{\sqrt{b}}\right)=1.$

128. $\int \frac{dx}{x\ln x} = \ln|\ln x|+C$. (Substitute $w=\ln x,\ dw=\frac{1}{x}\,dx$).
Thus

$$\int_{10}^\infty \frac{dx}{x\ln x} = \lim_{b\to\infty}\int_{10}^b \frac{dx}{x\ln x} = \lim_{b\to\infty}\ln|\ln x|\Big|_{10}^b = \lim_{b\to\infty}\ln(\ln b)-\ln(\ln 10).$$

As $b\to\infty, \ln(\ln b)\to\infty$, so this diverges.

129. To find $\int we^{-w}\,dw$, integrate by parts, with $u=w$ and $v'=e^{-w}$. Then $u'=1$ and $v=-e^{-w}$.
Then

$$\int we^{-w}\,dw = -we^{-w}+\int e^{-w}\,dw = -we^{-w}-e^{-w}+C.$$

Thus

$$\int_0^\infty we^{-w}\,dw = \lim_{b\to\infty}\int_0^b we^{-w}\,dw = \lim_{b\to\infty}\left(-we^{-w}-e^{-w}\right)\Big|_0^b = 1.$$

130. The trouble spot is at $x=0$, so we write

$$\int_{-1}^1 \frac{1}{x^4}\,dx = \int_{-1}^0 \frac{1}{x^4}\,dx + \int_0^1 \frac{1}{x^4}\,dx.$$

However, both these integrals diverge. For example,

$$\int_0^1 \frac{1}{x^4}\,dx = \lim_{a\to0^+}\int_a^1 \frac{1}{x^4}\,dx = \lim_{a\to0^+} -\frac{x^{-3}}{3}\Big|_a^1 = \lim_{a\to0^+}\left(\frac{1}{3a^3}-\frac{1}{3}\right).$$

Since this limit does not exist, $\int_0^1 \frac{1}{x^4}\,dx$ diverges and so the original integral diverges.

131. Since the value of $\tan\theta$ is between -1 and 1 on the interval $-\pi/4 \le \theta \le \pi/4$, our integral is not improper and so converges. Moreover, since $\tan\theta$ is an odd function, we have

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan\theta\, d\theta &= \int_{-\frac{\pi}{4}}^{0} \tan\theta\, d\theta + \int_{0}^{\frac{\pi}{4}} \tan\theta\, d\theta \\ &= -\int_{-\frac{\pi}{4}}^{0} \tan(-\theta)\, d\theta + \int_{0}^{\frac{\pi}{4}} \tan\theta\, d\theta \\ &= -\int_{0}^{\frac{\pi}{4}} \tan\theta\, d\theta + \int_{0}^{\frac{\pi}{4}} \tan\theta\, d\theta = 0.\end{aligned}$$

132. It is easy to see that this integral converges:

$$\frac{1}{4+z^2} < \frac{1}{z^2}, \quad \text{and so} \quad \int_2^\infty \frac{1}{4+z^2}\, dz < \int_2^\infty \frac{1}{z^2}\, dz = \frac{1}{2}.$$

We can also find its exact value.

$$\begin{aligned}\int_2^\infty \frac{1}{4+z^2}\, dz &= \lim_{b\to\infty} \int_2^b \frac{1}{4+z^2}\, dz \\ &= \lim_{b\to\infty} \left(\frac{1}{2} \arctan \frac{z}{2} \bigg|_2^b \right) \\ &= \lim_{b\to\infty} \left(\frac{1}{2} \arctan \frac{b}{2} - \frac{1}{2} \arctan 1 \right) \\ &= \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}.\end{aligned}$$

Note that $\frac{\pi}{8} < \frac{1}{2}$.

133. We find the exact value:

$$\begin{aligned}\int_{10}^\infty \frac{1}{z^2-4}\, dz &= \int_{10}^\infty \frac{1}{(z+2)(z-2)}\, dz \\ &= \lim_{b\to\infty} \int_{10}^b \frac{1}{(z+2)(z-2)}\, dz \\ &= \lim_{b\to\infty} \frac{1}{4} \left(\ln|z-2| - \ln|z+2| \right) \bigg|_{10}^b \\ &= \frac{1}{4} \lim_{b\to\infty} \left[(\ln|b-2| - \ln|b+2|) - (\ln 8 - \ln 12) \right] \\ &= \frac{1}{4} \lim_{b\to\infty} \left[\left(\ln \frac{b-2}{b+2} \right) + \ln \frac{3}{2} \right] \\ &= \frac{1}{4} (\ln 1 + \ln 3/2) = \frac{\ln 3/2}{4}.\end{aligned}$$

134. Substituting $w = t + 5$, we see that our integral is just $\int_0^{15} \frac{dw}{\sqrt{w}}$. This will converge, since $\int_0^b \frac{dw}{w^p}$ converges for $0 < p < 1$. We find its exact value:

$$\int_0^{15} \frac{dw}{\sqrt{w}} = \lim_{a\to 0^+} \int_a^{15} \frac{dw}{\sqrt{w}} = \lim_{a\to 0^+} 2w^{\frac{1}{2}} \bigg|_a^{15} = 2\sqrt{15}.$$

135. Since $\sin\phi < \phi$ for $\phi > 0$,

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin\phi}\, d\phi > \int_0^{\frac{\pi}{2}} \frac{1}{\phi}\, d\phi,$$

