CHAPTER NINE

Solutions for Section 9.1

Exercises

1. Yes, $a = 1$, ratio $= -1/2$.

2. No. Ratio between successive terms is not constant: $\dfrac{1/3}{1/2} = 0.66\ldots$, while $\dfrac{1/4}{1/3} = 0.75$.

3. Yes, $a = 5$, ratio $= -2$.

4. Yes, $a = 2$, ratio $= 1/2$.

5. No. Ratio between successive terms is not constant: $\dfrac{2x^2}{x} = 2x$, while $\dfrac{3x^3}{2x^2} = \dfrac{3}{2}x$.

6. Yes, $a = y^2$, ratio $= y$.

7. Yes, $a = 1$, ratio $= -x$.

8. Yes, $a = 1$, ratio $= -y^2$.

9. No. Ratio between successive terms is not constant: $\dfrac{6z^2}{3z} = 2z$, while $\dfrac{9z^3}{6z^2} = \dfrac{3}{2}z$.

10. Yes, $a = 1$, ratio $= 2z$.

11. Sum $= \dfrac{y^2}{1-y}$, $|y| < 1$

12. Sum $= \dfrac{1}{1-(-x)} = \dfrac{1}{1+x}$, $|x| < 1$

13. Sum $= \dfrac{1}{1-(-y^2)} = \dfrac{1}{1+y^2}$, $|y| < 1$.

14. Sum $= \dfrac{1}{1-2z}$, $|z| < 1/2$

15. $-2 + 1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \dfrac{1}{16} - \cdots = \displaystyle\sum_{n=0}^{\infty} (-2)\left(-\frac{1}{2}\right)^n$, a geometric series.

Let $a = -2$ and $x = -\frac{1}{2}$. Then

$$\sum_{n=0}^{\infty} (-2)\left(-\frac{1}{2}\right)^n = \frac{a}{1-x} = \frac{-2}{1-(-\frac{1}{2})} = -\frac{4}{3}.$$

16. $3 + \dfrac{3}{2} + \dfrac{3}{4} + \dfrac{3}{8} \cdots + \dfrac{3}{2^{10}} = 3\left(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{2^{10}}\right) = \dfrac{3\left(1 - \frac{1}{2^{11}}\right)}{1 - \frac{1}{2}} = \dfrac{3\left(2^{11} - 1\right)}{2^{10}}$

17. Using the formula for the sum of an infinite geometric series,

$$\sum_{n=4}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \cdots = \left(\frac{1}{3}\right)^4 \left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right) = \frac{(\frac{1}{3})^4}{1 - \frac{1}{3}} = \frac{1}{54}$$

18. Using the formula for the sum of a finite geometric series,

$$\sum_{n=4}^{20} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^4 + \left(\frac{1}{3}\right)^5 + \cdots + \left(\frac{1}{3}\right)^{20} = \left(\frac{1}{3}\right)^4 \left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots \left(\frac{1}{3}\right)^{16}\right) = \frac{(1/3)^4(1-(1/3)^{17})}{1-(1/3)} = \frac{3^{17}-1}{2\cdot 3^{20}}.$$

Problems

19. Since the amount of ampicillin excreted during the time interval between tablets is 250 mg, we have

$$\text{Amount of ampicillin excreted} = \text{Original quantity} - \text{Final quantity}$$
$$250 = Q - (0.04)Q.$$

Solving for Q gives, as before,

$$Q = \frac{250}{1 - 0.04} \approx 260.42.$$

20. **(a)** The amount of atenolol in the blood is given by $Q(t) = Q_0 e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 6.3 hours,

$$\frac{1}{2} = e^{-6.3k}, \quad k = -\frac{1}{6.3}\ln\frac{1}{2} \approx 0.11.$$

After 24 hours

$$Q = Q_0 e^{-k(24)} \approx Q_0 e^{-0.11(24)} \approx Q_0(0.07).$$

Thus, the percentage of the atenolol that remains after 24 hours $\approx 7\%$.

(b)

$$\begin{aligned}
Q_0 &= 50\\
Q_1 &= 50 + 50(0.07)\\
Q_2 &= 50 + 50(0.07) + 50(0.07)^2\\
Q_3 &= 50 + 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\
&\vdots\\
Q_n &= 50 + 50(0.07) + 50(0.07)^2 + \cdots + 50(0.07)^n = \frac{50(1-(0.07)^{n+1})}{1-0.07}
\end{aligned}$$

(c)

$$\begin{aligned}
P_1 &= 50(0.07)\\
P_2 &= 50(0.07) + 50(0.07)^2\\
P_3 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3\\
P_4 &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + 50(0.07)^4\\
&\vdots\\
P_n &= 50(0.07) + 50(0.07)^2 + 50(0.07)^3 + \cdots + 50(0.07)^n\\
&= 50(0.07)\left(1 + (0.07) + (0.07)^2 + \cdots + (0.07)^{n-1}\right) = \frac{0.07(50)(1-(0.07)^n)}{1-0.07}
\end{aligned}$$

21. **(a)**

$$\begin{aligned}
P_1 &= 0\\
P_2 &= 250(0.04)\\
P_3 &= 250(0.04) + 250(0.04)^2\\
P_4 &= 250(0.04) + 250(0.04)^2 + 250(0.04)^3\\
&\vdots\\
P_n &= 250(0.04) + 250(0.04)^2 + 250(0.04)^3 + \cdots + 250(0.04)^{n-1}
\end{aligned}$$

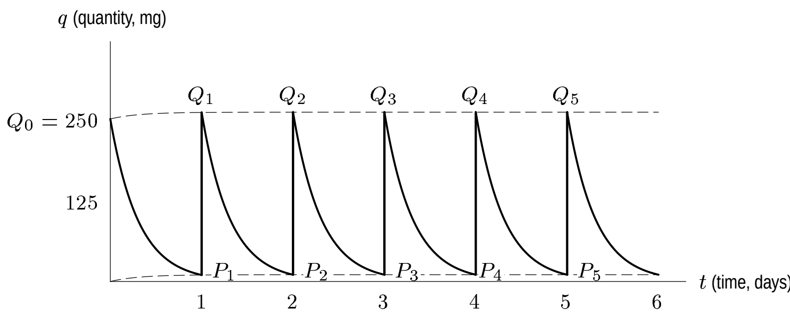
(b) $P_n = 250(0.04)\left(1 + (0.04) + (0.04)^2 + (0.04)^3 + \cdots + (0.04)^{n-2}\right) = 250\dfrac{0.04(1-(0.04)^{n-1})}{1-0.04}$

(c)

$$\begin{aligned} P &= \lim_{n\to\infty} P_n \\ &= \lim_{n\to\infty} 250\frac{0.04(1-(0.04)^{n-1})}{1-0.04} \\ &= \frac{(250)(0.04)}{0.96} = 0.04Q \approx 10.42 \end{aligned}$$

Thus, $\lim_{n\to\infty} P_n = 10.42$ and $\lim_{n\to\infty} Q_n = 260.42$. We would expect these limits to differ because one is right before taking a tablet, one is right after. We would expect the difference between them to be 250 mg, the amount of ampicillin in one tablet.

22.



23. (a) Let h_n be the height of the n^{th} bounce after the ball hits the floor for the n^{th} time. Then from Figure 9.1,

$$\begin{aligned} h_0 &= \text{height before first bounce } = 10 \text{ feet}, \\ h_1 &= \text{height after first bounce } = 10\left(\frac{3}{4}\right) \text{ feet}, \\ h_2 &= \text{height after second bounce } = 10\left(\frac{3}{4}\right)^2 \text{ feet}. \end{aligned}$$

Generalizing gives

$$h_n = 10\left(\frac{3}{4}\right)^n.$$

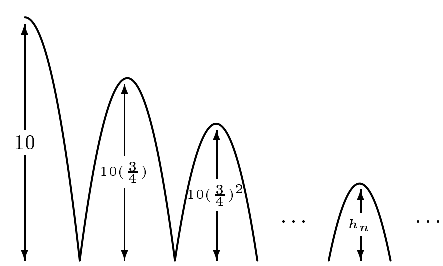


Figure 9.1

(b) When the ball hits the floor for the first time, the total distance it has traveled is just $D_1 = 10$ feet. (Notice that this is the same as $h_0 = 10$.) Then the ball bounces back to a height of $h_1 = 10\left(\frac{3}{4}\right)$, comes down and hits the floor for the second time. See Figure 9.1. The total distance it has traveled is

$$D_2 = h_0 + 2h_1 = 10 + 2\cdot 10\left(\frac{3}{4}\right) = 25 \text{ feet}.$$

Then the ball bounces back to a height of $h_2 = 10\left(\frac{3}{4}\right)^2$, comes down and hits the floor for the third time. It has traveled

$$D_3 = h_0 + 2h_1 + 2h_2 = 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 = 25 + 2\cdot 10\left(\frac{3}{4}\right)^2 = 36.25 \text{ feet}.$$

Similarly,

$$\begin{aligned} D_4 &= h_0 + 2h_1 + 2h_2 + 2h_3 \\ &= 10 + 2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 \\ &= 36.25 + 2\cdot 10\left(\frac{3}{4}\right)^3 \\ &\approx 44.69 \text{ feet.} \end{aligned}$$

(c) When the ball hits the floor for the n^{th} time, its last bounce was of height h_{n-1}. Thus, by the method used in part (b), we get

$$\begin{aligned} D_n &= h_0 + 2h_1 + 2h_2 + 2h_3 + \cdots + 2h_{n-1} \\ &= 10 + \underbrace{2\cdot 10\left(\frac{3}{4}\right) + 2\cdot 10\left(\frac{3}{4}\right)^2 + 2\cdot 10\left(\frac{3}{4}\right)^3 + \cdots + 2\cdot 10\left(\frac{3}{4}\right)^{n-1}}_{\text{finite geometric series}} \\ &= 10 + 2\cdot 10\cdot\left(\frac{3}{4}\right)\left(1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \cdots + \left(\frac{3}{4}\right)^{n-2}\right) \\ &= 10 + 15\left(\frac{1-\left(\frac{3}{4}\right)^{n-1}}{1-\left(\frac{3}{4}\right)}\right) \\ &= 10 + 60\left(1-\left(\frac{3}{4}\right)^{n-1}\right). \end{aligned}$$

24. (a) The acceleration of gravity is 32 ft/sec^2 so acceleration $= 32$ and velocity $v = 32t + C$. Since the ball is dropped, its initial velocity is 0 so $v = 32t$. Thus the position is $s = 16t^2 + C$. Calling the initial position $s = 0$, we have $s = 6t$. The distance traveled is h so $h = 16t$. Solving for t we get $t = \frac{1}{4}\sqrt{h}$.

(b) The first drop from 10 feet takes $\frac{1}{4}\sqrt{10}$ seconds. The first full bounce (to $10\cdot(\frac{3}{4})$ feet) takes $\frac{1}{4}\sqrt{10\cdot(\frac{3}{4})}$ seconds to rise, therefore the same time to come down. Thus, the full bounce, up and down, takes $2(\frac{1}{4})\sqrt{10\cdot(\frac{3}{4})}$ seconds. The next full bounce takes $2(\frac{1}{4})10\cdot(\frac{3}{4})^2 = 2(\frac{1}{4})\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^2$ seconds. The n^{th} bounce takes $2(\frac{1}{4})\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^n$ seconds. Therefore the

$$\begin{aligned} &\text{Total amount of time} \\ &= \frac{1}{4}\sqrt{10} + \underbrace{\frac{2}{4}\sqrt{10}\sqrt{\frac{3}{4}} + \frac{2}{4}\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^2 + \frac{2}{4}\sqrt{10}\left(\sqrt{\frac{3}{4}}\right)^3}_{\text{Geometric series with } a = \frac{2}{4}\sqrt{10}\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}} \text{ and } x = \sqrt{\frac{3}{4}}} + \cdots \\ &= \frac{1}{4}\sqrt{10} + \frac{1}{2}\sqrt{10}\sqrt{\frac{3}{4}}\left(\frac{1}{1-\sqrt{3/4}}\right) \text{ seconds.} \end{aligned}$$

25. (a)

$$\begin{aligned} \text{Total amount of money deposited} &= 100 + 92 + 84.64 + \cdots \\ &= 100 + 100(0.92) + 100(0.92)^2 + \cdots \\ &= \frac{100}{1-0.92} = 1250 \quad \text{dollars} \end{aligned}$$

(b) Credit multiplier $= 1250/100 = 12.50$
The 12.50 is the factor by which the bank has increased its deposits, from $\$100$ to $\$1250$.

