CHAPTER TEN

Solutions for Section 10.1

Exercises

1. Let $\frac{1}{1+x} = (1+x)^{-1}$. Then $f(0) = 1$.

$$\begin{aligned}
f'(x) &= -1!(1+x)^{-2} & f'(0) &= -1,\\
f''(x) &= 2!(1+x)^{-3} & f''(0) &= 2!,\\
f'''(x) &= -3!(1+x)^{-4} & f'''(0) &= -3!,\\
f^{(4)}(x) &= 4!(1+x)^{-5} & f^{(4)}(0) &= 4!,\\
f^{(5)}(x) &= -5!(1+x)^{-6} & f^{(5)}(0) &= -5!,\\
f^{(6)}(x) &= 6!(1+x)^{-7} & f^{(6)}(0) &= 6!,\\
f^{(7)}(x) &= -7!(1+x)^{-8} & f^{(7)}(0) &= -7!,\\
f^{(8)}(x) &= 8!(1+x)^{-9} & f^{(8)}(0) &= 8!.
\end{aligned}$$

$$\begin{aligned}
P_4(x) &= 1 - x + x^2 - x^3 + x^4,\\
P_6(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6,\\
P_8(x) &= 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8.
\end{aligned}$$

2. Let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$. Then $f(0) = 1$.

$$\begin{aligned}
f'(x) &= 1!(1-x)^{-2} & f'(0) &= 1!,\\
f''(x) &= 2!(1-x)^{-3} & f''(0) &= 2!,\\
f'''(x) &= 3!(1-x)^{-4} & f'''(0) &= 3!,\\
f^{(4)}(x) &= 4!(1-x)^{-5} & f^{(4)}(0) &= 4!,\\
f^{(5)}(x) &= 5!(1-x)^{-6} & f^{(5)}(0) &= 5!,\\
f^{(6)}(x) &= 6!(1-x)^{-7} & f^{(6)}(0) &= 6!,\\
f^{(7)}(x) &= 7!(1-x)^{-8} & f^{(7)}(0) &= 7!.
\end{aligned}$$

$$\begin{aligned}
P_3(x) &= 1 + x + x^2 + x^3,\\
P_5(x) &= 1 + x + x^2 + x^3 + x^4 + x^5,\\
P_7(x) &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7.
\end{aligned}$$

3. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Then $f(0) = 1$, and

$$\begin{aligned}
f'(x) &= \tfrac{1}{2}(1+x)^{-1/2} & f'(0) &= \tfrac{1}{2},\\
f''(x) &= -\tfrac{1}{4}(1+x)^{-3/2} & f''(0) &= -\tfrac{1}{4},\\
f'''(x) &= \tfrac{3}{8}(1+x)^{-5/2} & f'''(0) &= \tfrac{3}{8},\\
f^{(4)}(x) &= -\tfrac{15}{16}(1+x)^{-7/2} & f^{(4)}(0) &= -\tfrac{15}{16}.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_2(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2,\\
P_3(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3,\\
P_4(x) &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4.
\end{aligned}$$

4. Let $f(x)=\cos x$. Then $f(0)=\cos(0)=1$, and

$$\begin{aligned} f'(x) &= -\sin x & f'(0) &= 0,\\ f''(x) &= -\cos x & f''(0) &= -1,\\ f'''(x) &= \sin x & f'''(0) &= 0,\\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1,\\ f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0,\\ f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -1. \end{aligned}$$

Thus,

$$\begin{aligned} P_2(x) &= 1-\frac{x^2}{2!},\\ P_4(x) &= 1-\frac{x^2}{2!}+\frac{x^4}{4!},\\ P_6(x) &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}. \end{aligned}$$

5. Let $f(x)=\arctan x$. Then $f(0)=\arctan 0=0$, and

$$\begin{aligned} f'(x) &= 1/(1+x^2)=(1+x^2)^{-1} & f'(0) &= 1,\\ f''(x) &= (-1)(1+x^2)^{-2}2x & f''(0) &= 0,\\ f'''(x) &= 2!(1+x^2)^{-3}2^2x^2+(-1)(1+x^2)^{-2}2 & f'''(0) &= -2,\\ f^{(4)}(x) &= -3!(1+x^2)^{-4}2^3x^3+2!(1+x^2)^{-3}2^3x & &\\ &\quad +2!(1+x^2)^{-3}2^2x & f^{(4)}(0) &= 0. \end{aligned}$$

Therefore,

$$P_3(x)=P_4(x)=x-\frac{1}{3}x^3.$$

6. Let $f(x)=\tan x$. So $f(0)=\tan 0=0$, and

$$\begin{aligned} f'(x) &= 1/\cos^2 x & f'(0) &= 1,\\ f''(x) &= 2\sin x/\cos^3 x & f''(0) &= 0,\\ f'''(x) &= (2/\cos^2 x)+(6\sin^2 x/\cos^4 x) & f'''(0) &= 2,\\ f^{(4)}(x) &= (16\sin x/\cos^3 x)+(24\sin^3 x/\cos^5 x) & f^{(4)}(0) &= 0. \end{aligned}$$

Thus,

$$P_3(x)=P_4(x)=x+\frac{x^3}{3}.$$

7. Let $f(x)=\sqrt[3]{1-x}=(1-x)^{1/3}$. Then $f(0)=1$, and

$$\begin{aligned} f'(x) &= -\tfrac{1}{3}(1-x)^{-2/3} & f'(0) &= -\tfrac{1}{3},\\ f''(x) &= -\tfrac{2}{3^2}(1-x)^{-5/3} & f''(0) &= -\tfrac{2}{3^2},\\ f'''(x) &= -\tfrac{10}{3^3}(1-x)^{-8/3} & f'''(0) &= -\tfrac{10}{3^3},\\ f^{(4)}(x) &= -\tfrac{80}{3^4}(1-x)^{-11/3} & f^{(4)}(0) &= -\tfrac{80}{3^4}. \end{aligned}$$

Then,

$$\begin{aligned} P_2(x) &= 1-\frac{1}{3}x-\frac{1}{2!}\frac{2}{3^2}x^2=1-\frac{1}{3}x-\frac{1}{9}x^2,\\ P_3(x) &= P_2(x)-\frac{1}{3!}\left(\frac{10}{3^3}\right)x^3=1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3,\\ P_4(x) &= P_3(x)-\frac{1}{4!}\frac{80}{3^4}x^4=1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3-\frac{10}{243}x^4. \end{aligned}$$