The integral on the right diverges, so the integral on the left must also. Alternatively, we use IV-20 in the integral table to get

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{1}{\sin\phi}\,d\phi &= \lim_{b\to 0^+}\int_b^{\frac{\pi}{2}} \frac{1}{\sin\phi}\,d\phi \\ &= \lim_{b\to 0^+} \frac{1}{2}\ln\left|\frac{\cos\phi-1}{\cos\phi+1}\right|\Bigg|_b^{\frac{\pi}{2}} \\ &= -\frac{1}{2}\lim_{b\to 0^+}\ln\left|\frac{\cos b-1}{\cos b+1}\right|.\end{aligned}$$

As $b\to 0^+$, $\cos b - 1 \to 0$ and $\cos b + 1 \to 2$, so $\ln|\frac{\cos b-1}{\cos b+1}| \to -\infty$. Thus the integral diverges.

136. Let $\phi = 2\theta$. Then $d\phi = 2\,d\theta$, and

$$\begin{aligned}\int_0^{\pi/4} \tan 2\theta\,d\theta &= \int_0^{\pi/2} \frac{1}{2}\tan\phi\,d\phi = \int_0^{\pi/2} \frac{1}{2}\frac{\sin\phi}{\cos\phi}\,d\phi \\ &= \lim_{b\to(\pi/2)^-}\int_0^b \frac{1}{2}\frac{\sin\phi}{\cos\phi}\,d\phi = \lim_{b\to(\pi/2)^-} -\frac{1}{2}\ln|\cos\phi|\Bigg|_0^b.\end{aligned}$$

As $b\to\pi/2$, $\cos\phi\to 0$, so $\ln|\cos\phi|\to-\infty$. Thus the integral diverges.

One could also see this by noting that $\cos x \approx \pi/2 - x$ and $\sin x \approx 1$ for x close to $\pi/2$: therefore, $\tan x \approx 1/(\frac{\pi}{2}-x)$, the integral of which diverges.

137. The integrand $\dfrac{x}{x+1} \to 1$ as $x\to\infty$, so there's no way $\displaystyle\int_1^\infty \frac{x}{x+1}\,dx$ can converge.

138. This function is difficult to integrate, so instead we try to compare it with some other function. Since $\frac{\sin^2\theta}{\theta^2+1} \ge 0$, we see that $\int_0^\infty \frac{\sin^2\theta}{\theta^2+1}\,d\theta \ge 0$. Also, since $\sin^2\theta \le 1$,

$$\int_0^\infty \frac{\sin^2\theta}{\theta^2+1}\,d\theta \le \int_0^\infty \frac{1}{\theta^2+1}\,d\theta = \lim_{b\to\infty}\arctan\theta\Big|_0^b = \frac{\pi}{2}.$$

Thus $\int_0^\infty \frac{\sin^2\theta}{\theta^2+1}\,d\theta$ converges, and its value is between 0 and $\frac{\pi}{2}$.

139. $\displaystyle\int_0^\pi \tan^2\theta d\theta = \tan\theta - \theta + C$, by formula IV-23. The integrand blows up at $\theta = \frac{\pi}{2}$, so

$$\int_0^\pi \tan^2\theta d\theta = \int_0^{\frac{\pi}{2}} \tan^2\theta d\theta + \int_{\frac{\pi}{2}}^\pi \tan^2\theta d\theta = \lim_{b\to\frac{\pi}{2}}[\tan\theta-\theta]_0^b + \lim_{a\to\frac{\pi}{2}}[\tan\theta-\theta]_a^\pi$$

which is undefined.

140. Since $0 \le \sin x < 1$ for $0 \le x \le 1$, we have

$$\begin{aligned} &(\sin x)^{\frac{3}{2}} < (\sin x) \\ \text{so}\quad &\frac{1}{(\sin x)^{\frac{3}{2}}} > \frac{1}{(\sin x)} \\ \text{or}\quad &(\sin x)^{-\frac{3}{2}} > (\sin x)^{-1}\end{aligned}$$

Thus $\displaystyle\int_0^1 (\sin x)^{-1}dx = \lim_{a\to 0}\ln\left|\frac{1}{\sin x} - \frac{1}{\tan x}\right|\Bigg|_a^1$, which is infinite.

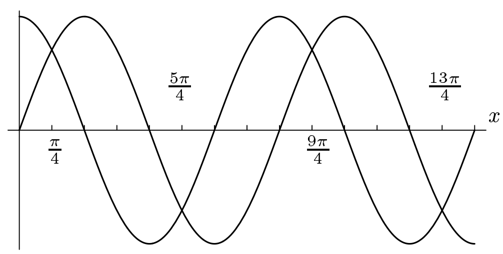
Hence, $\int_0^1 (\sin x)^{-\frac{3}{2}}dx$ is infinite.

Problems

141. Since the definition of f is different on $0 \le t \le 1$ than it is on $1 \le t \le 2$, break the definite integral at $t = 1$.

$$\begin{aligned}\int_0^2 f(t)\,dt &= \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \\ &= \int_0^1 t^2\,dt + \int_1^2 (2-t)\,dt \\ &= \frac{t^3}{3}\Big|_0^1 + \left(2t - \frac{t^2}{2}\right)\Big|_1^2 \\ &= 1/3 + 1/2 = 5/6 \approx 0.833\end{aligned}$$

142.