26. The amount of additional income generated directly by people spending their extra money is $\$100(0.8) = \$80$ million. This additional money in turn is spent, generating another $(\$100(0.8))\,(0.8) = \$100(0.8)^2$ million. This continues indefinitely, resulting in

$$\text{Total additional income} = 100(0.8) + 100(0.8)^2 + 100(0.8)^3 + \cdots = \frac{100(0.8)}{1-0.8} = \$400 \text{ million}$$

27. The total of the spending and respending of the additional income is given by the series: Total additional income $=$ $100(0.9) + 100(0.9)^2 + 100(0.9)^3 + \cdots = \frac{100(0.9)}{1-0.9} = \$900$ million.
Notice the large effect of changing the assumption about the fraction of money spent has: the additional spending more than doubles.

Solutions for Section 9.2

Exercises

1. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|0.2| < 1$, we have $\lim_{n\to\infty} (0.2)^n = 0$.

2. Since 2^n increases without bound as n increases, the limit does not exist.

3. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|-0.3| < 1$, we have $\lim_{n\to\infty} (-0.3)^n = 0$.

4. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $|e^{-2}| < 1$, we have $\lim_{n\to\infty} (e^{-2n}) = \lim_{n\to\infty} (e^{-2})^n = 0$, so $\lim_{n\to\infty} (3+e^{-2n}) = 3+0 = 3$.

5. Since $S_n = \cos(\pi n) = 1$ if n is even and $S_n = \cos(\pi n) = -1$ if n is odd, the values of S_n oscillate between 1 and -1, so the limit does not exist.

6. Since $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$ and $\left|\frac{2}{3}\right| < 1$, we have $\lim_{n\to\infty} \left(\frac{2^n}{3^n}\right) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

7. As n increases, the term $4n$ is much larger than 3 and $7n$ is much larger than 5. Thus dividing the numerator and denominator by n and using the fact that $\lim_{n\to\infty} 1/n = 0$, we have

$$\lim_{n\to\infty} \frac{3+4n}{5+7n} = \lim_{n\to\infty} \frac{(3/n)+4}{(5/n)+7} = \frac{4}{7}.$$

8. As n increases, the term $2n$ is much larger in magnitude than $(-1)^n 5$ and the term $4n$ is much larger in magnitude than $(-1)^n 3$. Thus dividing the numerator and denominator by n and using the fact that $\lim_{n\to\infty} 1/n = 0$, we have

$$\lim_{n\to\infty} \frac{2n+(-1)^n 5}{4n-(-1)^n 3} = \lim_{n\to\infty} \frac{2+(-1)^n 5/n}{4-(-1)^n 3/n} = \frac{1}{2}.$$

9. We use the integral test to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_1^\infty \frac{1}{x^3} dx$ converges or diverges:

$$\int_1^\infty \frac{1}{x^3} dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3} dx = \lim_{b\to\infty} \left.\frac{-1}{2x^2}\right|_1^b = \lim_{b\to\infty} \left(\frac{-1}{2b^2} + \frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\int_1^\infty \frac{1}{x^3} dx$ converges, we conclude from the integral test that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

10. We use the integral test to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_1^\infty \frac{x}{x^2+1} dx$ converges or diverges:

$$\int_1^\infty \frac{x}{x^2+1} dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1} dx = \lim_{b\to\infty} \left.\frac{1}{2}\ln(x^2+1)\right|_1^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2\right) = \infty.$$

Since the integral $\int_1^\infty \frac{x}{x^2+1} dx$ diverges, we conclude from the integral test that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

11. We use the integral test to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_0^\infty xe^{-x^2}\,dx$ converges or diverges:

$$\int_0^\infty xe^{-x^2}dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}dx = \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^b = \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

Since the integral $\int_0^\infty xe^{-x^2}\,dx$ converges, we conclude from the integral test that the series $\sum_{n=0}^{\infty} ne^{-n^2}$ converges.

12. We use the integral test to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges or diverges:

$$\int_2^\infty \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^2}dx = \lim_{b\to\infty}\frac{-1}{\ln x}\Big|_2^b = \lim_{b\to\infty}\left(\frac{-1}{\ln b}+\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the integral $\int_2^\infty \frac{1}{x(\ln x)^2}dx$ converges, we conclude from the integral test that the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ converges.

Problems

13. The series $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is a convergent geometric series, but $\sum_{n=1}^{\infty}\frac{1}{n}$ is the divergent harmonic series.
If $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n+\frac{1}{n}\right)$ converged, then $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n+\frac{1}{n}\right)-\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n}$ would converge by Theorem 9.2. Therefore $\sum_{n=1}^{\infty}\left(\left(\frac{3}{4}\right)^n+\frac{1}{n}\right)$ diverges.

14. Writing $a_n = n/(n+1)$, we have $\lim_{n\to\infty} a_n = 1$ so the series diverges by Property 3 of Theorem 9.2.

15. Using the integral test, we compare the series with

$$\int_0^\infty \frac{3}{x+2}\,dx = \lim_{b\to\infty}\int_0^b \frac{3}{x+2}\,dx = 3\ln|x+2|\Big|_0^b.$$

Since $\ln(b+2)$ is unbounded as $b\to\infty$, the integral diverges and therefore so does the series.

16. The series can be written as

$$\sum_{n=1}^{\infty}\frac{n+2^n}{n2^n} = \sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n}\right).$$

If this series converges, then $\sum_{n=1}^{\infty}\left(\frac{1}{2^n}+\frac{1}{n}\right)-\sum_{n=1}^{\infty}\frac{1}{2n} = \sum_{n=1}^{\infty}\frac{1}{n}$ would converge by Theorem 9.2. Since this is the harmonic series, which diverges, then the series $\sum_{n=1}^{\infty}\frac{n+2^n}{n}$ diverges.

17. We use the integral test and calculate the corresponding improper integral, $\int_1^\infty 3/(2x-1)^2dx$:

$$\int_1^\infty \frac{3\,dx}{(2x-1)^2} = \lim_{b\to\infty}\int_1^b \frac{3\,dx}{(2x-1)^2} = \lim_{b\to\infty}\frac{-3/2}{(2x-1)}\Big|_1^b = \lim_{b\to\infty}\left(\frac{-3/2}{(2b-1)}+\frac{3}{2}\right) = \frac{3}{2}.$$

Since the integral converges, the series $\sum_{n=1}^{\infty}\frac{3}{(2n-1)^2}$ converges.

18. We use the integral test and calculate the corresponding improper integral, $\int_0^\infty 2/\sqrt{2+x}\,dx$:

$$\int_0^\infty \frac{2}{\sqrt{2+x}}\,dx = \lim_{b\to\infty}\int_0^b \frac{2\,dx}{\sqrt{2+x}} = \lim_{b\to\infty} 4(2+x)^{1/2}\Big|_0^b = \lim_{b\to\infty} 4\left((2+b)^{1/2}-2^{1/2}\right).$$

Since the limit does not exist (it is ∞), the integral diverges, so the series $\sum_{n=1}^{\infty}\frac{2}{\sqrt{2+n}}$ diverges.

19. Let $a_n = (\ln n)/n$ and $f(x) = (\ln x)/x$. We use the integral test and consider the improper integral

$$\int_c^\infty \frac{\ln x}{x} dx.$$

Since

$$\int_c^R \frac{\ln x}{x} dx = \frac{1}{2}(\ln x)^2 \Big|_c^R = \frac{1}{2}\left((\ln R)^2 - (\ln c)^2\right),$$

and $\ln R$ grows without bound as $R \to \infty$, the integral diverges. Therefore, the integral test tells us that the series, $\sum_{n=1}^\infty \frac{\ln n}{n}$, also diverges.

20. We use the integral test and calculate the corresponding improper integral, $\int_3^\infty (x+1)/(x^2+2x+2)\,dx$:

$$\int_3^\infty \frac{x+1}{x^2+2x+2}\,dx = \lim_{b\to\infty}\int_3^b \frac{x+1}{x^2+2x+2}\,dx = \lim_{b\to\infty}\frac{1}{2}\ln|x^2+2x+2|\Big|_3^b = \lim_{b\to\infty}\frac{1}{2}(\ln(b^2+2b+2) - \ln 17).$$

Since the limit does not exist (it is ∞), the integral diverges, so the series $\sum_{n=3}^\infty \frac{n+1}{n^2+2n+2}$ diverges.

21. Using left-hand sums for the integral of $f(x) = 1/(4x-3)$ over the interval $1 \le x \le n+1$ with uniform subdivisions of length 1 gives a lower bound on the partial sum:

$$S_n = 1 + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{4n-3} > \int_1^{n+1} \frac{dx}{4x-3} = \frac{1}{4}\ln(4x-3)\Big|_1^{n+1} = \frac{1}{4}\ln(4n+1).$$

Since $\ln(4n+1)$ increases without bound as $n \to \infty$, the partial sums of the series are unbounded. Thus, this is not a convergent series.

22. Using right-hand sums for the integral of $f(x) = x^{-3/2}$ over the interval $1 \le x \le n$ with uniform subdivisions of length 1 gives:

$$\frac{1}{2^{3/2}} + \cdots + \frac{1}{n^{3/2}} < \int_1^n x^{-3/2}\,dx = -2(n^{-1/2} - 1).$$

Adding 1 to both sides gives an upper bound on the partial sum

$$S_n = 1 + \frac{1}{2^{3/2}} + \cdots + \frac{1}{n^{3/2}} < 1 - 2(n^{-1/2} - 1)$$

Thus, as $n \to \infty$, the sequence of partial sums is bounded. Each successive partial sum is obtained from the previous one by adding one more term in the series. Since all the terms are positive, the sequence of partial sums is increasing. Hence the series converges.

23. **(a)** We compare $\sum_{n=1}^\infty 1/n^p$ with the integral $\int_1^\infty (1/x^p)dx$. For $p \ne 1$, we have

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}dx = \lim_{b\to\infty}\frac{x^{-p+1}}{-p+1}\Big|_1^b = \lim_{b\to\infty}\frac{b^{-p+1}-1}{-p+1}.$$

If $p > 1$, the power of b is negative, so this limit exists. Thus the integral converges, so the series converges.

(b) If $p < 1$, then the power of b is positive and the limit does not exist. Thus, the integral diverges, so the series diverges.

We have to look at the case $p = 1$ separately, since the form of the antiderivative is different in that case. If $p = 1$, we compare $\sum_{n=1}^\infty 1/n$ with $\int_1^\infty (1/x)dx$. Since

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}\ln|x|\Big|_1^b = \lim_{b\to\infty}\ln b,$$

and since $\lim_{b\to\infty}\ln b$ does not exist, the integral diverges, so the series diverges. Combining these results shows that $\sum_{n=1}^\infty 1/n^p$ diverges if $p \le 1$.

24. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n+5}{4^n}=\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n+\sum_{n=0}^{\infty}\frac{5}{4^n},$ a sum of two geometric series.

$$\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n=\frac{1}{1-\frac{3}{4}}=4$$

$$\sum_{n=0}^{\infty}\frac{5}{4^n}=\frac{5}{1-\frac{1}{4}}=\frac{20}{3}$$

so $\displaystyle\sum_{n=0}^{\infty}\frac{3^n+5}{4^n}=4+\frac{20}{3}=\frac{32}{3}.$

25. We want to define $\lim_{n\to\infty} S_n = L$ so that S_n is as close to L as we please for all sufficiently large n. Thus, the definition says that for any positive ϵ, there is a value N such that

$$|S_n - L| < \epsilon \quad \text{whenever} \quad n \geq N.$$

26. Let S_n be the n^{th} partial sum for $\sum a_n$ and let T_n be the n^{th} partial sum for $\sum b_n$. Then the n^{th} partial sums for $\sum(a_n+b_n)$, $\sum(a_n-b_n)$, and $\sum ka_n$ are S_n+T_n, S_n-T_n, and kS_n, respectively. To show that these series converge, we have to show that the limits of their partial sums exist. By the properties of limits,

$$\lim_{n\to\infty}(S_n+T_n)=\lim_{n\to\infty}S_n+\lim_{n\to\infty}T_n$$
$$\lim_{n\to\infty}(S_n-T_n)=\lim_{n\to\infty}S_n-\lim_{n\to\infty}T_n$$
$$\lim_{n\to\infty}kS_n=k\lim_{n\to\infty}S_n.$$

This proves that the limits of the partial sums exist, so the series converge.