8. Let $f(x) = \ln(1+x)$. Then $f(0) = \ln 1 = 0$, and

$$\begin{aligned}
f'(x) &= (1+x)^{-1} & f'(0) &= 1,\\
f''(x) &= (-1)(1+x)^{-2} & f''(0) &= -1,\\
f'''(x) &= 2(1+x)^{-3} & f'''(0) &= 2,\\
f^{(4)}(x) &= -3!(1+x)^{-4} & f^{(4)}(0) &= -3!,\\
f^{(5)}(x) &= 4!(1+x)^{-5} & f^{(5)}(0) &= 4!,\\
f^{(6)}(x) &= -5!(1+x)^{-6} & f^{(6)}(0) &= -5!,\\
f^{(7)}(x) &= 6!(1+x)^{-7} & f^{(7)}(0) &= 6!,\\
f^{(8)}(x) &= -7!(1+x)^{-8} & f^{(8)}(0) &= -7!,\\
f^{(9)}(x) &= 8!(1+x)^{-9} & f^{(9)}(0) &= 8!.
\end{aligned}$$

So,

$$\begin{aligned}
P_5(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5},\\
P_7(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7},\\
P_9(x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \frac{x^9}{9}.
\end{aligned}$$

9. Let $f(x) = \dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$. Then $f(0) = 1$.

$$\begin{aligned}
f'(x) &= -\tfrac{1}{2}(1+x)^{-3/2} & f'(0) &= -\tfrac{1}{2},\\
f''(x) &= \tfrac{3}{2^2}(1+x)^{-5/2} & f''(0) &= \tfrac{3}{2^2},\\
f'''(x) &= -\tfrac{3\cdot 5}{2^3}(1+x)^{-7/2} & f'''(0) &= -\tfrac{3\cdot 5}{2^3},\\
f^{(4)}(x) &= \tfrac{3\cdot 5\cdot 7}{2^4}(1+x)^{-9/2} & f^{(4)}(0) &= \tfrac{3\cdot 5\cdot 7}{2^4}
\end{aligned}$$

Then,

$$\begin{aligned}
P_2(x) &= 1 - \frac{1}{2}x + \frac{1}{2!}\frac{3}{2^2}x^2 = 1 - \frac{1}{2}x + \frac{3}{8}x^2,\\
P_3(x) &= P_2(x) - \frac{1}{3!}\frac{3\cdot 5}{2^3}x^3 = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3,\\
P_4(x) &= P_3(x) + \frac{1}{4!}\frac{3\cdot 5\cdot 7}{2^4}x^4 = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4.
\end{aligned}$$

10. Let $f(x) = (1+x)^p$.

(a) Suppose that $p = 0$. Then $f(x) = 1$ and $f^{(k)}(x) = 0$ for any $k \geq 1$. Thus $P_2(x) = P_3(x) = P_4(x) = 1$.

(b) If $p = 1$ then $f(x) = 1 + x$, so

$$\begin{aligned}
f(0) &= 1,\\
f'(x) &= 1,\\
f^{(k)}(x) &= 0 \qquad k \geq 2.
\end{aligned}$$

Thus $P_2(x) = P_3(x) = P_4(x) = 1 + x$.

(c) In general:

$$\begin{aligned}
f(x) &= (1+x)^p,\\
f'(x) &= p(1+x)^{p-1},\\
f''(x) &= p(p-1)(1+x)^{p-2},\\
f'''(x) &= p(p-1)(p-2)(1+x)^{p-3},\\
f^{(4)}(x) &= p(p-1)(p-2)(p-3)(1+x)^{p-4}.
\end{aligned}$$

$$f(0) = 1,$$

$$\begin{aligned} f'(0) &= p, \\ f''(0) &= p(p-1), \\ f'''(0) &= p(p-1)(p-2), \\ f^{(4)}(0) &= p(p-1)(p-2)(p-3). \end{aligned}$$

$$\begin{aligned} P_2(x) &= 1 + px + \frac{p(p-1)}{2}x^2, \\ P_3(x) &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3, \\ P_4(x) &= 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 \\ &\quad + \frac{p(p-1)(p-2)(p-3)}{24}x^4. \end{aligned}$$

11. Let $f(x) = \sin x$. $f(\frac{\pi}{2}) = 1$.

$$\begin{aligned} f'(x) &= \cos x & f'(\tfrac{\pi}{2}) &= 0, \\ f''(x) &= -\sin x & f''(\tfrac{\pi}{2}) &= -1, \\ f'''(x) &= -\cos x & f'''(\tfrac{\pi}{2}) &= 0, \\ f^{(4)}(x) &= \sin x & f^{(4)}(\tfrac{\pi}{2}) &= 1. \end{aligned}$$

So,

$$\begin{aligned} P_4(x) &= 1 + 0 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + 0 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 \\ &= 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4. \end{aligned}$$

12. Let $f(x) = \cos x$. Then $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.
Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, and $f'''(x) = \sin x$, so the Taylor polynomial for $\cos x$ of degree three about $x = \pi/4$ is

$$\begin{aligned} P_3(x) &= \cos\frac{\pi}{4} + \left(-\sin\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{-\cos\frac{\pi}{4}}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sin\frac{\pi}{4}}{3!}\left(x - \frac{\pi}{4}\right)^3 \\ &= \frac{\sqrt{2}}{2}\left(1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right). \end{aligned}$$

13. Let $f(x) = e^x$. Since $f^{(k)}(x) = e^x = f(x)$ for all $k \geq 1$, the Taylor polynomial of degree 4 for $f(x) = e^x$ about $x = 1$ is

$$\begin{aligned} P_4(x) &= e^1 + e^1(x-1) + \frac{e^1}{2!}(x-1)^2 + \frac{e^1}{3!}(x-1)^3 + \frac{e^1}{4!}(x-1)^4 \\ &= e\left[1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4\right]. \end{aligned}$$

14. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$.
Then $f'(x) = \dfrac{1}{2}(1+x)^{-1/2}$, $f''(x) = -\dfrac{1}{4}(1+x)^{-3/2}$, and $f'''(x) = \dfrac{3}{8}(1+x)^{-5/2}$. The Taylor polynomial of degree three about $x = 1$ is thus

$$\begin{aligned} P_3(x) &= (1+1)^{1/2} + \frac{1}{2}(1+1)^{-1/2}(x-1) + \frac{-\frac{1}{4}(1+1)^{-3/2}}{2!}(x-1)^2 \\ &\quad + \frac{\frac{3}{8}(1+1)^{-5/2}}{3!}(x-1)^3 \\ &= \sqrt{2}\left(1 + \frac{x-1}{4} - \frac{(x-1)^2}{32} + \frac{(x-1)^3}{128}\right). \end{aligned}$$

Problems

15. Since $P_2(x)$ is the second degree Taylor polynomial for $f(x)$ about $x = 0$, $P_2(0) = f(0)$, which says $a = f(0)$. Since

$$\frac{d}{dx}P_2(x)\Big|_{x=0} = f'(0),$$

$b = f'(0)$; and since

$$\frac{d^2}{dx^2}P_2(x)\Big|_{x=0} = f''(0),$$

$2c = f''(0)$. In other words, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$. So $c < 0$ since f is concave down; $b > 0$ since f is increasing; $a > 0$ since $f(0) > 0$.

16. As we can see from Problem 15, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.
So $a > 0$, $b < 0$ and $c < 0$.

17. As we can see from Problem 15, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.
So $a < 0$, $b > 0$ and $c > 0$.

18. As we can see from Problem 15, a is the y-intercept of $f(x)$, b is the slope of the tangent line to $f(x)$ at $x = 0$ and c tells us the concavity of $f(x)$ near $x = 0$.
So $a < 0$, $b < 0$ and $c > 0$.

19. Using the fact that

$$f(x) \approx P_6(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \frac{f^{(6)}(0)}{6!}x^6$$

and identifying coefficients with those given for $P_6(x)$, we obtain the following:

(a) $f(0)$ = constant term which equals 0, so $f(0) = 0$.
(b) $f'(0)$ = coefficient of x which equals 3, so $f'(0) = 3$.
(c) $\frac{f'''(0)}{3!}$ = coefficient of x^3 which equals -4, so $f'''(0) = -24$.
(d) $\frac{f^{(5)}(0)}{5!}$ = coefficient of x^5 which equals 0, so $f^{(5)}(0) = 0$.
(e) $\frac{f^{(6)}(0)}{6!}$ = coefficient of x^6 which equals 5, so $f^{(6)}(0) = 5(6!) = 3600$.

20. (a) We have

$$g(x) = g(5) + g'(5)(x-5) + \frac{g''(5)}{2!}(x-5)^2 + \frac{g'''(5)}{3!}(x-5)^3 + \ldots$$

Substituting gives

$$g(x) = 3 - 2(x-5) + \frac{1}{2!}(x-5)^2 - \frac{3}{3!}(x-5)^3 + \ldots$$

The degree 2 Taylor polynomial, $P_2(x)$, is obtained by truncating after the $(x-5)^2$ term:

$$P_2(x) = 3 - 2(x-5) + \frac{1}{2}(x-5)^2.$$

The degree 3 Taylor polynomial, $P_3(x)$, is obtained by truncating after the $(x-5)^3$ term:

$$P_3(x) = 3 - 2(x-5) + \frac{1}{2}(x-5)^2 - \frac{1}{2}(x-5)^3.$$

(b) Substitute $x = 4.9$ into the Taylor polynomial of degree 2:

$$P_2(4.9) = 3 - 2(4.9-5) + \frac{1}{2}(4.9-5)^2 = 3.205.$$

From the Taylor polynomial of degree 3, we obtain

$$P_3(4.9) = 3 - 2(4.9-5) + \frac{1}{2}(4.9-5)^2 - \frac{1}{2}(4.9-5)^3 = 3.2055.$$

21.

$$\begin{aligned} f(x) &= 4x^2 - 7x + 2 & f(0) &= 2 \\ f'(x) &= 8x - 7 & f'(0) &= -7 \\ f''(x) &= 8 & f''(0) &= 8, \end{aligned}$$

so $P_2(x) = 2 + (-7)x + \frac{8}{2}x^2 = 4x^2 - 7x + 2$. We notice that $f(x) = P_2(x)$ in this case.

22. $f'(x) = 3x^2 + 14x - 5$, $f''(x) = 6x + 14$, $f'''(x) = 6$. Thus, about $a = 0$,

$$\begin{aligned} P_3(x) &= 1 + \frac{-5}{1!}x + \frac{14}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 - 5x + 7x^2 + x^3 \\ &= f(x). \end{aligned}$$

23. **(a)** We'll make the following conjecture:
"If $f(x)$ is a polynomial of degree n, i.e.

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n,$$

then $P_n(x)$, the n^{th} degree Taylor polynomial for $f(x)$ about $x = 0$, is $f(x)$ itself."

(b) All we need to do is to calculate $P_n(x)$, the n^{th} degree Taylor polynomial for f about $x = 0$ and see if it is the same as $f(x)$.

$$\begin{aligned} f(0) &= a_0; \\ f'(0) &= \left(a_1 + 2a_2x + \cdots + na_nx^{n-1}\right)\Big|_{x=0} \\ &= a_1; \\ f''(0) &= \left(2a_2 + 3\cdot 2a_3x + \cdots + n(n-1)a_nx^{n-2}\right)\Big|_{x=0} \\ &= 2!a_2. \end{aligned}$$

If we continue doing this, we'll see in general

$$f^{(k)}(0) = k!a_k, \qquad k = 1, 2, 3, \cdots, n.$$

Therefore,

$$\begin{aligned} P_n(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\ &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \\ &= f(x). \end{aligned}$$

24.