As is evident from the accompanying figure of the graphs of $y = \sin x$ and $y = \cos x$, the crossings occur at $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$, and the regions bounded by any two consecutive crossings have the same area. So picking two consecutive crossings, we get an area of

$$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,dx \\ &= 2\sqrt{2}.\end{aligned}$$

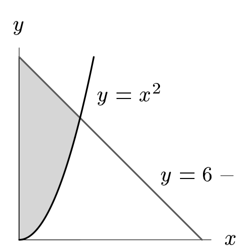
(Note that we integrated $\sin x - \cos x$ here because for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$, $\sin x \ge \cos x$.)

143. The point of intersection of the two curves $y = x^2$ and $y = 6 - x$ is at (2,4). The average height of the shaded area is the average value of the difference between the functions:

$$\frac{1}{(2-0)} \int_0^2 ((6-x) - x^2)\,dx = (3x - \frac{x^2}{4} - \frac{x^3}{6})\Big|_0^2 = \frac{11}{3}.$$

144. The average width of the shaded area in the figure below is the average value of the horizontal distance between the two functions. If we call this horizontal distance $h(y)$, then the average width is

$$\frac{1}{(6-0)} \int_0^6 h(y)\,dy.$$

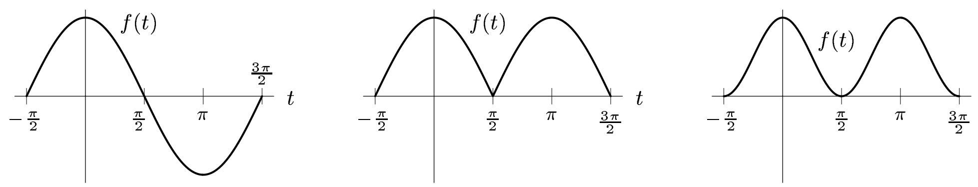


We could compute this integral if we wanted to, but we don't need to. We can simply note that the integral (without the $\frac{1}{6}$ term) is just the area of the shaded region; similarly, the integral in Problem 143 is *also* just the area of the shaded region. So they are the same. Now we know that our average width is just $\frac{1}{3}$ as much as the average height, since we divide by 6 instead of 2. So the answer is $\frac{11}{9}$.

145. (a) i. 0 ii. $\frac{2}{\pi}$ iii. $\frac{1}{2}$

(b) Average value of $f(t)$ < Average value of $k(t)$ < Average value of $g(t)$

We can look at the three functions in the range $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, since they all have periods of 2π ($|\cos t|$ and $(\cos t)^2$ also have a period of π, but that doesn't hurt our calculation). It is clear from the graphs of the three functions below that the average value for $\cos t$ is 0 (since the area above the x-axis is equal to the area below it), while the average values for the other two are positive (since they are everywhere positive, except where they are 0).



It is also fairly clear from the graphs that the average value of $g(t)$ is greater than the average value of $k(t)$; it is also possible to see this algebraically, since

$$(\cos t)^2 = |\cos t|^2 \leq |\cos t|$$

because $|\cos t| \leq 1$ (and both of these $\leq$'s are $<$'s at all the points where the functions are not 0 or 1).

146. Since $f(x)$ is decreasing on $[a, b]$, the left-hand Riemann sums are all overestimates and the right-hand sums are all underestimates. Because increasing the number of subintervals generally brings an approximation closer to the actual value, LEFT(10) is closer to the actual value (i.e., smaller, since the left sums are overestimates) than LEFT(5), and analogously for RIGHT(10) and RIGHT(5). Since the graph of $f(x)$ is concave down, a secant line lies below the curve and a tangent line lies above the curve. Therefore, TRAP is an underestimate and MID is an overestimate. Putting these observations together, we have

$$\text{RIGHT}(5) < \text{RIGHT}(10) < \text{TRAP}(10) < \text{Exact value} < \text{MID}(10) < \text{LEFT}(10) < \text{LEFT}(5).$$

147. Let's assume that $\text{TRAP}(10)$ and $\text{TRAP}(50)$ are either both overestimates or both underestimates. Since $\text{TRAP}(50)$ is more accurate, and it is bigger than $\text{TRAP}(10)$, both are underestimates. Since $\text{TRAP}(50)$ is 25 times more accurate, we have

$$I - \text{TRAP}(10) = 25(I - \text{TRAP}(50)),$$

where I is the value of the integral. Solving for I, we have

$$I \approx \frac{25\ \text{TRAP}(50) - \text{TRAP}(10)}{24} \approx 4.6969$$

Thus the error for $\text{TRAP}(10)$ is approximately 0.0078.

148. If $I(t)$ is average per capita income t years after 1987, then $I'(t) = r(t)$.

(a) Since $t = 8$ in 1995, by the Fundamental Theorem,

$$I(8) - I(0) = \int_0^8 r(t)\,dt = \int_0^8 480(1.024)^t\,dt$$
$$= \left.\frac{480(1.024)^t}{\ln(1.024)}\right|_0^8 = 4228$$

so $I(8) = 26{,}000 + 4228 = 30{,}228$.