27. Let S_n be the n-th partial sum for $\sum a_n$ and let T_n be the n-th partial sum for $\sum b_n$. Suppose that $S_N = T_N + k$. Since $a_n = b_n$ for $n \geq N$, we have $S_n = T_n + k$ for $n \geq N$. Hence if S_n converges to a limit, so does T_n, and vice versa.

28. We have $a_n = S_n - S_{n-1}$. If $\sum a_n$ converges, then $S = \lim_{n\to\infty} S_n$ exists. Hence $\lim_{n\to\infty} S_{n-1}$ exists and is equal to S also. Thus

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}(S_n-S_{n-1})=\lim_{n\to\infty}S_n-\lim_{n\to\infty}S_{n-1}=S-S=0.$$

29. From Property 1 in Theorem 9.2, we know that if $\sum a_n$ converges, then so does $\sum ka_n$.

Now suppose that $\sum a_n$ diverges and $\sum ka_n$ converges for $k \neq 0$. Thus using Property 1 and replacing $\sum a_n$ by $\sum ka_n$, we know that the following series converges:

$$\sum\frac{1}{k}(ka_n)=\sum a_n.$$

Thus, we have arrived at a contradiction, which means our original assumption, that $\displaystyle\sum_{n=1}^{\infty}ka_n$ converged, must be wrong.

30. **(a)** Show that the sum of each group of fractions is more than $1/2$.
(b) Explain why this shows that the harmonic series does not converge.

(a) Notice that

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$
$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$$
$$\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}=\frac{8}{16}=\frac{1}{2}.$$

In the same way, we can see that the sum of the fractions in each grouping is greater than $1/2$.

(b) Since the sum of the first n groups is greater than $n/2$, it follows that the harmonic series does not converge.

31. We want to estimate $\displaystyle\sum_{k=1}^{100{,}000}\frac{1}{k}$ using left and right Riemann sum approximations to $f(x) = 1/x$ on the interval $1 \le x \le 100{,}000$. Figure 9.2 shows a left Riemann sum approximation with 99,999 terms. Since $f(x)$ is decreasing, the left Riemann sum overestimates the area under the curve. Figure 9.2 shows that the first term in the sum is $f(1)\cdot 1$ and the last is $f(99{,}999)\cdot 1$, so we have

$$\int_1^{100{,}000}\frac{1}{x}\,dx < \text{ LHS } = f(1)\cdot 1 + f(2)\cdot 1 + \cdots + f(99{,}999)\cdot 1.$$

Since $f(x) = 1/x$, the left Riemann sum is

$$\text{LHS } = \frac{1}{1}\cdot 1 + \frac{1}{2}\cdot 1 + \cdots + \frac{1}{99{,}999}\cdot 1 = \sum_{k=1}^{99{,}999}\frac{1}{k},$$

so

$$\int_1^{100{,}000}\frac{1}{x}\,dx < \sum_{k=1}^{99{,}999}\frac{1}{k}.$$

Since we want the sum to go $k = 100{,}000$ rather than $k = 99{,}999$, we add $1/100{,}000$ to both sides:

$$\int_1^{100{,}000}\frac{1}{x}\,dx + \frac{1}{100{,}000} < \sum_{k=1}^{99{,}999}\frac{1}{k} + \frac{1}{100{,}000} = \sum_{k=1}^{100{,}000}\frac{1}{k}.$$

The left Riemann sum has therefore given us an underestimate for our sum. We now use the right Riemann sum in Figure 9.3 to get an overestimate for our sum.

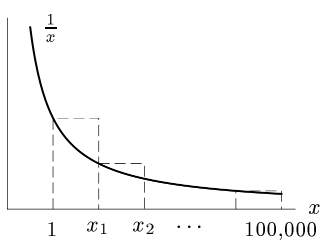


Figure 9.2

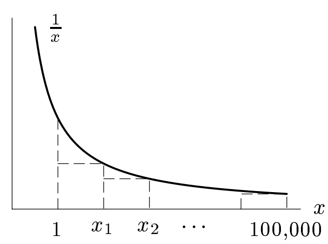


Figure 9.3

The right Riemann sum again has 99,999 terms, but this time the sum underestimates the area under the curve. Figure 9.3 shows that the first rectangle has area $f(2)\cdot 1$ and the last $f(100{,}000)\cdot 1$, so we have

$$\text{RHS } = f(2)\cdot 1 + f(3)\cdot 1 + \cdots + f(100{,}000)\cdot 1 < \int_1^{100{,}000}\frac{1}{x}\,dx.$$

Since $f(x) = 1/x$, the right Riemann sum is

$$\text{RHS } = \frac{1}{2}\cdot 1 + \frac{1}{3}\cdot 1 + \cdots + \frac{1}{100{,}000}\cdot 1 = \sum_{k=2}^{100{,}000}\frac{1}{k}.$$

So

$$\sum_{k=2}^{100{,}000}\frac{1}{k} < \int_1^{100{,}000}\frac{1}{x}\,dx.$$

Since we want the sum to start at $k = 1$, we add 1 to both sides:

$$\sum_{k=1}^{100{,}000}\frac{1}{k} = \frac{1}{1} + \sum_{k=2}^{100{,}000}\frac{1}{k} < 1 + \int_1^{100{,}000}\frac{1}{x}\,dx.$$

Putting these under- and overestimates together, we have

$$\int_1^{100,000} \frac{1}{x}\,dx + \frac{1}{100{,}000} < \sum_{k=1}^{100,000} \frac{1}{k} < 1 + \int_1^{100,000} \frac{1}{x}\,dx.$$

Since $\int_1^{100,000} \frac{1}{x}\,dx = \ln 100{,}000 - \ln 1 = 11.513$, we have

$$11.513 < \sum_{k=1}^{100,000} \frac{1}{k} < 12.513.$$

Therefore we have $\sum_{k=1}^{100,000} \frac{1}{k} \approx 12.$

32. Using a right-hand sum, we have

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} < \int_1^n \frac{dx}{x} = \ln n.$$

If a computer could add a million terms in one second, then it could add

$$60\frac{\text{sec}}{\text{min}} \cdot 60\frac{\text{min}}{\text{hour}} \cdot 24\frac{\text{hour}}{\text{day}} \cdot 365\frac{\text{days}}{\text{year}} \cdot 1 \text{ million } \frac{\text{terms}}{\text{sec}}$$

terms per year. Thus,

$$1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} < 1 + \ln n = 1 + \ln(60 \cdot 60 \cdot 24 \cdot 365 \cdot 10^6) \approx 32.082 < 33.$$

So the sum after one year is about 32.

33. (a) Let N an integer with $N \geq c$. Consider the series $\sum_{i=N+1}^{\infty} a_i$. The partial sums of this series are increasing because all the terms in the series are positive. We show the partial sums are bounded using the right-hand sum in Figure 9.4. We see that for each positive integer k

$$f(N+1) + f(N+2) + \cdots + f(N+k) \leq \int_N^{N+k} f(x)\,dx.$$

Since $f(n) = a_n$ for all n, and $c \leq N$, we have

$$a_{N+1} + a_{N+2} + \cdots + a_{N+k} \leq \int_c^{N+k} f(x)\,dx.$$

Since $f(x)$ is a positive function, $\int_c^{N+k} f(x)\,dx \leq \int_c^b f(x)\,dx$ for all $b \geq N + k$. Since f is positive and $\int_c^{\infty} f(x)\,dx$ is convergent, $\int_c^{N+k} f(x)\,dx < \int_c^{\infty} f(x)\,dx$, so we have

$$a_{N+1} + a_{N+2} + \cdots + a_{N+k} \leq \int_c^{\infty} f(x)\,dx \quad \text{for all } k.$$

Thus, the partial sums of the series $\sum_{i=N+1}^{\infty} a_i$ are all bounded by the same number, so this series converges. Now use Theorem 9.2, property 2, to conclude that $\sum_{i=1}^{\infty} a_i$ converges.

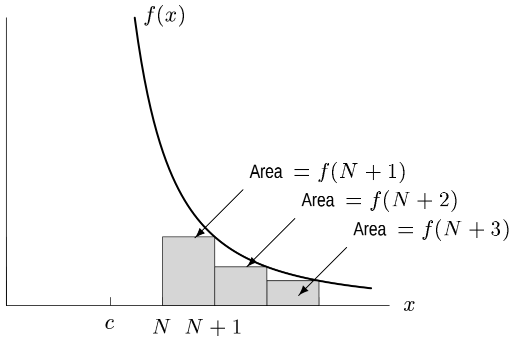


Figure 9.4

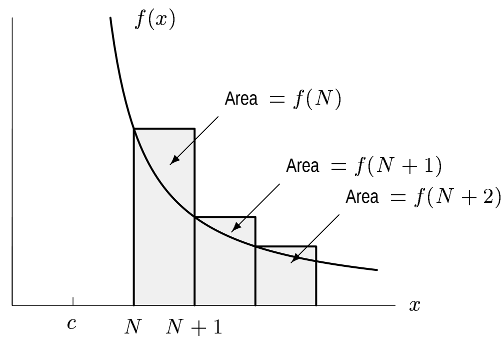


Figure 9.5

(b) We now suppose $\displaystyle\int_c^\infty f(x)\,dx$ diverges. In Figure 9.5 we see that for each positive integer k

$$\int_N^{N+k+1} f(x)\,dx \le f(N) + f(N+1) + \cdots + f(N+k).$$

Since $f(n) = a_n$ for all n, we have

$$\int_N^{N+k+1} f(x)\,dx \le a_N + a_{N+1} + \cdots + a_{N+k}.$$

Since $f(x)$ is defined for all $x \ge c$, if $\int_c^\infty f(x)\,dx$ is divergent, then $\int_N^\infty f(x)\,dx$ is divergent. So as $k \to \infty$, the the integral $\int_N^{N+k+1} f(x)\,dx$ diverges, so the partial sums of the series $\displaystyle\sum_{i=N}^{\infty} a_i$ diverge. Thus, the series $\displaystyle\sum_{i=1}^{\infty} a_i$ diverges.

More precisely, suppose the series converged. Then the partial sums would be bounded. (The partial sums would be less than the sum of the series, since all the terms in the series are positive.) But that would imply that the integral converged, by Theorem 9.1 on Convergence of Increasing Bounded Sequences. This contradicts the assumption that $\int_N^\infty f(x)\,dx$ is divergent.

Solutions for Section 9.3

Exercises

1. Let $a_n = 1/(n^2+2)$. Since $n^2+2 > n^2$, we have $1/(n^2+2) < 1/n^2$, so

$$0 < a_n < \frac{1}{n^2}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ also converges.

2. Let $a_n = 1/(n-3)$, for $n \ge 4$. Since $n-3 < n$, we have $1/(n-3) > 1/n$, so

$$a_n > \frac{1}{n}.$$

The harmonic series $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n}$ diverges, so the comparison test tells us that the series $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n-3}$ also diverges.

3. Let $a_n = e^{-n}/n^2$. Since $e^{-n} < 1$, for $n \geq 1$,we have $\dfrac{e^{-n}}{n^2} < \dfrac{1}{n^2}$, so
$$0 < a_n < \frac{1}{n^2}.$$
The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ also converges.

4. Since $n^3 \geq n^2$, we have $1/n^3 \leq 1/n^2$. Hence the series converges by comparison with $1/n^2$, which we showed converges on page 415 of the text.

5. Since $\ln n \leq n$ for $n \geq 2$, we have $1/\ln n \geq 1/n$, so the series diverges by comparison with the harmonic series, $\sum 1/n$.

6. Let $a_n = 1/(3^n + 1)$. Since $3^n + 1 > 3^n$, we have $1/(3^n + 1) < 1/3^n = \left(\dfrac{1}{3}\right)^n$, so
$$0 < a_n < \left(\frac{1}{3}\right)^n.$$
Thus we can compare the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 1}$ with the geometric series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$. This geometric series converges since $|1/3| < 1$, so the comparison test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 1}$ also converges.

7. Let $a_n = 1/(n^4 + e^n)$. Since $n^4 + e^n > n^4$, we have
$$\frac{1}{n^4 + e^n} < \frac{1}{n^4},$$
so
$$0 < a_n < \frac{1}{n^4}.$$
Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges, the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4 + e^n}$ also converges.