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x - \frac{x^3}{3!}}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!}\right) = 1.$$

25.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1 - (1 - \frac{x^2}{2!} + \frac{x^4}{4!})}{x^2} = \lim_{x\to 0}\left(\frac{1}{2} - \frac{x^2}{4!}\right) = \frac{1}{2}.$$

26. For $f(h) = e^h$, $P_4(h) = 1 + h + \dfrac{h^2}{2} + \dfrac{h^3}{3!} + \dfrac{h^4}{4!}$. So

(a)

$$\begin{aligned} \lim_{h\to 0}\frac{e^h - 1 - h}{h^2} &= \lim_{h\to 0}\frac{P_4(h) - 1 - h}{h^2} \\ &= \lim_{h\to 0}\frac{\frac{h^2}{2} + \frac{h^3}{3!} + \frac{h^4}{4!}}{h^2} \\ &= \lim_{h\to 0}\left(\frac{1}{2} + \frac{h}{3!} + \frac{h^2}{4!}\right) \\ &= \frac{1}{2}. \end{aligned}$$

(b)

$$\begin{aligned}\lim_{h\to 0}\frac{e^h-1-h-\frac{h^2}{2}}{h^3} &= \lim_{h\to 0}\frac{P_4(h)-1-h-\frac{h^2}{2}}{h^3}\\ &= \lim_{h\to 0}\frac{\frac{h^3}{3!}+\frac{h^4}{4!}}{h^3} = \lim_{h\to 0}\left(\frac{1}{3!}+\frac{h}{4!}\right)\\ &= \frac{1}{3!} = \frac{1}{6}.\end{aligned}$$

Using Taylor polynomials of higher degree would not have changed the results since the terms with higher powers of h all go to zero as $h \to 0$.

27. (a) We use the Taylor polynomial of degree two for f and h about $x = 2$.

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \frac{3}{2}(x-2)^2$$
$$h(x) \approx h(2) + h'(2)(x-2) + \frac{h''(2)}{2!}(x-2)^2 = \frac{7}{2}(x-2)^2$$

Thus, using the fact that near $x = 2$ we can approximate a function by Taylor polynomials

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{\frac{3}{2}(x-2)^2}{\frac{7}{2}(x-2)^2} = \frac{3}{7}.$$

(b) We use the Taylor polynomial of degree two for f and g about $x = 2$.

$$f(x) \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 = \frac{3}{2}(x-2)^2$$
$$g(x) \approx g(2) + g'(2)(x-2) + \frac{g''(2)}{2!}(x-2)^2 = 22(x-2) + \frac{5}{2}(x-2)^2.$$

Thus,

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\left(\frac{\frac{3}{2}(x-2)^2}{22(x-2)+5(x-2)^2}\right) = \lim_{x\to 2}\left(\frac{\frac{3}{2}(x-2)}{22+5(x-2)}\right) = \frac{0}{22} = 0.$$

28. Let $f(x)$ be a function that has derivatives up to order n at $x = a$. Let

$$P_n(x) = C_0 + C_1(x-a) + \cdots + C_n(x-a)^n$$

be the polynomial of degree n that approximates $f(x)$ about $x = a$. We require that $P_n(x)$ and all of its first n derivatives agree with those of the function $f(x)$ at $x = a$, i.e., we want

$$\begin{aligned}f(a) &= P_n(a),\\ f'(a) &= P_n'(a),\\ f''(a) &= P_n''(a),\\ &\vdots\\ f^{(n)}(a) &= P_n^{(n)}(a).\end{aligned}$$

When we substitute $x = a$ in $P_n(x)$, all the terms except the first drop out, so

$$f(a) = C_0.$$

Now differentiate $P_n(x)$:

$$P_n'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \cdots + nC_n(x-a)^{n-1}.$$

Substitute $x = a$ again, which yields

$$f'(a) = P_n'(a) = C_1.$$

Differentiate $P_n'(x)$:

$$P_n''(x) = 2C_2 + 3\cdot 2C_3(x-a) + \cdots + n(n-1)C_n(x-a)^{n-2}$$

and substitute $x = a$ again:

$$f''(a) = P_n''(a) = 2C_2.$$

Differentiating and substituting again gives

$$f'''(a) = P_n'''(a) = 3 \cdot 2C_3.$$

Similarly,

$$f^{(k)}(a) = P_n{}^{(k)}(a) = k!C_k.$$

So, $C_0 = f(a)$, $C_1 = f'(a)$, $C_2 = \frac{f''(a)}{2!}$, $C_3 = \frac{f'''(a)}{3!}$, and so on.

If we adopt the convention that $f^{(0)}(a) = f(a)$ and $0! = 1$, then

$$C_k = \frac{f^{(k)}(a)}{k!},\ k = 0, 1, 2, \cdots, n.$$

Therefore,

$$\begin{aligned} f(x) \approx P_n(x) &= C_0 + C_1(x-a) + C_2(x-a)^2 \cdots + C_n(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

29. **(a)** $f(x) = e^{x^2}$.
$f'(x) = 2xe^{x^2}$, $f''(x) = 2(1+2x^2)e^{x^2}$, $f'''(x) = 4(3x+2x^3)e^{x^2}$,
$f^{(4)}(x) = 4(3+6x^2)e^{x^2} + 4(3x+2x^3)2xe^{x^2}$.
The Taylor polynomial about $x = 0$ is

$$\begin{aligned} P_4(x) &= 1 + \frac{0}{1!}x + \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 \\ &= 1 + x^2 + \frac{1}{2}x^4. \end{aligned}$$

(b) $f(x) = e^x$. The Taylor polynomial of degree 2 is

$$Q_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} = 1 + x + \frac{1}{2}x^2.$$

If we substitute x^2 for x in the Taylor polynomial for e^x of degree 2, we will get $P_4(x)$, the Taylor polynomial for e^{x^2} of degree 4:

$$\begin{aligned} Q_2(x^2) &= 1 + x^2 + \frac{1}{2}(x^2)^2 \\ &= 1 + x^2 + \frac{1}{2}x^4 \\ &= P_4(x). \end{aligned}$$

(c) Let $Q_{10}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!}$ be the Taylor polynomial of degree 10 for e^x about $x = 0$. Then

$$\begin{aligned} P_{20}(x) &= Q_{10}(x^2) \\ &= 1 + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \cdots + \frac{(x^2)^{10}}{10!} \\ &= 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots + \frac{x^{20}}{10!}. \end{aligned}$$

(d) Let $e^x \approx Q_5(x) = 1 + \frac{x}{1!} + \cdots + \frac{x^5}{5!}$. Then

$$\begin{aligned} e^{-2x} &\approx Q_5(-2x) \\ &= 1 + \frac{-2x}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} + \frac{(-2x)^5}{5!} \\ &= 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 - \frac{4}{15}x^5. \end{aligned}$$

30. **(a)** The equation $\sin x = 0.2$ has one solution near $x = 0$ and infinitely many others, one near each multiple of π. See Figure 10.1. The equation $x - \dfrac{x^3}{3!} = 0.2$ has three solutions, one near $x = 0$ and two others. See Figure 10.2.