(b)

$$I(t) - I(0) = \int_0^t r(t)\,dt = \int_0^t 480(1.024)^t\,dt$$
$$= \left.\frac{480(1.024)^t}{\ln(1.024)}\right|_0^t$$
$$= \frac{480}{\ln(1.024)}\left((1.024)^t - 1\right)$$
$$= 20{,}239\left((1.024)^t - 1\right)$$

Thus, since $I(0) = 26{,}000$,

$$I(t) = 26{,}000 + 20{,}239(1.024^t - 1) = 20{,}239(1.024)^t + 5761.$$

149. **(a)** Since the rate is given by $r(t) = 2te^{-2t}$ ml/sec, by the Fundamental Theorem of Calculus, the total quantity is given by the definite integral:

$$\text{Total quantity} \approx \int_0^\infty 2te^{-2t}\,dt = 2\lim_{b\to\infty}\int_0^b te^{-2t}\,dt.$$

Integration by parts with $u = t$, $v' = e^{-2t}$ gives

$$\begin{aligned}\text{Total quantity} &\approx 2\lim_{b\to\infty}\left(-\frac{t}{2}e^{-2t} - \frac{1}{4}e^{-2t}\right)\Big|_0^b \\ &= 2\lim_{b\to\infty}\left(\frac{1}{4} - \left(\frac{b}{2}+\frac{1}{4}\right)e^{-2b}\right) = 2\cdot\frac{1}{4} = 0.5 \text{ ml.}\end{aligned}$$

(b) At the end of 5 seconds,

$$\text{Quantity received} = \int_0^5 2te^{-2t}\,dt \approx 0.49975 \text{ ml.}$$

Since $0.49975/0.5 = 0.9995 = 99.95\%$, the patient has received 99.95% of the dose in the first 5 seconds.

150. The rate at which petroleum is being used t years after 1990 is given by

$$r(t) = 1.4\cdot 10^{20}(1.02)^t \text{ joules/year.}$$

Between 1990 and M years later

$$\begin{aligned}\text{Total quantity of petroleum used } &= \int_0^M 1.4\cdot 10^{20}(1.02)^t\,dt = 1.4\cdot 10^{20}\left.\frac{(1.02)^t}{\ln(1.02)}\right|_0^M \\ &= \frac{1.4\cdot 10^{20}}{\ln(1.02)}((1.02)^M - 1) \text{ joules.}\end{aligned}$$

Setting the total quantity used equal to 10^{22} gives

$$\begin{aligned}\frac{1.4\cdot 10^{20}}{\ln(1.02)}\left((1.02)^M - 1\right) &= 10^{22} \\ (1.02)^M &= \frac{100\ln(1.02)}{1.4} + 1 = 2.41 \\ M &= \frac{\ln(2.41)}{\ln(1.02)} \approx 45 \text{ years.}\end{aligned}$$

So we will run out of petroleum in 2035.

CAS Challenge Problems

151. **(a)** A CAS gives

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \frac{(\ln x)^2}{2} \\ \int \frac{(\ln x)^2}{x}\,dx &= \frac{(\ln x)^3}{3} \\ \int \frac{(\ln x)^3}{x}\,dx &= \frac{(\ln x)^4}{4}\end{aligned}$$

(b) Looking at the answers to part (a),

$$\int \frac{(\ln x)^n}{x}\,dx = \frac{(\ln x)^{n+1}}{n+1} + C.$$

(c) Let $w = \ln x$. Then $dw = (1/x)dx$, and

$$\int \frac{(\ln x)^n}{x}\,dx = \int w^n\,dw = \frac{w^{n+1}}{n+1} + C = \frac{(\ln x)^{n+1}}{n} + C.$$

152. (a) A CAS gives

$$\begin{aligned}\int \ln x\,dx &= -x + x\ln x\\ \int (\ln x)^2\,dx &= 2x - 2x\ln x + x(\ln x)^2\\ \int (\ln x)^3\,dx &= -6x + 6x\ln x - 3x(\ln x)^2 + x(\ln x)^3\\ \int (\ln x)^4\,dx &= 24x - 24x\ln x + 12x(\ln x)^2 - 4x(\ln x)^3 + x(\ln x)^4\end{aligned}$$

(b) In each of the cases in part (a), the expression for the integral $\int(\ln x)^n\,dx$ has two parts. The first part is simply a multiple of the expression for $\int(\ln x)^{n-1}\,dx$. For example, $\int(\ln x)^2\,dx$ starts out with $2x-2x\ln x = -2\int \ln x\,dx$. Similarly, $\int(\ln x)^3\,dx$ starts out with $-6x+6x\ln x-3(\ln x)^2 = -3\int(\ln x)^2\,dx$, and $\int(\ln x)^4\,dx$ starts out with $-4\int(\ln x)^3\,dx$. The remaining part of each antiderivative is a single term: it's $x(\ln x)^2$ in the case $n=2$, it's $x(\ln x)^3$ for $n=3$, and it's $x(\ln x)^4$ for $n=4$. The general pattern is

$$\int(\ln x)^n\,dx = -n\int(\ln x)^{n-1}\,dx + x(\ln x)^n.$$

To check this formula, we use integration by parts. Let $u=(\ln x)^n$ so $u' = n(\ln x)^{n-1}/x$ and $v'=1$ so $v=x$. Then

$$\begin{aligned}\int(\ln x)^n\,dx &= x(\ln x)^n - \int n\frac{(\ln x)^{n-1}}{x}\cdot x\,dx\\ \int(\ln x)^n\,dx &= x(\ln x)^n - n\int(\ln x)^{n-1}\,dx.\end{aligned}$$

This is the result we obtained before.