8. Let $a_n = 2^{-n}\dfrac{(n+1)}{(n+2)} = \left(\dfrac{n+1}{n+2}\right)\left(\dfrac{1}{2^n}\right)$. Since $\dfrac{(n+1)}{(n+2)} < 1$ and $\dfrac{1}{2^n} = \left(\dfrac{1}{2}\right)^n$, we have
$$0 < a_n < \left(\frac{1}{2}\right)^n,$$
so that we can compare the series $\displaystyle\sum_{n=1}^{\infty} 2^{-n}\frac{(n+1)}{(n+2)}$ with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. The comparison test tells us that
$$\sum_{n=1}^{\infty} 2^{-n}\frac{(n+1)}{(n+2)}$$
also converges.

9. Let $a_n = n^2/(n^4 + 1)$. Since $n^4 + 1 > n^4$, we have $\dfrac{1}{n^4 + 1} < \dfrac{1}{n^4}$, so
$$a_n = \frac{n^2}{n^4 + 1} < \frac{n^2}{n^4} = \frac{1}{n^2},$$
therefore
$$0 < a_n < \frac{1}{n^2}.$$
Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1}$ converges also.

10. Let $a_n = (2^n+1)/(n2^n-1)$. Since $n2^n - 1 < n2^n + n = n(2^n+1)$, we have

$$\frac{2^n+1}{n2^n-1} > \frac{2^n+1}{n(2^n+1)} = \frac{1}{n}.$$

Therefore, we can compare the series $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. The comparison test tells us that $\sum_{n=1}^{\infty} \frac{2^n+1}{n2^n-1}$ also diverges.

11. We know that $|\sin n| < 1$, so

$$\left|\frac{n \sin n}{n^3+1}\right| \le \frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, comparison gives that $\sum_{n=1}^{\infty} \left|\frac{n \sin n}{n^3+1}\right|$ converges. Thus, by Theorem 9.5, $\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+1}$ converges.

12. Since $a_n = 1/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 1/(2n+2)!$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n)!}} = \frac{(2n)!}{(2n+2)!} = \frac{(2n)!}{(2n+2)(2n+1)(2n)!} = \frac{1}{(2n+2)(2n+1)},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0.$$

Since $L = 0$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges.

13. Since $a_n = (n!)^2/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = ((n+1)!)^2/(2n+2)!$. Thus,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2}.$$

However, since $(n+1)! = (n+1)n!$ and $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2(n!)^2(2n)!}{(2n+2)(2n+1)(2n)!(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n+1}{4n+2},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{4}.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges.

14. Since $a_n = 2^n/(n^3+1)$, replacing n by $n+1$ gives $a_{n+1} = 2^{n+1}/((n+1)^3+1)$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{2^{n+1}}{(n+1)^3+1}}{\frac{2^n}{n^3+1}} = \frac{2^{n+1}}{(n+1)^3+1} \cdot \frac{n^3+1}{2^n} = 2\frac{n^3+1}{(n+1)^3+1},$$

so

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2.$$

Since $L > 1$ the ratio test tells us that the series $\sum_{n=0}^{\infty} \frac{2^n}{n^3+1}$ diverges.

15. Since $a_n = 1/(ne^n)$, replacing n by $n+1$ gives $a_{n+1} = 1/(n+1)e^{n+1}$. Thus

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)e^{n+1}}}{\frac{1}{ne^n}} = \frac{ne^n}{(n+1)e^{n+1}} = \left(\frac{n}{n+1}\right)\frac{1}{e}.$$

Therefore

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \frac{1}{e} < 1.$$

Since $L < 1$, the ratio test tells us that $\sum_{n=1}^{\infty}\frac{1}{ne^n}$ converges.

16. Let $a_n = 1/(2n+1)$. Then replacing n by $n+1$ gives $a_{n+1} = 1/(2n+3)$. Since $2n+3 > 2n+1$, we have

$$0 < a_{n+1} = \frac{1}{2n+3} < \frac{1}{2n+1} = a_n.$$

We also have $\lim_{n\to\infty} a_n = 0$. Therefore, the alternating series test tells us that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n+1}$ converges.

17. Let $a_n = 1/\sqrt{n}$. Then replacing n by $n+1$ we have $a_{n+1} = 1/\sqrt{n+1}$. Since $\sqrt{n+1} > \sqrt{n}$, we have $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, hence $a_{n+1} < a_n$. In addition, $\lim_{n\to\infty} a_n = 0$ so $\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

Problems

18. The partial sums look like: $S_1 = 1$, $S_2 = 0.9$, $S_3 = 0.91$, $S_4 = 0.909$, $S_5 = 0.9091$, $S_6 = 0.90909$. The series appears to be converging to $0.909090\ldots$ or $10/11$.

Since $a_n = 10^{-k}$ is positive and decreasing and $\lim_{n\to\infty} 10^{-n} = 0$, the alternating series test confirms the convergence of the series.

19. The partial sums are $S_1 = 1$, $S_2 = -1$, $S_3 = 2$, $S_{10} = -5$, $S_{11} = 6$, $S_{100} = -50$, $S_{101} = 51$, $S_{1000} = -500$, $S_{1001} = 501$, which appear to be oscillating further and further from 0. This series does not converge.

20. The partial sums look like: $S_1 = 1$, $S_2 = 0$, $S_3 = 0.5$, $S_4 = 0.3333$, $S_5 = 0.375$, $S_{10} = 0.3679$, $S_{20} = 0.3679$, and higher partial sums agree with these first 4 decimal places. The series appears to be converging to about 0.3679.

Since $a_n = 1/n!$ is positive and decreasing and $\lim_{n\to\infty} 1/n! = 0$, the alternating series test confirms the convergence of this series.

21. The first few terms of the series may be written

$$1 + e^{-1} + e^{-2} + e^{-3} + \cdots;$$

this is a geometric series with $a = 1$ and $x = e^{-1} = 1/e$. Since $|x| < 1$, the geometric series converges to $S = \frac{1}{1-x} = \frac{1}{1-e^{-1}} = \frac{e}{e-1}$.

22. We use the ratio test and calculate

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(0.1)^{n+1}/(n+1)!}{(0.1)^n/n!} = \lim_{n\to\infty}\frac{0.1}{n+1} = 0.$$

Since the limit is less than 1, the series converges.

23. We use the ratio test and calculate

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{n!/(n+1)^2}{(n-1)!/n^2} = \lim_{n\to\infty}\left(\frac{n!}{(n-1)!}\cdot\frac{n^2}{(n+1)^2}\right) = \lim_{n\to\infty}\left(n\cdot\frac{n^2}{(n+1)^2}\right).$$

Since the limit does not exist (it is ∞), the series diverges.

24. The first few terms of the series may be written

$$e + e^2 + e^3 + \cdots = e + e\cdot e + e\cdot e^2 + \cdots;$$

this is a geometric series with $a = e$ and $x = e$. Since $|x| > 1$, this geometric series diverges.

25. Let $a_n = 1/\sqrt{3n-1}$. Then replacing n by $n+1$ gives $a_{n+1} = 1/\sqrt{3(n+1)-1}$. Since

$$\sqrt{3(n+1)-1} > \sqrt{3n-1},$$

we have

$$a_{n+1} < a_n.$$

In addition, $\lim_{n\to\infty} a_n = 0$ so the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{3n-1}}$ converges.

26. Since the exponential, 2^n, grows faster than the power, n^2, the terms are growing in size. Thus, $\displaystyle\lim_{n\to\infty} a_n \neq 0$. We conclude that this series diverges.

27. Let $a_n = n(n+1)/\sqrt{n^3+2n^2}$. Since $n^3 + 2n^2 = n^2(n+2)$, we have

$$a_n = \frac{n(n+1)}{n\sqrt{n+2}} = \frac{n+1}{\sqrt{n+2}}$$

so a_n grows without bound as $n \to \infty$, therefore the series $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{n^3+2n^2}}$ diverges.

28. Let $a_n = 1/\sqrt{n^2(n+2)}$. Since $n^2(n+2) = n^3 + 2n^2 > n^3$, we have

$$0 < a_n < \frac{1}{n^{3/2}}.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, the comparison test tells us that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2(n+2)}}$$

also converges.

29. **(a)** Assume that n is even. Then

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{n} &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n}. \end{aligned}$$

(b) The given series $\dfrac{1}{1\cdot 2} + \dfrac{1}{3\cdot 4} + \dfrac{1}{5\cdot 6} + \cdots$ is term by term less than the series

$$\frac{1}{1\cdot 1} + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \cdots = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots.$$

Since this second series, $\sum 1/n^2$, converges by the integral test, the first series converges.

(c) By parts (a) and (b), the sequence of partial sums for even n converges. The partial sum for odd n equals $1/n$ plus the partial sum for even $n-1$. Thus the partial sums for odd n approach the partial sums for even n, as $n \to \infty$. Therefore the sequence of all partial sums converges, and hence the series converges.

30. The argument is false. Property 1 of Theorem 9.2 only applies to convergent series. Furthermore, since $n(n+1) > n^2$ we can compare $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ with the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ and deduce that it converges.

31. Suppose we let $c_n = (-1)^n a_n$. (We have just given the terms of the series $\sum (-1)^n a_n$ a new name.) Then

$$|c_n| = |(-1)^n a_n| = |a_n|.$$

Thus $\sum |c_n|$ converges, and by Theorem 9.5,

$$\sum c_n = \sum (-1)^n a_n \quad \text{converges.}$$

32. **(a)** Since $b_n = a_n$ if a_n is positive or zero and $b_n = 0$ if a_n is negative, we have

$$0 \le b_n \le |a_n| \qquad \text{for all } n.$$

Thus, by the comparison test, $\sum b_n$ converges.

(b) Since $c_n = 0$ if a_n is positive or zero and $c_n = -a_n$ if a_n is negative, c_n is never negative and

$$0 \le c_n \le |a_n| \qquad \text{for all } n.$$

Thus, by the comparison test, $\sum c_n$ converges.

(c) The b_ns are the positive terms in $\sum a_n$, and the c_ns are the negative terms. For each n, either b_n or c_n is 0, and

$$a_n = b_n - c_n.$$

Thus, $\sum a_n = \sum b_n - \sum c_n$ is the difference of convergent sequences and hence converges.

Solutions for Section 9.4

Exercises

1. Yes.

2. No, because it contains negative powers of x.

3. No, each term is a power of a different quantity.

4. Yes. It's a polynomial, or a series with all coefficients beyond the 7th being zero.

5. The general term can be written as $\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\cdot n!}x^n$ for $n \ge 1$.

6. The general term can be written as $\dfrac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n$ for $n \ge 1$.

7. The general term can be written as $(-1)^k(x-1)^{2k}/(2k)!$ for $k \ge 0$.

8. The general term can be written as $(-1)^k(x-1)^{2k+3}/(2k)!$ for $k \ge 0$.

9. The general term can be written as $\dfrac{(x-a)^n}{2^{n-1}\cdot n!}$ for $n \ge 1$.

10. The general term can be written as $\dfrac{(k+2)(x+5)^{2k+3}}{k!}$ for $k \ge 0$.

11. This series may be written as

$$1 + 5x + 25x^2 + \cdots$$

so $C_n = 5^n$. Using the ratio test, with $a_n = 5^n x^n$, we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|\lim_{n\to\infty}\frac{|C_{n+1}|}{|C_n|} = |x|\lim_{n\to\infty}\frac{5^{n+1}}{5^n} = 5|x|.$$

Thus the radius of convergence is $R = 1/5$.

12. Since $C_n = n^3$, replacing n by $n+1$ gives $C_{n+1} = (n+1)^3$. Using the ratio test, with $a_n = n^3x^n$, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(n+1)^3}{n^3} = |x|\left(\frac{n+1}{n}\right)^3.$$

We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|.$$

Thus the radius of convergence is $R = 1$.

13. Since $C_n = (n+1)/(2^n+n)$, replacing n by $n+1$ gives $C_{n+1} = (n+2)/(2^{n+1}+n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{(n+2)/(2^{n+1}+n+1)}{(n+1)/(2^n+n)} = |x|\frac{n+2}{2^{n+1}+n+1}\cdot\frac{2^n+n}{n+1} = |x|\frac{n+2}{n+1}\cdot\frac{2^n+n}{2^{n+1}+n+1}.$$

Since

$$\lim_{n\to\infty}\frac{n+2}{n+1} = 1$$

and

$$\lim_{n\to\infty}\left(\frac{2^n+n}{2^{n+1}+n+1}\right) = \frac{1}{2}\lim_{n\to\infty}\left(\frac{2^n+n}{2^n+(n+1)/2}\right) = \frac{1}{2},$$

because 2^n dominates n as $n\to\infty$, we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \frac{1}{2}|x|.$$

Thus the radius of convergence is $R = 2$.