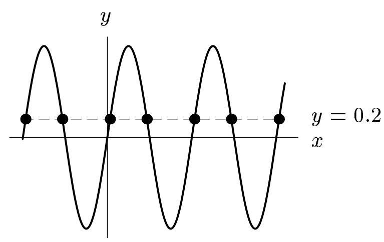


Figure 10.1: Graph of $y = \sin x$ and $y = 0.2$

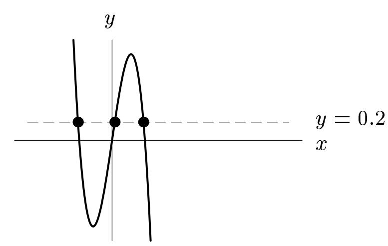


Figure 10.2: Graph of $y = x - \frac{x^3}{3!}$ and $y = 0.2$

(b) Near $x = 0$, the cubic Taylor polynomial $x - x^3/3! \approx \sin x$. Thus, the solutions to the two equations near $x = 0$ are approximately equal. The other solutions are not close. The reason is that $x - x^3/3!$ only approximates $\sin x$ near $x = 0$ but not further away. See Figure 10.3.

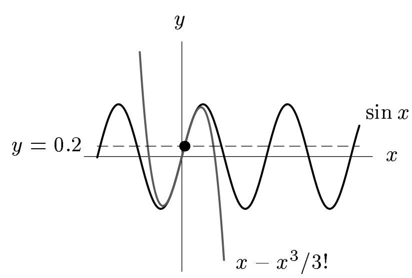


Figure 10.3

31. **(a)** $\dfrac{\sin t}{t} \approx \dfrac{t - \frac{t^3}{3!}}{t} = 1 - \dfrac{t^2}{6}$

$$\int_0^1 \frac{\sin t}{t}\,dt \approx \int_0^1 \left(1 - \frac{t^2}{6}\right)\,dt = t - \frac{t^3}{18}\bigg|_0^1 = 0.94444\cdots$$

(b) $\dfrac{\sin t}{t} \approx \dfrac{t - \frac{t^3}{3!} + \frac{t^5}{5!}}{t} = 1 - \dfrac{t^2}{6} + \dfrac{t^4}{120}$

$$\int_0^1 \frac{\sin t}{t}\,dt \approx \int_0^1 \left(1 - \frac{t^2}{6} + \frac{t^4}{120}\right)\,dt = t - \frac{t^3}{18} + \frac{t^5}{600}\bigg|_0^1 = 0.94611\cdots$$

32. **(a)** Since the coefficient of the x-term of each f is 1, we know $f_1'(0) = f_2'(0) = f_3'(0) = 1$. Thus, each of the fs slopes upward near 0, and are in the second figure.

The coefficient of the x-term in g_1 and in g_2 is 1, so $g_1'(0) = g_2'(0) = 1$. For g_3 however, $g_3'(0) = -1$. Thus, g_1 and g_2 slope up near 0, but g_3 slopes down. The gs are in the first figure.

(b) Since $g_1(0) = g_2(0) = g_3(0) = 1$, the point A is $(0, 1)$.
Since $f_1(0) = f_2(0) = f_3(0) = 2$, the point B is $(0, 2)$.

(c) Since g_3 slopes down, g_3 is I. Since the coefficient of x^2 for g_1 is 2, we know

$$\frac{g_1''(0)}{2!} = 2 \qquad \text{so} \qquad g_1''(0) = 4.$$

By similar reasoning $g_2''(0) = 2$. Since g_1 and g_2 are concave up, and g_1 has a larger second derivative, g_1 is III and g_2 is II.

Calculating the second derivatives of the fs from the coefficients x^2, we find

$$f_1''(0) = 4 \qquad f_2''(0) = -2 \qquad f_3''(0) = 2.$$

Thus, f_1 and f_3 are concave up, with f_1 having the larger second derivative, so f_1 is III and f_3 is II. Then f_2 is concave down and is I.

Solutions for Section 10.2

Exercises

1.

$$\begin{aligned} f(x) &= \tfrac{1}{1-x} = (1-x)^{-1} & f(0) &= 1, \\ f'(x) &= -(1-x)^{-2}(-1) = (1-x)^{-2} & f'(0) &= 1, \\ f''(x) &= -2(1-x)^{-3}(-1) = 2(1-x)^{-3} & f''(0) &= 2, \\ f'''(x) &= -6(1-x)^{-4}(-1) = 6(1-x)^{-4} & f'''(0) &= 6. \end{aligned}$$

$$\begin{aligned} f(x) = \frac{1}{1-x} &= 1 + 1\cdot x + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \cdots \\ &= 1 + x + x^2 + x^3 + \cdots \end{aligned}$$

2.

$$\begin{aligned} f(x) &= \tfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} & f(0) &= 1 \\ f'(x) &= -\tfrac{1}{2}(1+x)^{-\frac{3}{2}} & f'(0) &= -\tfrac{1}{2} \\ f''(x) &= \tfrac{3}{4}(1+x)^{-\frac{5}{2}} & f''(0) &= \tfrac{3}{4} \\ f'''(x) &= -\tfrac{15}{8}(1+x)^{-\frac{7}{2}} & f'''(0) &= -\tfrac{15}{8} \end{aligned}$$

$$\begin{aligned} f(x) = \frac{1}{\sqrt{1+x}} &= 1 + \left(-\frac{1}{2}\right)x + \frac{(\frac{3}{4})x^2}{2!} + \frac{(-\frac{15}{8})x^3}{3!} + \cdots \\ &= 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots \end{aligned}$$

3.