Alternatively, we can check our result by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(-n\int(\ln x)^{n-1}\,dx + x(\ln x)^n\right) &= -n(\ln x)^{n-1} + \frac{d}{dx}(x(\ln x)^n)\\ &= -n(\ln x)^{n-1} + (\ln x)^n + x\cdot n(\ln x)^{n-1}\frac{1}{x}\\ &= -n(\ln x)^{n-1} + (\ln x)^n + n(\ln x)^{n-1} = (\ln x)^n.\end{aligned}$$

Therefore,

$$\int(\ln x)^n\,dx = -n\int(\ln x)^{n-1}\,dx + x(\ln x)^n.$$

153. (a) A possible answer from the CAS is

$$\int \sin^3 x\,dx = \frac{-9\,\cos(x)+\cos(3\,x)}{12}.$$

(b) Differentiating

$$\frac{d}{dx}\left(\frac{-9\,\cos(x)+\cos(3\,x)}{12}\right) = \frac{9\,\sin(x)-3\,\sin(3\,x)}{12} = \frac{3\sin x - \sin(3x)}{4}.$$

(c) Using the identities, we get

$$\begin{aligned}\sin(3x) = \sin(x+2x) &= \sin x\cos 2x + \cos x\sin 2x\\ &= \sin x(1-2\sin^2 x) + \cos x(2\sin x\cos x)\\ &= \sin x - 2\sin^3 x + 2\sin x(1-\sin^2 x)\\ &= 3\sin x - 4\sin^3 x.\end{aligned}$$

Thus,

$$3\sin x - \sin(3x) = 3\sin x - (3\sin x - 4\sin^3 x) = 4\sin^3 x,$$

so

$$\frac{3\sin x - \sin(3x)}{4} = \sin^3 x.$$

154. (a) A possible answer is

$$\int \sin x \cos x \cos(2x)\, dx = -\frac{\cos(4x)}{16}.$$

Different systems may give the answer in a different form.

(b)

$$\frac{d}{dx}\left(-\frac{\cos(4x)}{16}\right) = \frac{\sin(4x)}{4}.$$

(c) Using the double angle formula $\sin 2A = 2\sin A\cos A$ twice, we get

$$\frac{\sin(4x)}{4} = \frac{2\sin(2x)\cos(2x)}{4} = \frac{2\cdot 2\sin x\cos x\cos(2x)}{4} = \sin x\cos x\cos(2x).$$

155. (a) A possible answer from the CAS is

$$\int \frac{x^4}{(1+x^2)^2}\,dx = x + \frac{x}{2\ (1+x^2)} - \frac{3}{2}\arctan(x).$$

Different systems may give the answer in different form.

(b) Differentiating gives

$$\frac{d}{dx}\left(x + \frac{x}{2\ (1+x^2)} - \frac{3}{2}\arctan(x)\right) = 1 - \frac{x^2}{(1+x^2)^2} - \frac{1}{1+x^2}.$$

(c) Putting the result of part (b) over a common denominator, we get

$$\begin{aligned} 1 - \frac{x^2}{(1+x^2)^2} - \frac{1}{1+x^2} &= \frac{\left(1+x^2\right)^2 - x^2 - (1+x^2)}{(1+x^2)^2} \\ &= \frac{1+2x^2+x^4-x^2-1-x^2}{(1+x^2)^2} = \frac{x^4}{(1+x^2)^2}. \end{aligned}$$

CHECK YOUR UNDERSTANDING

1. False. The subdivision size $\Delta x = (1/10)(6-2) = 4/10$.
2. True, since $\Delta x = (6-2)/n = 4/n$.
3. False. If f is decreasing, then on each subinterval the value of $f(x)$ at the left endpoint is larger than the value at the right endpoint, which means that LEFT(n) >RIGHT(n) for any n.
4. False. As n approaches infinity, LEFT(n) approaches the value of the integral $\int_2^6 f(x)dx$, which is generally not 0.
5. True. We have

$$\text{LEFT}(n) - \text{RIGHT}(n) = (f(x_0) + f(x_1) + \cdots + f(x_{n-1}))\Delta x - (f(x_1) + f(x_2) + \cdots + f(x_n))\Delta x.$$

On the right side of the equation, all terms cancel except the first and last, so:

$$\text{LEFT}(n) - \text{RIGHT}(n) = (f(x_0) - f(x_n))\Delta x = (f(2) - f(6))\Delta x.$$

This is also discussed in Section 5.1.

6. True. This follows from the fact that $\Delta x = (6-2)/n = 4/n$.
7. False. Since LEFT(n) − RIGHT$(n) = (f(2) - f(6))\Delta x$, we have LEFT$(n)$ = RIGHT(n) for any function such that $f(2) = f(6)$. Such a function, for example $f(x) = (x-4)^2$, need not be a constant function.
8. False. Although TRAP(n) is usually a better estimate, it is not always better. If $f(2) = f(6)$, then LEFT(n) = RIGHT(n) and hence TRAP(n) = LEFT(n) = RIGHT(n), so in this case TRAP(n) is no better.

9. False. This is true if f is an increasing function or if f is a decreasing function, but it is not true in general. For example, suppose that $f(2) = f(6)$. Then LEFT(n) = RIGHT(n) for all n, which means that if $\int_2^6 f(x)dx$ lies between LEFT(n) and RIGHT(n), then it must equal LEFT(n), which is not always the case.