14. Since $C_n = 2^n/n$, replacing n by $n+1$ gives $C_{n+1} = 2^{n+1}/(n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x-1|\frac{|C_{n+1}|}{|C_n|} = |x-1|\frac{2^{n+1}/(n+1)}{2^n/n} = |x-1|\frac{2^{n+1}}{(n+1)}\cdot\frac{n}{2^n} = 2|x-1|\left(\frac{n}{n+1}\right),$$

so

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x-1|.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

15. To find R, we consider the following limit, where the coefficient of the n^{th} term is given by $C_n = n^2$.

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}}{n^2x^n}\right| = \lim_{n\to\infty}|x|\frac{n^2+2n+1}{n^2}$$
$$= |x|\lim_{n\to\infty}\left(\frac{1+(2/n)+(1/n^2)}{1}\right) = |x|.$$

Thus, the radius of convergence is $R = 1$.

16. The coefficient of the n^{th} term is $C_n = (-1)^{n+1}/n^2$. Now consider the ratio

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n^2x^{n+1}}{(n+1)^2x^n}\right| \to |x| \quad \text{as} \quad n\to\infty.$$

Thus, the radius of convergence is $R = 1$.

17. Here the coefficient of the n^{th} term is $C_n = (2^n/n!)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2^{n+1}/(n+1)!)x^{n+1}}{(2^n/n!)x^n}\right| = \frac{2|x|}{n+1} \to 0 \text{ as } n\to\infty.$$

Thus, the radius of convergence is $R = \infty$, and the series converges for all x.

18. Here the coefficient of the n^{th} term is $C_n = n/(2n+1)$. Now we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)/(2n+3))x^{n+1}}{(n/(2n+1))x^n}\right| = \frac{(n+1)(2n+1)}{n(2n+3)}|x| \to |x| \text{ as } n\to\infty.$$

Thus, by the ratio test, the radius of convergence is $R = 1$.

19. Here $C_n = (2n)!/(n!)^2$. We have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1))!/((n+1)!)^2x^{n+1}}{(2n)!/(n!)^2x^n}\right| = \frac{(2(n+1))!}{(2n)!}\cdot\frac{(n!)^2}{((n+1)!)^2}|x|$$
$$= \frac{(2n+2)(2n+1)|x|}{(n+1)^2} \to 4|x| \text{ as } n\to\infty.$$

Thus, the radius of convergence is $R = 1/4$.

20. Here the coefficient of the n^{th} term is $C_n = (2n+1)/n$. Applying the ratio test, we consider:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((2n+3)/(n+1))x^{n+1}}{((2n+1)/n)x^n}\right| = |x|\frac{2n+3}{2n+1}\cdot\frac{n}{n+1} \to |x| \text{ as } n \to \infty.$$

Thus, the radius of convergence is $R = 1$.

21. We write the series as

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots,$$

so

$$a_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}.$$

Replacing n by $n+1$, we have

$$a_{n+1} = (-1)^{n+1-1}\frac{x^{2(n+1)-1}}{2(n+1)-1} = (-1)^n\frac{x^{2n+1}}{2n+1}.$$

Thus

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(-1)^n x^{2n+1}}{2n+1}\right| \cdot \left|\frac{2n-1}{(-1)^{n-1}x^{2n-1}}\right| = \frac{2n-1}{2n+1}x^2,$$

so

$$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{2n-1}{2n+1}x^2 = x^2.$$

By the ratio test, this series converges if $L < 1$, that is, if $x^2 < 1$, so $R = 1$.

Problems

22. **(a)** The general term of the series is x^n/n if n is odd and $-x^n/n$ if n is even, so $C_n = (-1)^{n-1}/n$, and we can use the ratio test. We have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|\lim_{n\to\infty}\frac{|(-1)^n/(n+1)|}{|(-1)^{n-1}/n|} = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|.$$

Therefore the radius of convergence is $R = 1$. This tells us that the power series converges for $|x| < 1$ and does not converge for $|x| > 1$. Notice that the radius of convergence does not tell us what happens at the endpoints, $x = \pm 1$.

(b) The endpoints of the interval of convergence are $x = \pm 1$. At $x = 1$, we have the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots$$

This is an alternating series with $a_n = 1/n$, so by the alternating series test, it converges. At $x = -1$, we have the series

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots - \frac{1}{n} - \cdots$$

This is the negative of the harmonic series, so it does not converge. Therefore the right endpoint is included, and the left endpoint is not included in the interval of convergence, which is $-1 < x \leq 1$.

23. Let $C_n = 2^n/n$. Then replacing n by $n+1$ gives $C_{n+1} = 2^{n+1}/(n+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{2^{n+1}/(n+1)}{2^n/n} = |x|\frac{2^{n+1}}{n+1}\cdot\frac{n}{2^n} = 2|x|\left(\frac{n}{n+1}\right).$$

Thus

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 2|x|.$$

The radius of convergence is $R = 1/2$.

For $x = 1/2$ the series becomes the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges.

For $x = -1/2$ the series becomes the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges. See Example 7 on page 421.

24. The coefficient of the n^{th} term of the binomial power series is given by

$$C_n = \frac{p(p-1)(p-2)\cdots(p-(n-1))}{n!}.$$

To apply the ratio test, consider

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= |x|\left|\frac{p(p-1)(p-2)\cdots(p-(n-1))(p-n)/(n+1)!}{p(p-1)(p-2)\cdots(p-(n-1))/n!}\right| \\ &= |x|\left|\frac{p-n}{n+1}\right| = |x|\left|\frac{p}{n+1} - \frac{n}{n+1}\right| \to |x| \text{ as } n \to \infty.\end{aligned}$$

Thus, the radius of convergence is $R = 1$.

25. The k^{th} coefficient in the series $\sum kC_kx^k$ is $D_k = k \cdot C_k$. We are given that the series $\sum C_kx^k$ has radius of convergence R by the ratio test, so

$$|x| \lim_{k\to\infty} \frac{|C_{k+1}|}{|C_k|} = \frac{|x|}{R}.$$

Thus, applying the ratio test to the new series, we have

$$\lim_{k\to\infty}\left|\frac{D_{k+1}x^{k+1}}{D_kx^k}\right| = \lim_{k\to\infty}\left|\frac{(k+1)C_{k+1}}{kC_k}\right||x| = \frac{|x|}{R}.$$

Hence the new series has radius of convergence R.

26. The radius of convergence of the series, R, is at least 4 but no larger than 7.

(a) False. Since $10 > R$ the series diverges.
(b) True. Since $3 < R$ the series converges.
(c) False. Since $1 < R$ the series converges.
(d) Not possible to determine since the radius of convergence may be more or less than 6.

Solutions for Chapter 9 Review

Exercises

1. Let $a_n = n^2/(3n^2+4)$. Since $3n^2+4 > 3n^2$, we have $\dfrac{n^2}{3n^2+4} < \dfrac{1}{3}$, so

$$0 < a_n < \left(\frac{1}{3}\right)^n.$$

The geometric series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2}{3n^2+4}\right)^n$ also converges.

2. Let $a_n = 1/(n\sin^2 n)$. Since $0 < \sin^2 n < 1$, for any integer n, we have $n\sin^2 n < n$, so $\dfrac{1}{n\sin^2 n} > \dfrac{1}{n}$, thus

$$a_n > \frac{1}{n}.$$

The harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so the comparison test tells us that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sin^2 n}$ also diverges.

3. The first few terms of this series $\displaystyle\sum_{n=1}^{\infty}\frac{4-n}{n^3+1}$ are

$$\frac{3}{2} + \frac{2}{9} + \frac{1}{28} + 0 - \frac{1}{126} - \frac{2}{217} - \cdots.$$

Note that we cannot use the comparison test directly since $a_n = \dfrac{4-n}{n^3+1}$ is negative for $n > 4$. However

$$\sum_{n=1}^{\infty} \frac{4-n}{n^3+1} = \frac{3}{2} + \frac{2}{9} + \frac{1}{28} - \sum_{n=5}^{\infty} \frac{n-4}{n^3+1}.$$

Since $n^3 + 1 > n^3$, we have

$$\frac{1}{n^3+1} < \frac{1}{n^3},$$

therefore

$$\frac{n-4}{n^3+1} < \frac{n-4}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}, \quad \text{for } n > 4,$$

so we can compare the series $\sum_{n=5}^{\infty} \dfrac{n-4}{n^3+1}$ with $\sum_{n=5}^{\infty} \dfrac{1}{n^2}$, which converges. The comparison test tells us that the series

$$\sum_{n=1}^{\infty} \frac{4-n}{n^3+1}$$

also converges.

4. The series can be written as

$$\sum_{n=1}^{\infty} \frac{3n^2+n+1}{n^5+1} = \sum_{n=1}^{\infty} \left(3\frac{n^2}{n^5+1} + \frac{n}{n^5+1} + \frac{1}{n^5+1} \right).$$

To show that the original series converges we show that each of the series $\sum_{n=1}^{\infty} \dfrac{n^2}{n^5+1}$, $\sum_{n=1}^{\infty} \dfrac{n}{n^5+1}$, and $\sum_{n=1}^{\infty} \dfrac{1}{n^5+1}$ converges. (See Property 1 of Theorem 9.2.)

Firstly, consider the series $\sum_{n=1}^{\infty} \dfrac{n^2}{n^5+1}$. Let $a_n = \dfrac{n^2}{n^5+1}$, then since $n^5 + 1 > n^5$ we have

$$\frac{n^2}{n^5+1} < \frac{n^2}{n^5} = \frac{1}{n^3},$$

so we can compare the first series with the convergent series $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$. The comparison test tells us that the series $\sum_{n=1}^{\infty} \dfrac{n^2}{n^5+1}$ converges.

Now consider the series, $\sum_{n=1}^{\infty} \dfrac{n}{n^5+1}$. Let $a_n = \dfrac{n}{n^5+1}$, then we have

$$\frac{n}{n^5+1} < \frac{n}{n^5} = \frac{1}{n^4},$$

so we can compare this series with the convergent series $\sum_{n=1}^{\infty} \dfrac{1}{n^4}$. The comparison test tells us that the series $\sum_{n=1}^{\infty} \dfrac{n}{n^5+1}$ converges.

Finally, consider the series $\sum_{n=1}^{\infty} \dfrac{1}{n^5+1}$. Let $a_n = \dfrac{1}{n^5+1}$, then

$$\frac{1}{n^5+1} < \frac{1}{n^5},$$

so we can compare this series with the convergent series $\sum_{n=1}^{\infty} \dfrac{1}{n^5}$. The comparison test tells us that the series $\sum_{n=1}^{\infty} \dfrac{1}{n^5+1}$ also converges.

From Property 1 of Theorem 9.2, since the series $\sum_{n=1}^{\infty} \frac{n^2}{n^5+1}$, $\sum_{n=1}^{\infty} \frac{n}{n^5+1}$ and $\sum_{n=1}^{\infty} \frac{1}{n^5+1}$ converge, the series $\sum_{n=1}^{\infty} \frac{3n^2+n+1}{n^5+1}$ also converges.

5. To show that the original series converges, we show that the series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge. The first of these is a convergent geometric series, since $|3/4| < 1$. The integral test tells us that series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by comparing it with the convergent integral $\int_0^1 1/x^2 dx$. Theorem 9.2 then tells us that the series $\sum_{n=1}^{\infty} \left(\frac{3}{4}^n + \frac{1}{n^2}\right)$ also converges.

6. The series can be written as

$$\sum_{n=0}^{\infty} \frac{2+3^n}{5^n} = \sum_{n=0}^{\infty} \left(\frac{2}{5^n} + \frac{3^n}{5^n}\right) = \sum_{n=0}^{\infty} \left(2\left(\frac{1}{5}\right)^n + \left(\frac{3}{5}\right)^n\right).$$

The series $\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n$ is a geometric series which converges because $|\frac{1}{5}| < 1$. Likewise, the geometric series $\sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n$ converges because $|\frac{3}{5}| < 1$. Since both series converge, Property 1 of Theorem 9.2 tells us that the series $\sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$ also converges.

7. Writing $a_n = 1/(2+\sin n)$, we have $\lim_{n\to\infty} a_n \neq 0$ so the series diverges by Property 3 of Theorem 9.2.

8. Since $a_n = 3^n/(2n)!$, replacing n by $n+1$ gives $a_{n+1} = 3^{n+1}/(2n+2)!$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}/(2n+2)!}{3^n/(2n)!} = \frac{3^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{3^n}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{3}{(2n+2)(2n+1)},$$

so

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0.$$

The ratio test tells us that the series $\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$ converges.