$$\begin{aligned} f(x) &= \sqrt{1+x} = (1+x)^{\frac{1}{2}} & f(0) &= 1 \\ f'(x) &= \tfrac{1}{2}(1+x)^{-\frac{1}{2}} & f'(0) &= \tfrac{1}{2} \\ f''(x) &= -\tfrac{1}{4}(1+x)^{-\frac{3}{2}} & f''(0) &= -\tfrac{1}{4} \\ f'''(x) &= \tfrac{3}{8}(1+x)^{-\frac{5}{2}} & f'''(0) &= \tfrac{3}{8} \end{aligned}$$

$$\begin{aligned} f(x) = \sqrt{1+x} &= 1 + \frac{1}{2}x + \frac{(-\frac{1}{4})x^2}{2!} + \frac{(\frac{3}{8})x^3}{3!} + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \cdots \end{aligned}$$

4.

$$\begin{aligned} f(y) &= \sqrt[3]{1-y} = (1-y)^{\frac{1}{3}} & f(0) &= 1 \\ f'(y) &= \tfrac{1}{3}(1-y)^{-\frac{2}{3}}(-1) = -\tfrac{1}{3}(1-y)^{-\frac{2}{3}} & f'(0) &= -\tfrac{1}{3} \\ f''(y) &= \tfrac{2}{9}(1-y)^{-\frac{5}{3}}(-1) = -\tfrac{2}{9}(1-y)^{-\frac{5}{3}} & f''(0) &= \tfrac{2}{9} \\ f'''(y) &= \tfrac{10}{27}(1-y)^{-\frac{8}{3}}(-1) = -\tfrac{10}{27}(1-y)^{-\frac{8}{3}} & f'''(0) &= -\tfrac{10}{27} \end{aligned}$$

$$\begin{aligned} f(y) = \sqrt[3]{1-y} &= 1 + \left(-\frac{1}{3}\right)y + \frac{(-\frac{2}{9})y^2}{2!} + \frac{(-\frac{10}{27})y^3}{3!} + \cdots \\ &= 1 - \frac{y}{3} - \frac{y^2}{9} - \frac{5y^3}{81} - \cdots \end{aligned}$$

5.

$$\begin{aligned} f(x) &= \sin x & f(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'(x) &= \cos x & f'(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f''(x) &= -\sin x & f''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'''(x) &= -\cos x & f'''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned}\sin x &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^2}{2!} - \frac{\sqrt{2}}{2}\frac{(x-\frac{\pi}{4})^3}{3!} - \cdots \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 - \cdots\end{aligned}$$

6.

$$\begin{aligned} f(\theta) &= \cos\theta & f(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'(\theta) &= -\sin\theta & f'(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f''(\theta) &= -\cos\theta & f''(\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'''(\theta) &= \sin\theta & f'''(\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned}\cos\theta &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\theta - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2}\frac{(\theta-\frac{\pi}{4})^2}{2!} + \frac{\sqrt{2}}{2}\frac{(\theta-\frac{\pi}{4})^3}{3!} - \cdots \\ &= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(\theta - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(\theta - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12}\left(\theta - \frac{\pi}{4}\right)^3 - \cdots\end{aligned}$$

7.

$$\begin{aligned} f(\theta) &= \sin\theta & f(-\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}, \\ f'(\theta) &= \cos\theta & f'(-\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f''(\theta) &= -\sin\theta & f''(-\tfrac{\pi}{4}) &= \tfrac{\sqrt{2}}{2}, \\ f'''(\theta) &= -\cos\theta & f'''(-\tfrac{\pi}{4}) &= -\tfrac{\sqrt{2}}{2}. \end{aligned}$$

$$\begin{aligned}\sin\theta &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\theta + \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\frac{(\theta+\frac{\pi}{4})^2}{2!} - \frac{\sqrt{2}}{2}\frac{(\theta+\frac{\pi}{4})^3}{3!} + \cdots \\ &= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\theta + \frac{\pi}{4}\right) + \frac{\sqrt{2}}{4}\left(\theta + \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(\theta + \frac{\pi}{4}\right)^3 + \cdots.\end{aligned}$$

8.

$$\begin{aligned} f(x) &= \tan x & f(\tfrac{\pi}{4}) &= 1, \\ f'(x) &= \tfrac{1}{\cos^2 x} & f'(\tfrac{\pi}{4}) &= 2, \\ f''(x) &= \tfrac{-2(-\sin x)}{\cos^3 x} = \tfrac{2\sin x}{\cos^3 x} & f''(\tfrac{\pi}{4}) &= 4, \\ f'''(x) &= \tfrac{-6\sin x(-\sin x)}{\cos^4 x} + \tfrac{2}{\cos^2 x} & f'''(\tfrac{\pi}{4}) &= 16. \end{aligned}$$

$$\begin{aligned}\tan x &= 1 + 2\left(x - \frac{\pi}{4}\right) + 4\frac{(x-\frac{\pi}{4})^2}{2!} + 16\frac{\left(x-\frac{\pi}{4}\right)^3}{3!} + \cdots \\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \cdots\end{aligned}$$

9.

$$\begin{aligned} f(x) &= \tfrac{1}{x} & f(1) &= 1 \\ f'(x) &= -\tfrac{1}{x^2} & f'(1) &= -1 \\ f''(x) &= \tfrac{2}{x^3} & f''(1) &= 2 \\ f'''(x) &= -\tfrac{6}{x^4} & f'''(1) &= -6 \end{aligned}$$

$$\begin{aligned}\frac{1}{x} &= 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{6(x-1)^3}{3!} + \cdots \\ &= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots.\end{aligned}$$

10. Again using the derivatives found in Problem 9, we have

$$f(2)=\frac{1}{2},\qquad f'(2)=-\frac{1}{4},\qquad f''(2)=\frac{1}{4},\qquad f'''(2)=-\frac{3}{8}.$$

$$\begin{aligned}\frac{1}{x}&=\frac{1}{2}-\frac{x-2}{4}+\frac{(x-2)^2}{4\cdot 2!}-\frac{3(x-2)^3}{8\cdot 3!}+\cdots\\&=\frac{1}{2}-\frac{(x-2)}{4}+\frac{(x-2)^2}{8}-\frac{(x-2)^3}{16}+\cdots\end{aligned}$$

11. Using the derivatives from Problem 9, we have

$$f(-1)=-1,\quad f'(-1)=-1,\quad f''(-1)=-2,\quad f'''(-1)=-6.$$

Hence,

$$\begin{aligned}\frac{1}{x}&=-1-(x+1)-\frac{2(x+1)^2}{2!}-\frac{6(x+1)^3}{3!}-\cdots\\&=-1-(x+1)-(x+1)^2-(x+1)^3-\cdots\end{aligned}$$