For example, if $f(x) = (x-4)^2$ and $n = 1$, then $f(2) = f(6) = 4$, so

$$\text{LEFT}(1) = \text{RIGHT}(1) = 4 \cdot (6-2) = 16.$$

However

$$\int_2^6 (x-4)^2 dx = \left.\frac{(x-4)^3}{3}\right|_2^6 = \frac{2^3}{3} - \left(-\frac{2^3}{3}\right) = \frac{16}{3}.$$

In this example, since LEFT(n) = RIGHT(n), we have TRAP(n) = LEFT(n). However trapezoids overestimate the area, since the graph of f is concave up. This is also discussed in Section 7.5.

10. True. Let $w = f(x)$, so $dw = f'(x)\,dx$, then

$$\int f'(x)\cos(f(x))\,dx = \int \cos w\,dw = \sin w + C = \sin(f(x)) + C.$$

11. False. Differentiating gives

$$\frac{d}{dx}\ln|f(x)| = \frac{1}{f(x)} \cdot f'(x),$$

so, in general

$$\int \frac{1}{f(x)}\,dx \neq \ln|f(x)| + C.$$

12. True. Let $w = 5 - t^2$, then $dw = -2t\,dt$.

13. True. Rewrite $\sin^7\theta = \sin\theta\sin^6\theta = \sin\theta(1-\cos^2\theta)^3$. Expanding, substituting $w = \cos\theta, dw = -\sin\theta\,d\theta$, and integrating gives a polynomial in w, which is a polynomial in $\cos\theta$.

14. False. Completing the square gives

$$\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \arctan(x+2) + C.$$

15. False. Factoring gives

$$\int \frac{dx}{x^2+4x-5} = \int \frac{dx}{(x+5)(x-1)} = \frac{1}{6}\int\left(\frac{1}{x-1} - \frac{1}{x+5}\right)dx = \frac{1}{6}(\ln|x-1| - \ln|x+5|) + C.$$

16. True. Let $w = \ln x, dw = x^{-1}\,dx$. Then

$$\int x^{-1}((\ln x)^2 + (\ln x)^3)\,dx = \int (w^2 + w^3)\,dw = \frac{w^3}{3} + \frac{w^4}{4} + C = \frac{(\ln x)^3}{3} + \frac{(\ln x)^4}{4} + C.$$

17. True. Let $u = t, v' = \sin(5-t)$, so $u' = 1, v = \cos(5-t)$. Then the integral $\int 1 \cdot \cos(5-t)\,dt$ can be done by guess-and-check or by substituting $w = 5 - t$.

18. True. Since

$$\lim_{b\to\infty}\int_0^b f(x)dx = \int_0^a f(x)dx + \lim_{b\to\infty}\int_a^b f(x)dx,$$

the limit on the left side of the equation is finite exactly when the limit on the right side is finite. Thus, if $\int_0^\infty f(x)dx$ converges, then so does $\int_a^\infty f(x)dx$.

19. diverges.

True. Suppose that f has period p. Then $\int_0^p f(x)dx$, $\int_p^{2p} f(x)dx$, $\int_{2p}^{3p} f(x)dx$,…are all equal. If we let $k = \int_0^p f(x)dx$, then $\int_0^{np} f(x)dx = nk$, for any positive integer n. Since $f(x)$ is positive, so is k. Thus as n approaches ∞, the value of $\int_0^{np} f(x)dx = nk$ approaches ∞. That means that $\lim_{b\to\infty}\int_0^b f(x)dx$ is not finite; that is, the integral diverges.

20. False. Let $f(x) = 1/(x+1)$. Then

$$\int_0^\infty \frac{1}{x+1}dx = \lim_{b\to\infty} \ln|x+1|\Big|_0^b = \lim_{b\to\infty} \ln(b+1),$$

but $\lim_{b\to\infty} \ln(b+1)$ does not exist.

21. False. Let $f(x) = x+1$. Then

$$\int_0^\infty \frac{1}{x+1}dx = \lim_{b\to\infty} \ln|x+1|\Big|_0^b = \lim_{b\to\infty} \ln(b+1),$$

but $\lim_{b\to\infty} \ln(b+1)$ does not exist.

22. True. By properties of integrals and limits,

$$\lim_{b\to\infty}\int_0^b (f(x)+g(x))dx = \lim_{b\to\infty}\int_0^b f(x)dx + \lim_{b\to\infty}\int_0^b g(x)dx.$$

Since the two limits on the right side of the equation are finite, the limit on the left side is also finite, that is, $\int_0^\infty (f(x)+g(x))dx$ converges.

23. False. For example, let $f(x) = x$ and $g(x) = -x$. Then $f(x) + g(x) = 0$, so $\int_0^\infty (f(x)+g(x))\,dx$ converges, even though $\int_0^\infty f(x)\,dx$ and $\int_0^\infty g(x)\,dx$ diverge.

24. True. By properties of integrals and limits,

$$\lim_{b\to\infty}\int_0^b af(x)\,dx = a\lim_{b\to\infty}\int_0^b f(x)\,dx.$$

Thus, the limit on the left of the equation is finite exactly when the limit on the right side of the equation is finite. Thus $\int_0^\infty af(x)\,dx$ converges if $\int_0^\infty f(x)\,dx$ converges.

25. True. Make the substitution $w = ax$. Then $dw = a\,dx$, so

$$\int_0^b f(ax)\,dx = \frac{1}{a}\int_0^c f(w)\,dw,$$

where $c = ab$. As b approaches infinity, so does c, since a is constant. Thus the limit of the left side of the equation as b approaches infinity is finite exactly when the limit of the right side of the equation as c approaches infinity is finite. That is, $\int_0^\infty f(ax)\,dx$ converges exactly when $\int_0^\infty f(x)\,dx$ converges.