9. Since $a_n = (2n)!/(n!)^2$, replacing n by $n+1$ gives $a_{n+1} = (2n+2)!/((n+1)!)^2$. Thus

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{n!n!}{(2n)!}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$ and $(n+1)! = (n+1)n!$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)},$$

therefore

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 4.$$

As $L = 4$ the ratio test tells us that the series $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges.

10. This is an alternating series. Let $a_n = 1/(\sqrt{n}+1)$. Then $\lim_{n\to\infty} a_n = 0$. Now replace n by $n+1$ to give $a_{n+1} = 1/(\sqrt{n+1}+1)$. Since $\sqrt{n+1}+1 > \sqrt{n}+1$, we have $\dfrac{1}{\sqrt{n+1}+1} < \dfrac{1}{\sqrt{n}+1}$, so

$$a_{n+1} = \frac{1}{\sqrt{n+1}+1} < \frac{1}{\sqrt{n}+1} = a_n.$$

Therefore, the alternating series test tells us that the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+1}$ converges.

11. Since $\ln(1+1/k) = \ln((k+1)/k) = \ln(k+1) - \ln k$, the n^{th} partial sum of this series is

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) \\
&= \sum_{k=1}^{n} \ln(k+1) - \sum_{k=1}^{n} \ln k \\
&= (\ln 2 + \ln 3 + \cdots + \ln(n+1)) - (\ln 1 + \ln 2 + \cdots + \ln n) \\
&= \ln(n+1) - \ln 1 \\
&= \ln(n+1).
\end{aligned}$$

Thus, the partial sums, S_n, grow without bound as $n \to \infty$, so the series diverges by the definition.

12. Since $C_n = n$, replacing n by $n+1$ gives $C_{n+1} = n+1$. Using the ratio test with $a_n = nx^n$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{n+1}{n} = |x|.$$

Thus the radius of convergence is $R = 1$.

13. Let $C_n = \dfrac{(2n)!}{(n!)^2}$. Then replacing n by $n+1$, we have $C_{n+1} = \dfrac{(2n+2)!}{((n+1)!)^2}$. Thus, with $a_n = (2n)!x^n/(n!)^2$, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x| \frac{|C_{n+1}|}{|C_n|} = |x| \frac{(2n+2)!/((n+1)!)^2}{(2n)!/(n!)^2} = |x| \frac{(2n+2)!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2}.$$

Since $(2n+2)! = (2n+2)(2n+1)(2n)!$ and $(n+1)! = (n+1)n!$ we have

$$\frac{|C_{n+1}|}{|C_n|} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)},$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = |x| \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = |x| \lim_{n\to\infty} \frac{4n+2}{n+1} = 4|x|,$$

so the radius of convergence of this series is $R = 1/4$.

14. Let $C_n = 2^n + n^2$. Then replacing n by $n+1$ gives $C_{n+1} = 2^{n+1} + (n+1)^2$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x| \frac{|C_{n+1}|}{|C_n|} = |x| \frac{2^{n+1} + (n+1)^2}{2^n + n^2} = 2|x| \left(\frac{2^n + \frac{1}{2}(n+1)^2}{2^n + n^2} \right).$$

Since 2^n dominates n^2 as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2|x|.$$

Thus the radius of convergence is $R = \frac{1}{2}$.

15. Let $C_n = 1/(n!+1)$. Then replacing n by $n+1$ gives $C_{n+1} = 1/((n+1)!+1)$. Using the ratio test, we have

$$\frac{|a_{n+1}|}{|a_n|} = |x|\frac{|C_{n+1}|}{|C_n|} = |x|\frac{1/((n+1)!+1)}{1/(n!+1)} = |x|\frac{n!+1}{(n+1)!+1}.$$

Since $n!$ and $(n+1)!$ dominate the constant term 1 as $n \to \infty$ and $(n+1)! = (n+1)\cdot n!$ we have

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = 0.$$

Thus the radius of convergence is $R = \infty$.

Problems

16. **(a)** $0.232323\ldots = 0.23 + 0.23(0.01) + 0.23(0.01)^2 + \cdots$ which is a geometric series with $a = 0.23$ and $x = 0.01$.

(b) The sum is $\dfrac{0.23}{1-0.01} = \dfrac{0.23}{0.99} = \dfrac{23}{99}$.

17. The amount of cephalexin in the body is given by $Q(t) = Q_0e^{-kt}$, where $Q_0 = Q(0)$ and k is a constant. Since the half-life is 0.9 hours,

$$\frac{1}{2} = e^{-0.9k}, \quad k = -\frac{1}{0.9}\ln\frac{1}{2} \approx 0.8.$$

(a) After 6 hours

$$Q = Q_0e^{-k(6)} \approx Q_0e^{-0.8(6)} = Q_0(0.01).$$

Thus, the percentage of the cephalexin that remains after 6 hours $\approx 1\%$.

(b)

$$\begin{aligned}
Q_1 &= 250\\
Q_2 &= 250 + 250(0.01)\\
Q_3 &= 250 + 250(0.01) + 250(0.01)^2\\
Q_4 &= 250 + 250(0.01) + 250(0.01)^2 + 250(0.01)^3
\end{aligned}$$

(c)

$$\begin{aligned}
Q_3 &= \frac{250(1-(0.01)^3)}{1-0.01}\\
&\approx 252.5\\
Q_4 &= \frac{250(1-(0.01)^4)}{1-0.01}\\
&\approx 252.5
\end{aligned}$$

Thus, by the time a patient has taken three cephalexin tablets, the quantity of drug in the body has leveled off to 252.5 mg.

(d) Looking at the answers to part (b) shows that

$$\begin{aligned}
Q_n &= 250 + 250(0.01) + 250(0.01)^2 + \cdots + 250(0.01)^{n-1}\\
&= \frac{250(1-(0.01)^n)}{1-0.01}.
\end{aligned}$$

(e) In the long run, $n \to \infty$. So,

$$Q = \lim_{n\to\infty} Q_n = \frac{250}{1-0.01} = 252.5.$$

18. **(a)** (i) On the night of December 31, 1999:

First deposit will have grown to $2(1.04)^7$ million dollars.
Second deposit will have grown to $2(1.04)^6$ million dollars.
...
Most recent deposit (Jan.1, 1999) will have grown to $2(1.04)$ million dollars.

Thus

$$\begin{aligned}\text{Total amount} &= 2(1.04)^7 + 2(1.04)^6 + \cdots + 2(1.04)\\ &= 2(1.04)\underbrace{(1 + 1.04 + \cdots + (1.04)^6)}_{\text{finite geometric series}}\\ &= 2(1.04)\left(\frac{1-(1.04)^7}{1-1.04}\right)\\ &= 16.43 \text{ million dollars.}\end{aligned}$$

(ii) Notice that if 10 payments are made, there are 9 years between the first and the last. On the day of the last payment:

First deposit will have grown to $2(1.04)^9$ million dollars.
Second deposit will have grown to $2(1.04)^8$ million dollars.
. . .
Last deposit will be 2 million dollars.

Therefore

$$\begin{aligned}\text{Total amount} &= 2(1.04)^9 + 2(1.04)^8 + \cdots + 2\\ &= 2\underbrace{(1 + 1.04 + (1.04)^2 + \cdots + (1.04)^9)}_{\text{finite geometric series}}\\ &= 2\left(\frac{1-(1.04)^{10}}{1-1.04}\right)\\ &= 24.01 \text{ million dollars.}\end{aligned}$$

(b) In part (a) (ii) we found the future value of the contract 9 years in the future. Thus

$$\text{Present Value} = \frac{24.01}{(1.04)^9} = 16.87 \text{ million dollars.}$$

Alternatively, we can calculate the present value of each of the payments separately:

$$\begin{aligned}\text{Present Value} &= 2 + \frac{2}{1.04} + \frac{2}{(1.04)^2} + \cdots + \frac{2}{(1.04)^9}\\ &= 2\left(\frac{1-(1/1.04)^{10}}{1-1/1.04}\right) = 16.87 \text{ million dollars.}\end{aligned}$$

Notice that the present value of the contract ($\$16.87$ million) is considerably less than the face value of the contract, $\$20$ million.

19.

$$\begin{aligned}\text{Total present value, in dollars} &= 1000 + 1000e^{-0.04} + 1000e^{-0.04(2)} + 1000e^{-0.04(3)} + \cdots\\ &= 1000 + 1000(e^{-0.04}) + 1000(e^{-0.04})^2 + 1000(e^{-0.04})^3 + \cdots\end{aligned}$$

This is an infinite geometric series with $a = 1000$ and $x = e^{(-0.04)}$, and sum

$$\text{Total present value, in dollars} = \frac{1000}{1-e^{-0.04}} = 25{,}503.$$

20. A person should expect to pay the present value of the bond on the day it is bought.

$$\begin{aligned}\text{Present value of first payment } &= \frac{10}{1.04}\\ \text{Present value of second payment } &= \frac{10}{(1.04)^2}, \text{ etc.}\end{aligned}$$

Therefore,

$$\text{Total present value } = \frac{10}{1.04} + \frac{10}{(1.04)^2} + \frac{10}{(1.04)^3} + \cdots.$$

This is a geometric series with $a = \dfrac{10}{1.04}$ and $x = \dfrac{1}{1.04}$, so

$$\text{Total present value } = \frac{\frac{10}{1.04}}{1-\frac{1}{1.04}} = £250.$$

21.

$$\begin{aligned}\text{Present value of first coupon } &= \frac{50}{1.06}\\ \text{Present value of second coupon } &= \frac{50}{(1.06)^2}, \text{ etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value } &= \underbrace{\frac{50}{1.06}+\frac{50}{(1.06)^2}+\cdots+\frac{50}{(1.06)^{10}}}_{\text{coupons}}+\underbrace{\frac{1000}{(1.06)^{10}}}_{\text{principal}}\\ &= \frac{50}{1.06}\left(1+\frac{1}{1.06}+\cdots+\frac{1}{(1.06)^9}\right)+\frac{1000}{(1.06)^{10}}\\ &= \frac{50}{1.06}\left(\frac{1-\left(\frac{1}{1.06}\right)^{10}}{1-\frac{1}{1.06}}\right)+\frac{1000}{(1.06)^{10}}\\ &= 368.004+558.395\\ &= \$926.40\end{aligned}$$

22.

$$\begin{aligned}\text{Present value of first coupon } &= \frac{50}{1.04}\\ \text{Present value of second coupon } &= \frac{50}{(1.04)^2}, \text{etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value } &= \underbrace{\frac{50}{1.04}+\frac{50}{(1.04)^2}+\cdots+\frac{50}{(1.04)^{10}}}_{\text{coupons}}+\underbrace{\frac{1000}{(1.04)^{10}}}_{\text{principal}}\\ &= \frac{50}{1.04}\left(1+\frac{1}{1.04}+\cdots+\frac{1}{(1.04)^9}\right)+\frac{1000}{(1.04)^{10}}\\ &= \frac{50}{1.04}\left(\frac{1-\left(\frac{1}{1.04}\right)^{10}}{1-\frac{1}{1.04}}\right)+\frac{1000}{(1.04)^{10}}\\ &= 405.545+675.564\\ &= \$1081.11\end{aligned}$$

23. (a)

$$\begin{aligned}\text{Present value of first coupon } &= \frac{50}{1.05}\\ \text{Present value of second coupon } &= \frac{50}{(1.05)^2}, \text{etc.}\end{aligned}$$

$$\begin{aligned}\text{Total present value } &= \underbrace{\frac{50}{1.05}+\frac{50}{(1.05)^2}+\cdots+\frac{50}{(1.05)^{10}}}_{\text{coupons}}+\underbrace{\frac{1000}{(1.05)^{10}}}_{\text{principal}}\\ &= \frac{50}{1.05}\left(1+\frac{1}{1.05}+\cdots+\frac{1}{(1.05)^9}\right)+\frac{1000}{(1.05)^{10}}\end{aligned}$$

$$= \frac{50}{1.05}\left(\frac{1-\left(\frac{1}{1.05}\right)^{10}}{1-\frac{1}{1.05}}\right)+\frac{1000}{(1.05)^{10}}$$
$$= 386.087 + 613.913$$
$$= \$1000$$

(b) When the interest rate is 5%, the present value equals the principal.

(c) When the interest rate is more than 5%, the present value is smaller than it is when interest is 5% and must therefore be less than the principal. Since the bond will sell for around its present value, it will sell for less than the principal; hence the description *trading at discount*.