12. The general term can be written as x^n for $n\geq 0$.

13. The general term can be written as $(-1)^n x^n$ for $n\geq 0$.

14. The general term can be written as $-x^n/n$ for $n\geq 1$.

15. The general term can be written as $(-1)^{n-1}x^n/n$ for $n\geq 1$.

16. The general term can be written as $(-1)^k x^{2k+1}/(2k+1)!$ for $k\geq 0$.

17. The general term can be written as $(-1)^k x^{2k+1}/(2k+1)$ for $k\geq 0$.

18. The general term can be written as $x^{2k}/k!$ for $k\geq 0$.

19. The general term can be written as $(-1)^k x^{2k+2}/(2k)!$ for $k\geq 0$.

Problems

20. **(a)**

$$\begin{aligned}f(x)&=\sin x^2\\f'(x)&=(\cos x^2)2x\\f''(x)&=(-\sin x^2)4x^2+(\cos x^2)2\\f'''(x)&=(-\cos x^2)8x^3+(-\sin x^2)8x+(-\sin x^2)4x\\&=(-\cos x^2)8x^3+(-\sin x^2)12x\\f^{(4)}(x)&=(\sin x^2)16x^4+(-\cos x^2)24x^2+(-\cos x^2)24x^2+(-\sin x^2)12\\&=(\sin x^2)16x^4+(-\cos x^2)48x^2+(-\sin x^2)12\\f^{(5)}(x)&=(\cos x^2)32x^5+(\sin x^2)64x^3+(\sin x^2)96x^3\\&\quad+(-\cos x^2)96x+(-\cos x^2)24x\\&=(\cos x^2)32x^5+(\sin x^2)160x^3+(-\cos x^2)120x\\f^{(6)}(x)&=(-\sin x^2)64x^6+(\cos x^2)160x^4+(\cos x^2)320x^4+(\sin x^2)480x^2\\&\quad+(\sin x^2)240x^2+(-\cos x^2)120\\&=(-\sin x^2)64x^6+(\cos x^2)480x^4+(\sin x^2)720x^2+(-\cos x^2)120\end{aligned}$$

So,

$$\begin{aligned}f(0)&=0 & f^{(4)}(0)&=0,\\f'(0)&=0 & f^{(5)}(0)&=0,\\f''(0)&=2 & f^{(6)}(0)&=-120,\\f'''(0)&=0.\end{aligned}$$

Thus

$$f(x) = \sin x^2 = \frac{2}{2!}x^2 - \frac{120}{6!}x^6 + \cdots$$
$$= x^2 - \frac{1}{3!}x^6 + \cdots.$$

As we can see, the amount of calculation in order to find the higher derivatives of $\sin x^2$ increases very rapidly. In fact, the next non-zero term in the Taylor expansion of $\sin x^2$ is the 10^{th} derivative term, which really requires a lot of work to get.

(b)

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

The first couple of coefficients of the above expansion are the same as those in part (a). If we substitute x^2 for x in the Taylor expansion of $\sin x$, we should get the Taylor expansion of $\sin x^2$.

$$\sin x^2 = x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \cdots$$
$$= x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \cdots.$$

21. (a)

$$\begin{aligned} f(x) &= \ln(1+2x) & f(0) &= 0 \\ f'(x) &= \tfrac{2}{1+2x} & f'(0) &= 2 \\ f''(x) &= -\tfrac{4}{(1+2x)^2} & f''(0) &= -4 \\ f'''(x) &= \tfrac{16}{(1+2x)^3} & f'''(0) &= 16 \end{aligned}$$

$$\ln(1+2x) = 2x - 2x^2 + \frac{8}{3}x^3 + \cdots$$

(b) To get the expression for $\ln(1+2x)$ from the series for $\ln(1+x)$, substitute $2x$ for x in the series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

to get

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$$
$$= 2x - 2x^2 + \frac{8x^3}{3} - 4x^4 + \cdots$$

(c) Since the interval of convergence for $\ln(1+x)$ is $-1 < x < 1$, substituting $2x$ for x suggests the interval of convergence of $\ln(1+2x)$ is $-1 < 2x < 1$, or $-\frac{1}{2} < x < \frac{1}{2}$.

22. By looking at Figure 10.4, we see that the Taylor polynomials are reasonable approximations for the function $f(x) = \sqrt{1+x}$ between $x = -1$ and $x = 1$. Thus a good guess is that the interval of convergence is $-1 < x < 1$.

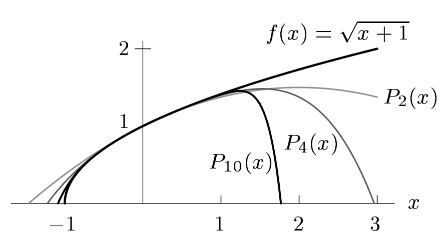


Figure 10.4

23. By looking at Figure 10.5 we can that the Taylor polynomials are reasonable approximations for the function $f(x) = \frac{1}{\sqrt{1+x}}$ between $x = -1$ and $x = 1$. Thus a good guess is that the interval of convergence is $-1 < x < 1$.

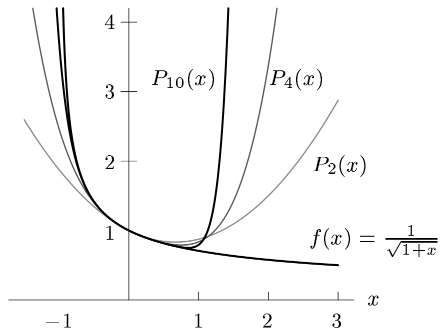


Figure 10.5

24. The graph suggests that the Taylor polynomials converge to $f(x) = \dfrac{1}{1-x}$ on the interval $-1 < x < 1$. See Figure 10.6. Since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots.$$

the ratio test gives

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x^{n+1}|}{|x^n|} = |x|.$$

Thus, the series converges if $|x| < 1|$; that is, $-1 < x < 1$.