26. True. Make the substitution $w = a+x$, so $dw = dx$. Then $w = a$ when $x = 0$, and $w = a+b$ when $x = b$, so

$$\int_0^b f(a+x)\,dx = \int_a^{b+a} f(w)\,dw = \int_a^c f(w)\,dw$$

where $c = b+a$. As b approaches infinity, so does c, since a is constant. Thus the limit of the left side of the equation as b approaches infinity is finite exactly when the limit of the right side of the equation as c approaches infinity is finite. Since $\int_0^\infty f(x)\,dx$ converges, we know that $\lim_{c\to\infty}\int_0^c f(w)\,dw$ is finite, so $\lim_{c\to\infty}\int_a^c f(w)\,dw$ is finite for any positive a. Thus, $\int_0^\infty f(a+x)\,dx$ converges.

27. False. We have

$$\int_0^b (a+f(x))\,dx = \int_0^b a\,dx + \int_0^b f(x)\,dx.$$

Since $\int_0^\infty f(x)\,dx$ converges, the second integral on the right side of the equation has a finite limit as b approaches infinity. But the first integral on the right side has an infinite limit as b approaches infinity, since $a \neq 0$. Thus the right side all together has an infinite limit, which means that $\int_0^\infty (a+f(x))\,dx$ diverges.

PROJECTS FOR CHAPTER SEVEN

1. (a) If $e^t \ge 1+t$, then

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\,dt \\ &\ge 1 + \int_0^x (1+t)\,dt = 1 + x + \frac{1}{2}x^2. \end{aligned}$$

We can keep going with this idea. Since $e^t \geq 1 + t + \frac{1}{2}t^2$,

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\,dt \\ &\geq 1 + \int_0^x (1 + t + \frac{1}{2}t^2)\,dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3. \end{aligned}$$

We notice that each term in our summation is of the form $\frac{x^n}{n!}$. Furthermore, we see that if we have a sum $1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}$ such that

$$e^x \geq 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!},$$

then

$$\begin{aligned} e^x &= 1 + \int_0^x e^t\,dt \\ &\geq 1 + \int_0^x \left(1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!}\right)\,dt \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^{n+1}}{(n+1)!}. \end{aligned}$$

Thus we can continue this process as far as we want, so

$$e^x \geq 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n = \sum_{j=0}^{n} \frac{x^j}{j!} \text{ for any } n.$$

(In fact, it turns out that if you let n get larger and larger and keep adding up terms, your values approach exactly e^x.)

(b) We note that $\sin x = \int_0^x \cos t\,dt$ and $\cos x = 1 - \int_0^x \sin t\,dt$. Thus, since $\cos t \leq 1$, we have

$$\sin x \leq \int_0^x 1\,dt = x.$$

Now using $\sin t \leq t$, we have

$$\cos x \leq 1 - \int_0^x t\,dt = 1 - \frac{1}{2}x^2.$$

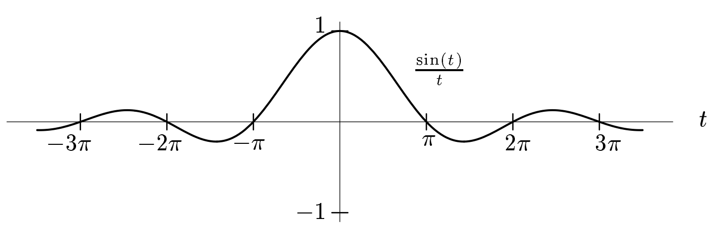
Then we just keep going:

$$\sin x \leq \int_0^x \left(1 - \frac{1}{2}t^2\right)\,dt = x - \frac{1}{6}x^3.$$

Therefore

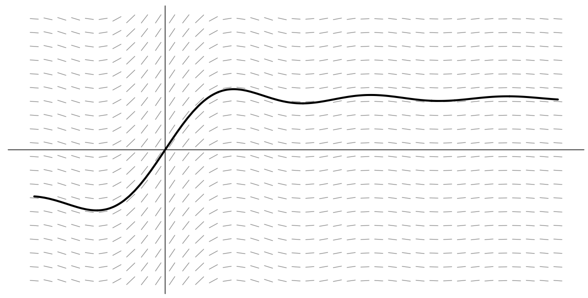
$$\cos x \leq 1 - \int_0^x \left(t - \frac{1}{6}t^3\right)\,dt = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

2. (a) (i)



(ii) $\text{Si}(x)$ neither always decreases nor always increases, since its derivative, $x^{-1}\sin x$, has both positive and negative values for $x > 0$. For positive x, $\text{Si}(x)$ is the area under the curve $\frac{\sin t}{t}$ and above the t-axis from $t = 0$ to $t = x$, minus the area above the curve and below the t-axis. Looking at the graph above, one can see that this difference of areas is going to always be positive.