(d) When the interest rate is less than 5%, the present value is more than the principal. Hence the bound will be selling for more than the principal, and is described as *trading at a premium*.

24. The series converges for $|x-2|=2$ and diverges for $|x-2|=4$, thus the radius of convergence of the series, R, is at least 2 but no larger than 4.

(a) False. If $x=7$ then $|x-2|=5$, so the series diverges.
(b) False. If $x=1$ then $|x-2|=1$, so the series converges.
(c) True. If $x=0.5$ then $|x-2|=1.5$, so the series converges.
(d) If $x=5$ then $|x-2|=3$ and it is not possible to determine whether or not the series converges at this point.
(e) False. If $x=-3$ then $|x-2|=5$, so the series diverges.

25. (a) Since

$$\begin{aligned} |a_n| &= a_n \qquad && \text{if } a_n \geq 0 \\ |a_n| &= -a_n && \text{if } a_n < 0, \end{aligned}$$

we have

$$\begin{aligned} a_n + |a_n| &= 2|a_n| \qquad && \text{if } a_n > 0 \\ a_n + |a_n| &= 0 && \text{if } a_n < 0. \end{aligned}$$

Thus, for all n,

$$0 \leq a_n + |a_n| \leq 2|a_n|.$$

(b) If $\sum |a_n|$ converges, then $\sum 2|a_n|$ is convergent, so, by comparison, $\sum (a_n + |a_n|)$ is convergent. Then

$$\sum ((a_n + |a_n|) - |a_n|) = \sum a_n$$

is convergent, as it is the difference of two convergent series.

CAS Challenge Problems

26. (a) Using a CAS, we get

$$S_1(x)T_1(x) = x(1+x) = x + x^2$$
$$S_2(x)T_2(x) = \left(x+2\,x^2\right)\left(1+x+\frac{x^2}{2}\right) = x + 3\,x^2 + \frac{5\,x^3}{2} + x^4$$
$$S_3(x)T_3(x) = \left(x+2\,x^2+3\,x^3\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right) = x + 3\,x^2 + \frac{11\,x^3}{2} + \frac{25\,x^4}{6} + \frac{11\,x^5}{6} + \frac{x^6}{2}$$
$$\begin{aligned} S_4(x)T_4(x) &= \left(x+2\,x^2+3\,x^3+4\,x^4\right)\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}\right) \\ &= x + 3\,x^2 + \frac{11\,x^3}{2} + \frac{49\,x^4}{6} + \frac{47\,x^5}{8} + \frac{31\,x^6}{12} + \frac{19\,x^7}{24} + \frac{x^8}{6} \end{aligned}$$

(b) The coefficient of x is always the same, namely 1. The coefficient of x^2 is 1 in the first line, and then 3 thereafter. The coefficient of x^3 changes twice, but then remains at 11/2 for the last two lines.

(c) Following the same pattern, we expect that the coefficient of x^4 to remain the same after $n=4$, and indeed we find that

$$S_5(x)T_5(x) = x + 3\,x^2 + \frac{11\,x^3}{2} + \frac{49\,x^4}{6} + \frac{87\,x^5}{8} + \frac{911\,x^6}{120} + \frac{397\,x^7}{120} + \frac{41\,x^8}{40} + \frac{29\,x^9}{120} + \frac{x^{10}}{24},$$

so the coefficient of x^4 stays at $49/6$.

(d) In general, the coefficient of x^k in the product can vary in $S_1T_1, S_2T_2, \ldots, S_kT_k$ and then stays the same after that. This is because the coefficient of x^k in the product depends on the coefficients of $1, x, x^2, \ldots x^k$ in $S_n(x)$ and $T_n(x)$, and these remain the same for $n \geq k$.

27. (a) Using a CAS, we get

$$T_1(S_1(x)) = 1 + x$$
$$T_2(S_2(x)) = 1 + x + \frac{5\,x^2}{2} + 2\,x^3 + 2\,x^4$$
$$T_3(S_3(x)) = 1 + x + \frac{5\,x^2}{2} + \frac{31\,x^3}{6} + 6\,x^4 + \frac{19\,x^5}{2} + \frac{71\,x^6}{6} + \frac{21\,x^7}{2} + 9\,x^8 + \frac{9\,x^9}{2}$$

(b) The coefficient of x stays the same, namely 1. The coefficient of x^2 is 0 in the first line, but after that stabilizes at 5/2.

(c) Thus we predict that the coefficient of x^3 will stabilize after $n = 3$ and will be 31/6 in $T_4(S_4(x))$. This is confirmed by

$$T_4(S_4(x)) = 1 + x + \frac{5\,x^2}{2} + \frac{31\,x^3}{6} + \frac{241\,x^4}{24} + \frac{83\,x^5}{6} + \frac{70\,x^6}{3} + \frac{71\,x^7}{2} + \frac{599\,x^8}{12} + \frac{127\,x^9}{2} + \cdots.$$

(d) In general, the coefficient of x^k in the composite can vary in $T_1(S_1(x)), T_2(S_2(x)), \ldots, T_k(S_k(x))$ and then stays the same after that. This is because the coefficient of x^k in the composite depends on the coefficients of $1, x, x^2, \ldots x^k$ in $S_n(x)$ and $T_n(x)$, and these remain the same for $n \geq k$.

28. (a) Both p and q are geometric series. The radius of convergence of p is 1 and that of q is 1/2.

(b) Using a CAS, we get

$$\begin{aligned} pq &= (1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + x^{10} - \cdots) \\ &\quad (1 + 2\,x + 4\,x^2 + 8\,x^3 + 16\,x^4 + 32\,x^5 + 64\,x^6 + 128\,x^7 + 256\,x^8 + 512\,x^9 + 1024\,x^{10} + \cdots) \\ &= 1 + x + 3\,x^2 + 5\,x^3 + 11\,x^4 + 21\,x^5 + 43\,x^6 + 85\,x^7 + 171\,x^8 + 341\,x^9 + 683\,x^{10} + \cdots \end{aligned}$$

(c) The following table gives the ratio C_{n+1}/C_n for $n = 0, \ldots, 9$, where $pq = \sum C_n x^n$.

n	0	1	2	3	4	5	6	7	8	9
C_{n+1}/C_n	1	3.000	1.667	2.200	1.910	2.048	1.977	2.012	1.994	2.003

The ratios look like they are approaching 2 so we guess that the radius of convergence is 1/2.

(d) A reasonable conjecture is that the radius of convergence of a product is the smaller of the radii of convergence of the two original series.

CHECK YOUR UNDERSTANDING

1. True. A geometric series, $a + ax + ax^2 + \cdots$, is a power series about $x = 0$ with all coefficients equal to a.

2. False. Writing out terms, we have
$$(x-1) + (x-2)^2 + (x-3)^3 + \cdots.$$
A power series is a sum of powers of $(x - a)$ for constant a. In this case, the value of a changes from term to term, so it is not a power series.

3. True. This power series has an interval of convergence centered on $x = 0$. If the power series converges for $x = 2$, the radius of convergence is 2 or more. Thus, $x = 1$ is well within the interval of convergence, so the series converges at $x = 1$.

4. False. This power series has an interval of convergence centered on $x = 0$. Knowing the power series converges for $x = 1$ does not tell us whether the series converges for $x = 2$. Since the series converges at $x = 1$, we know the radius of convergence is at least 1. However, we do not know whether the interval of convergence extends as far as $x = 2$, so we cannot say whether the series converges at $x = 2$. Since this statement is not true for all C_n, the statement is false.

5. True. This power series has an interval of convergence centered on $x = 0$. If the power series does not converge for $x = 1$, then the radius of convergence is less than or equal to 1. Thus, $x = 2$ lies outside the interval of convergence, so the series does not converge there.

6. False. It does not tell us anything to know that b_n is larger than a convergent series. For example, if $a_n = 1/n^2$ and $b_n = 1$, then $0 \le a_n \le b_n$ and $\sum a_n$ converges, but $\sum b_n$ diverges. Since this statement is not true for all a_n and b_n, the statement is false.

7. True. This is one of the statements of the comparison test.

8. True. Consider the series $\sum(-b_n)$ and $\sum(-a_n)$. The series $\sum(-b_n)$ converges, since $\sum b_n$ converges, and

$$0 \le -a_n \le -b_n.$$

By the comparison test, $\sum(-a_n)$ converges, so $\sum a_n$ converges.

9. False. It is true that if $\sum |a_n|$ converges, then we know that $\sum a_n$ converges. However, knowing that $\sum a_n$ converges does *not* tell us that $\sum |a_n|$ converges.

For example, if $a_n = (-1)^{n-1}/n$, then $\sum a_n$ converges by the alternating series test. However, $\sum |a_n|$ is the harmonic series which diverges.

10. False. For example, if $a_n = 1/n$ and $b_n = -1/n$, then $|a_n + b_n| = 0$, so $\sum |a_n + b_n|$ converges. However $\sum |a_n|$ and $\sum |b_n|$ are the harmonic series, which diverge.

11. False. For example, if $a_n = 1/n^2$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

However, $\sum 1/n^2$ converges.

12. False, since if we write out the terms of the series, using the fact that $\cos 0 = 1, \cos\pi = -1, \cos(2\pi) = 1, \cos(3\pi) = -1$, and so on, we have

$$\begin{aligned} &(-1)^0\cos 0 + (-1)^1\cos\pi + (-1)^2\cos 2\pi + (-1)^3\cos 3\pi + \cdots \\ &= (1)(1) + (-1)(-1) + (1)(1) + (-1)(-1) + \cdots \\ &= 1 + 1 + 1 + 1 + \cdots. \end{aligned}$$

This is not an alternating series.

13. True. Writing out the terms of this series, we have

$$\begin{aligned} &(1 + (-1)^1) + (1 + (-1)^2) + (1 + (-1)^3) + (1 + (-1)^4) + \cdots \\ &= (1 - 1) + (1 + 1) + (1 - 1) + (1 + 1) + \cdots \\ &= 0 + 2 + 0 + 2 + \cdots. \end{aligned}$$

14. False. This is an alternating series, but since the terms do not go to zero, it does not converge.

15. False. The terms in the series do not go to zero:

$$\begin{aligned} 2^{(-1)^1} + 2^{(-1)^2} + 2^{(-1)^3} + 2^{(-1)^4} + 2^{(-1)^5} + \cdots &= 2^{-1} + 2^1 + 2^{-1} + 2^1 + 2^{-1} + \cdots \\ &= 1/2 + 2 + 1/2 + 2 + 1/2 + \cdots. \end{aligned}$$

16. False. For example, if $a_n = (-1)^{n-1}/n$, then $\sum a_n$ converges by the alternating series test. But $(-1)^n a_n = (-1)^n(-1)^{n-1}/n = (-1)^{2n-1}/n = -1/n$. Thus, $\sum(-1)^n a_n$ is the negative of the harmonic series and does not converge.

17. True. Let $c_n = (-1)^n|a_n|$. Then $|c_n| = |a_n|$ so $\sum |c_n|$ converges, and therefore $\sum c_n = \sum(-1)^n|a_n|$ converges.

18. True. Since the series is alternating, Theorem 9.8 gives the error bound. Summing the first 100 terms gives S_{100}, and if the true sum is S,

$$|S - S_{100}| < a_{101} = \frac{1}{101} < 0.01.$$

19. True. The radius of convergence, R, is given by $\lim_{n\to\infty} |C_{n+1}|/|C_n| = 1/R$, if this limit exists, and since these series have the same coefficients, C_n, the radii of convergence are the same.

20. False. Two series can have the same radius of convergence without having the same coefficients. For example, $\sum x^n$ and $\sum nx^n$ both have radius of convergence of 1:

$$\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = \lim_{n\to\infty} \frac{1}{1} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{B_{n+1}}{B_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

21. False. Consider the power series

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} + \cdots,$$

whose interval of convergence is $0 < x \le 2$. This series converges at one endpoint, $x = 2$, but not at the other, $x = 0$.

22. True. If the terms do not tend to zero, the partial sums do not tend to a limit. For example, if the terms are all greater than 0.1, the partial sums will grow without bound.

23. False. Consider the series $\sum_{n=1}^{\infty} 1/n$. This series does not converge, but $1/n \to 0$ as $n \to \infty$.

24. False. If $a_n = b_n = 1/n$, then $\sum a_n$ and $\sum b_n$ do not converge. However, $a_n b_n = 1/n^2$, so $\sum a_n b_n$ does converge.

25. False. If $a_n b_n = 1/n^2$ and $a_n = b_n = 1/n$, then $\sum a_n b_n$ converges, but $\sum a_n$ and $\sum b_n$ do not converge.