(iii)

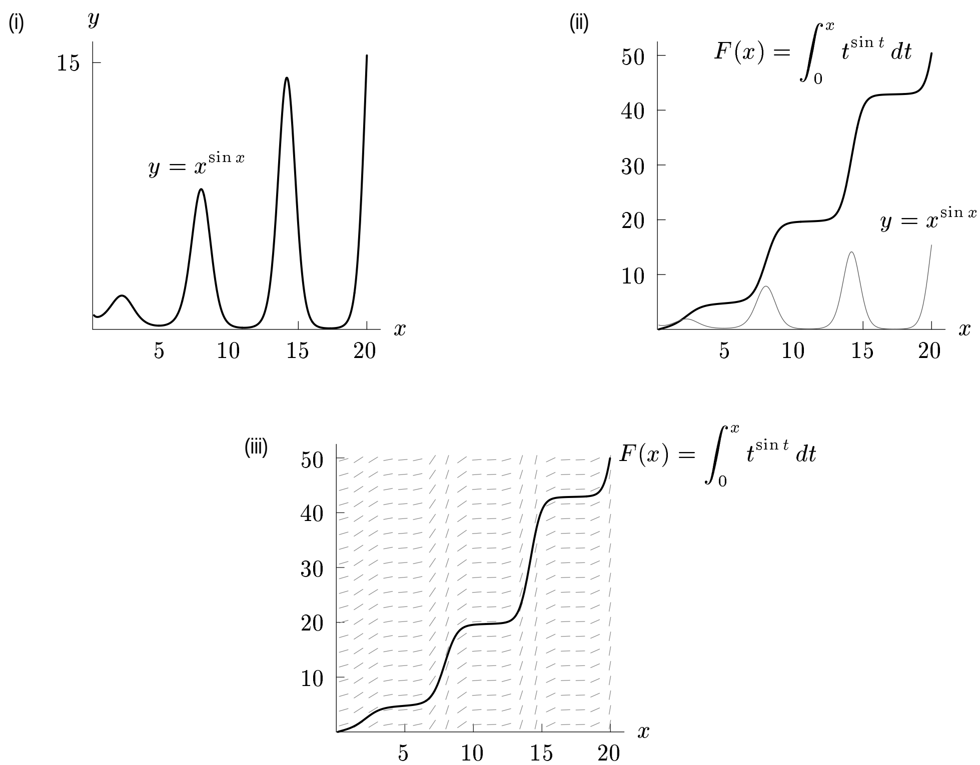


It seems that the limit exists: the curve drawn in the slope field,

$$y = Si(x) = \int_0^x \frac{\sin t}{t}\, dt,$$

seems to approach some limiting height as $x \to \infty$. (In fact, the limiting height is $\pi/2$.)

(b)



(c) (i) The most obvious feature of the graph of $y = \sin(x^2)$ is its symmetry about the y-axis. This means the function $g(x) = \sin(x^2)$ is an even function, i.e. for all x, we have $g(x) = g(-x)$. Since $\sin(x^2)$ is even, its antiderivative F must be odd, that is $F(-x) = -F(-x)$. To see this, set $F(t) = \int_0^t \sin(x^2)\, dx$, then

$$F(-t) = \int_0^{-t} \sin(x^2)\, dx = -\int_{-t}^{0} \sin(x^2)\, dx = -\int_0^{t} \sin(x^2)\, dx = -F(t),$$

since the area from $-t$ to 0 is the same as the area from 0 to t. Thus $F(t) = -F(-t)$ and F is odd.

The second obvious feature of the graph of $y = \sin(x^2)$ is that it oscillates between -1 and 1 with a "period" which goes to zero as $|x|$ increases. This implies that $F'(x)$ alternates between intervals where it is positive or negative, and increasing or decreasing, with frequency growing arbitrarily large as $|x|$ increases. Thus $F(x)$ itself similarly alternates between intervals where it is increasing or decreasing, and concave up or concave down.

Finally, since $y = \sin(x^2) = F'(x)$ passes through $(0, 0)$, and $F(0) = 0$, F is tangent to the x-axis at the origin.

(ii)

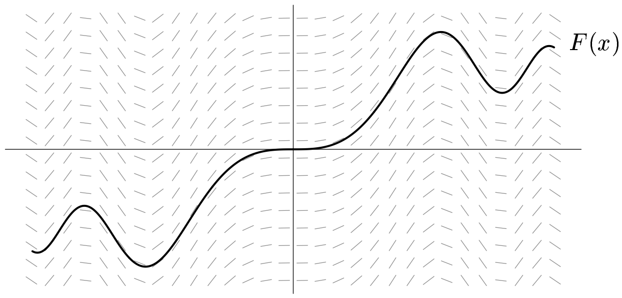


Figure 7.22

F never crosses the x-axis in the region $x > 0$, and $\lim_{x\to\infty} F(x)$ exists. One way to see these facts is to note that by the Construction Theorem,

$$F(x) = F(x) - F(0) = \int_0^x F'(t)\,dt.$$

So $F(x)$ is just the area between the curve $y = \sin(t^2)$ and the t-axis for $0 \leq t \leq x$ (with area above the t-axis counting positively, and area below the t-axis counting negatively). Now looking at the graph of curve, we see that this area will include alternating pieces above and below the t-axis. We can also see that the area of these pieces is approaching 0 as we go further out. So we add a piece, take a piece away, add another piece, take another piece away, and so on.

It turns out that this means that the sums of the pieces converge. To see this, think of walking from point A to point B. If you walk almost to B, then go a smaller distance toward A, then a yet smaller distance back toward B, and so on, you will eventually approach some point between A and B. So we can see that $\lim_{x\to\infty} F(x)$ exists. Also, since we always subtract a smaller piece than we just added, and the first piece is added instead of subtracted, we see that we never get a negative sum; thus $F(x)$ is never negative in the region $x > 0$, so $F(x)$ never crosses the x-axis there.