PROJECTS FOR CHAPTER NINE

1. (a) (i) p^2

(ii) There are two ways to do this. One way is to compute your opponent's probability of winning two in a row, which is $(1-p)^2$. Then the probability that neither of you win the next points is:

$$\begin{aligned} 1 - (&\text{Probability you win next two} + \text{Probability opponent wins next two}) \\ &= 1 - (p^2 + (1-p)^2) \\ &= 1 - (p^2 + 1 - 2p + p^2) \\ &= 2p^2 - 2p \\ &= 2p(1-p). \end{aligned}$$

The other way to compute this is to observe either you win the first point and lose the second or vice versa. Both have probability $p(1-p)$, so the probability you split the points is $2p(1-p)$.

(iii)

$$\begin{aligned} \text{Probability} &= (\text{Probability of splitting next two}) \cdot (\text{Probability of winning two after that}) \\ &= 2p(1-p)p^2 \end{aligned}$$

(iv)

$$\begin{aligned} \text{Probability} &= (\text{Probability of winning next two}) + (\text{Probability of splitting next two,} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{winning two after that}) \\ &= p^2 + 2p(1-p)p^2 \end{aligned}$$

(v) The probability is:

$$\begin{aligned} w &= (\text{Probability of winning first two}) \\ &\quad + (\text{Probability of splitting first two})\cdot(\text{Probability of winning next two}) \\ &\quad + (\text{Prob. of split. first two})\cdot(\text{Prob. of split. next two})\cdot(\text{Prob. of winning next two}) \\ &\quad + \cdots \\ &= p^2 + 2p(1-p)p^2 + (2p(1-p))^2\, p^2 + \cdots. \end{aligned}$$

This is an infinite geometric series with a first term of p^2 and a ratio of $2p(1-p)$. Therefore the probability of winning is

$$w = \frac{p^2}{1-2p(1-p)}.$$

(vi) For $p = 0.5$, $w = \frac{(0.5)^2}{1-2(0.5)(1-(0.5))} = 0.5$. This is what we would expect. If you and your opponent are equally likely to score the next point, you and your opponent are equally likely to win the next game.

For $p = 0.6$, $w = \frac{(0.6)^2}{1-2(0.6)(0.4)} = 0.69$. Here your probability of winning the next point has been magnified to a probability 0.69 of winning the game. Thus it gives the better player an advantage to have to win by two points, rather than the "sudden death" of winning by just one point. This makes sense: when you have to win by two, the stronger player always gets a second chance to overcome the weaker player's winning the first point on a "fluke."

For $p = 0.7$, $w = \frac{(0.7)^2}{1-2(0.7)(0.3)} = 0.84$. Again, the stronger player's probability of winning is magnified.

For $p = 0.4$, $w = \frac{(0.4)^2}{1-2(0.4)(0.6)} = 0.31$. We already computed that for $p = 0.6$, $w = 0.69$. Thus the value for w when $p = 0.4$, should be the same as the probability of your opponent winning for $p = 0.6$, namely $1 - 0.69 = 0.31$.

(b) (i)

$$\begin{aligned}
S = {} & \text{(Prob. you score first point)} \\
& +\text{(Prob. you lose first point, your opponent loses the next,} \\
& \quad \text{you win the next)} \\
& +\text{(Prob. you lose a point, opponent loses, you lose,} \\
& \quad \text{opponent loses, you win)} \\
& + \cdots \\
= {} & \text{(Prob. you score first point)} \\
& +\text{(Prob. you lose)}\cdot\text{(Prob. opponent loses)}\cdot\text{(Prob. you win)} \\
& +\text{(Prob. you lose)}\cdot\text{(Prob. opponent loses)}\cdot\text{(Prob. you lose)} \\
& \quad \cdot\text{(Prob. opponent loses)}\cdot\text{(Prob. you win)} + \cdots \\
= {} & p + (1-p)(1-q)p + ((1-p)(1-q))^2 p + \cdots \\
= {} & \frac{p}{1-(1-p)(1-q)}
\end{aligned}$$

(ii) Since S is your probability of winning the next point, we can use the formula computed in part (v) of (a) for winning two points in a row, thereby winning the game:

$$w = \frac{S^2}{1-2S(1-S)}.$$

- When $p = 0.5$ and $q = 0.5$,

$$S = \frac{0.5}{1-(0.5)(0.5)} = 0.67.$$

Therefore

$$w = \frac{S^2}{1-2S(1-S)} = \frac{(0.67)^2}{1-2(0.67)(1-0.67)} = 0.80.$$

- When $p = 0.6$ and $q = 0.5$,

$$S = \frac{0.6}{1-(0.4)(0.5)} = 0.75 \quad \text{and} \quad w = \frac{(0.75)^2}{1-2(0.75)(1-0.75)} = 0.9.$$

2. (a) Let k by the relative rate of decay, per minute, of quinine. Since quinine's half-life is 11.5 hours, we have

$$\frac{1}{2} = e^{-k(11.5)(60)},$$

so

$$k = \frac{\ln 2}{(11.5)(60)} \approx 0.001.$$

Hence, $k = 0.1\%$/min.

(b) Just prior to 8 am of the first day the patient has no quinine in her body. Assuming the drug mixes rapidly in the patient's body, she has about $50/70 \approx 0.714$ mg/kg of the drug soon after 8 am. Suppose we represent the concentration of quinine in the patient (in mg/kg) by x and represent time since 8 am (in minutes) by t. Then

$$x = Ae^{-0.001t},$$

where A is the initial concentration and $k = -0.001$ is the rate at which quinine is metabolized per minute. There are $24 \cdot 60 = 1440$ minutes in a day. On the first day, the patient begins with 0.714 mg/kg in her system, so just before 8 am of the second day the patient's system holds

$$0.714e^{-0.001 \cdot 1440} \approx 0.169 \text{ mg/kg}.$$

After the patient's second dose of quinine, her system contains $0.714 + 0.169 = 0.883$ mg/kg of quinine.

(c) By continuing in a similar manner, we see that just prior to 8 am on the third day, she has $0.883e^{-0.001 \cdot 1440} \approx 0.209$ mg/kg; just after 8 am, she has $0.209 + 0.714 = 0.923$ mg/kg. Just prior to 8 am on the fourth day, she has $0.923e^{-0.001 \cdot 1440} \approx 0.218$ mg/kg; just after 8 am, she has $0.228 + 0.714 = 0.932$ mg/kg. We can keep going with these calculations: just prior to 8 am on the fifth day, the concentration is 0.221 mg/kg; on the sixth day, it is 0.222 mg/kg; on the seventh day, it is 0.222 mg/kg, and so on forever.

We find a formula for the concentration just after the n^{th} dose as follows. The last dose contributes 0.714 mg/kg. The previous dose contributes $0.714e^{-0.001(1440)}$ mg/kg. The dose before that contributes $0.714e^{-0.001(2)(1440)}$ mg/kg, and so on, back to $0.714e^{-0.001(n-1)(1440)}$ mg/kg from the initial dose. So

$$\begin{array}{c}\text{Concentration just}\\ \text{after } n \text{ doses}\end{array} = 0.714 + 0.714e^{-1.44} + 0.714\left(e^{-1.44}\right)^2 + \cdots + 0.714\left(e^{-1.44}\right)^{n-1}.$$

We notice that this is a geometric series, with sum given by

$$\begin{array}{c}\text{Concentration just}\\ \text{after } n \text{ doses}\end{array} = 0.714\left(\frac{1 - e^{-1.44n}}{1 - e^{-1.44}}\right) = 0.936(1 - e^{-1.44n}).$$

Although the concentration of quinine does not reach an equilibrium it does fall into a steady-state pattern which repeats over and over again. This makes sense; at some point the patient must metabolize the daily dosage exactly. If we let $n \to \infty$ in our formula, we have $e^{-1.44n} \to 0$, which means that the concentration just after the n^{th} dose gets very close to 0.936. So the concentration just before the n^{th} dose is $0.936 - 0.714 = 0.222$, as we found in our calculations for the first few days.

(d)

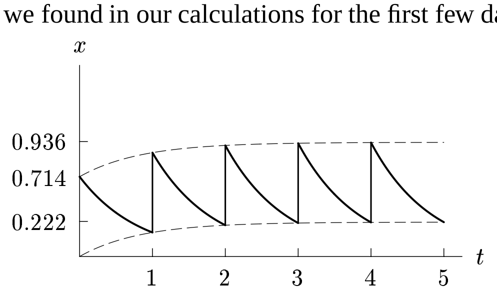


Figure 9.6

If we keep setting the clock back to 0 minutes each day at 8 am, then we have that at $t = 0$ each day, the concentration (starting on the fifth day or so) is 0.936 mg/kg. As the day progresses, we have

$$x = 0.936e^{-0.001 \cdot t}.$$

(e) The average concentration of quinine in the patient is given by the integral of the concentration over a day, divided by the time in a day:

$$\begin{aligned}\text{Average concentration } &= \frac{1}{1440}\int_0^{1440} x\,dt = \frac{1}{1440}\int_0^{1440} 0.936e^{-0.001t}dt \\ &= \frac{0.936}{1440}\left(\frac{-e^{-0.001t}}{0.001}\right)\Bigg|_0^{1440} = \frac{0.936}{1.44}(1-e^{-1.44}) \\ &\approx 0.496 \text{ mg/kg.}\end{aligned}$$

(f) Since the average concentration is 0.496 mg/kg and the minimum effective average concentration is 0.4 mg/kg, this treatment is effective. It is also safe—the highest concentration (0.936 mg/kg, achieved shortly after 8 am) is less than the toxic concentration of 3.0 mg/kg.

(g) Each dose of 25 mg corresponds to $25/70 = 0.357$ mg/kg. Let x_s be the steady-state concentration just before each 0.357 mg/kg dose. Then $x_s + 0.357$ will be the concentration just after the dose. Since we are in a steady-state, this concentration decays to exactly x_s just before the next dose. So

$$x_s = (x_s + 0.357)e^{-0.001(12)(60)}.$$

This means

$$x_s = \frac{0.357e^{-0.001(12)(60)}}{1-e^{-0.001(12)(60)}} \approx 0.339 \text{ mg/kg},$$

so $x_s + 0.357 = 0.696$ mg/kg is the concentration just after each dose. At t minutes after a dose, for $0 \le t \le (12)(60)$, there is a steady-state concentration of

$$x = 0.696e^{-0.001t} \text{ mg/kg.}$$

This means

$$\begin{aligned}\text{Average concentration } &= \frac{1}{720}\int_0^{720} x\,dt \approx \frac{1}{720}\int_0^{720} 0.696e^{-0.001t}dt \\ &= \frac{0.696}{720}\left[\frac{-e^{-0.001t}}{0.001}\right]\Bigg|_0^{720} = \frac{0.696}{0.72}\left[1-0.487\right] \\ &\approx 0.496 \text{ mg/kg.}\end{aligned}$$

This treatment is also effective and safe. The average concentration of 0.496 mg/kg is greater than 0.4 mg/kg, and the highest concentration of 0.696 mg/kg is less than 3 mg/kg.

(h) For an exponentially decaying function, the average value between two points (x_0, y_0) and (x_1, y_1) is $\frac{(y_0-y_1)}{(x_1-x_0)r}$, where r is the relative rate of decay and A_0 is the initial concentration. The reason is as follows.

$$\begin{aligned}\text{Average } &= \frac{1}{x_1-x_0}\int_{x_0}^{x_1} A_0e^{-rt}dt \\ &= \frac{A_0}{x_1-x_0}\left[\frac{e^{-rt}}{r}\right]\Bigg|_{x_0}^{x_1} \\ &= \frac{y_0-y_1}{(x_1-x_0)\cdot r}\end{aligned}$$

(i) Since a steady state has been reached, y_0 is the concentration right after a dose and y_1 is the concentration just prior to a dose. Thus, $y_0 - y_1$ represents the increase in concentration from each dose. Furthermore, $x_1 - x_0$ is the time between doses. When we go to the new protocol, we halve both the numerator and the denominator of the equation for the average concentration, and so the average remains unchanged. Similarly, if we were to double the dose to 100 mg and give it every 48 hours we would simply be doubling both the numerator and the denominator; again the average concentration would not change.

(j) We want the final concentration to be 10^{-10} kg/kg $= 10^{-4}$ mg/kg. We therefore need to solve for t in $10^{-4} = 0.883 \cdot e^{-0.001 \cdot t}$. Doing so yields $t \approx 9086$ min ≈ 6.3 days.