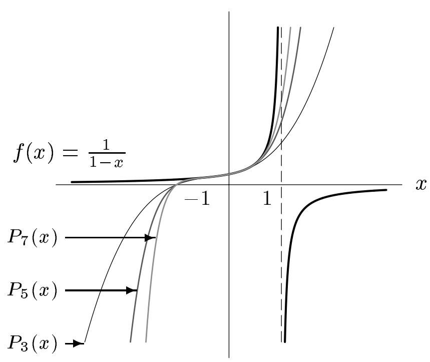


Figure 10.6

25. The Taylor series for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^n}{n} - \cdots,$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{1/(n+1)}{1/n} = |x| \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = |x|.$$

Thus the series converges for $|x| < 1$, and the radius of convergence is 1. Note: This series can be obtained from the series for $\ln(1+x)$ by replacing x by $-x$ and has the same radius of convergence as the series for $\ln(1+x)$.

26. **(a)** We have shown that the series is

$$1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

so the general term is

$$\frac{p(p-1)\ldots(p-(n-1))}{n!}x^n.$$

(b) We use the ratio test

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = |x|\lim_{n\to\infty}\left|\frac{p(p-1)\dots(p-(n-1))(p-n)\cdot n!}{(n+1)!p(p-1)\dots(p-(n-1))}\right| = |x|\lim_{n\to\infty}\left|\frac{p-n}{n+1}\right|.$$

Since p is fixed, we have

$$\lim_{n\to\infty}\left|\frac{p-n}{n+1}\right| = 1, \quad \text{so} \quad R = 1.$$

27. This is the series for e^x with x replaced by 2, so the series converges to e^2.

28. This is the series for $\sin x$ with x replaced by 1, so the series converges to $\sin 1$.

29. This is the series for $1/(1-x)$ with x replaced by $1/4$, so the series converges to $1/(1-(1/4)) = 4/3$.

30. This is the series for $\cos x$ with x replaced by 10, so the series converges to $\cos 10$.

31. This is the series for $\ln(1+x)$ with x replaced by $1/2$, so the series converges to $\ln(3/2)$.

32. The Taylor series for $f(x) = 1/(1+x)$ is

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

Substituting $x = 0.1$ gives

$$1 - 0.1 + (0.1)^2 - (0.1)^3 + \cdots = \frac{1}{1+0.1} = \frac{1}{1.1}.$$

Alternatively, this is a geometric series with $a = 1$, $x = -0.1$.

33. This is the series for e^x with $x = 3$ substituted. Thus

$$1 + 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots = e^3.$$

34. This is the series for $\cos x$ with $x = 1$ substituted. Thus

$$1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots = \cos 1.$$

35. This is the series for e^x with -0.1 substituted for x, so

$$1 - 0.1 + \frac{0.01}{2!} - \frac{0.001}{3!} + \cdots = e^{-0.1}.$$

36. Since $1 + x + x^2 + x^3 + \cdots = \dfrac{1}{1-x}$, a geometric series, we solve $\dfrac{1}{1-x} = 5$ giving $\dfrac{1}{5} = 1 - x$, so $x = \dfrac{4}{5}$.

37. Since $x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \cdots = \ln(1+x)$, we solve $\ln(1+x) = 0.2$, giving $1 + x = e^{0.2}$, so $x = e^{0.2} - 1$.

38. **(a)** From the coefficients of the $(x-1)$ terms of the fs, we see that

$$f_1'(1) = 1, \qquad f_2'(1) = -1 \qquad f_3'(1) = -2.$$

From the $(x-1)^2$ terms of the fs, we see that

$$\frac{f_1''(1)}{2!} = -1, \qquad \frac{f_2''(1)}{2!} = 1, \qquad \frac{f_3''(1)}{2!} = 1,$$

so $f_1''(1) = -2$, $f_2''(1) = 2$, $f_3''(1) = 2$.

Thus, f_1 slopes up at $x = 1$ and f_2 and f_3 slope down; f_3 slopes down more steeply than f_2. This means that the fs are in the first figure, since graphs II and III in the second figure have the same negative slope at point B.

By a similar argument, we find

$$g_1'(4) = -1, \quad g_2'(4) = -1, \quad g_3'(4) = 1, \text{ and } g_1''(4) = -2, \quad g_2''(4) = 2, \quad g_3''(4) = 2.$$

Thus, two of the gs slope down, one of which is concave up and one is concave down; the third g slopes up and is concave up. This confirms that the gs are in the second figure.

(b) Since $f_1(1) = f_2(1) = f_3(1) = 3$, the point A is $(1,3)$.
Since $g_1(4) = g_2(4) = g_3(4) = 5$, the point B is $(4,5)$.

(c) In the first figure, graph I is f_1 since it slopes up. Graph II is f_2 since it slopes down, but less steeply than graph III, which is f_3.

In the second figure, graph I is g_3, since it slopes up. Graph II is g_2 since it slopes down and is concave up. Graph III is g_1 since it slopes down and is concave down.

39. Let C_n be the coefficient of the n^{th} term in the series. Note that

$$0 = C_1 = \frac{d}{dx}(x^2 e^{x^2})\bigg|_{x=0},$$

and since

$$\frac{1}{2} = C_6 = \frac{\frac{d^6}{dx^6}(x^2 e^{x^2})\Big|_{x=0}}{6!},$$

we have

$$\frac{d^6}{dx^6}(x^2 e^{x^2})\bigg|_{x=0} = \frac{6!}{2} = 360.$$

40. Let C_n be the coefficient of the n^{th} term in the series. $C_1 = f'(0)/1!$, so $f'(0) = 1!C_1 = 1 \cdot 1 = 1$.
Similarly, $f''(0) = 2!C_2 = 2! \cdot \frac{1}{2} = 1$;
$f'''(0) = 3!C_3 = 3! \cdot \frac{1}{3} = 2! = 2$;
$f^{(10)}(0) = 10!C_{10} = 10! \cdot \frac{1}{10} = \frac{10!}{10} = 9! = 362880$.

41. We define $e^{i\theta}$ to be

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

Suppose we consider the expression $\cos\theta + i\sin\theta$, with $\cos\theta$ and $\sin\theta$ replaced by their Taylor series:

$$\cos\theta + i\sin\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

Reordering terms, we have

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \cdots$$

Using the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$, we can rewrite the series as

$$\cos\theta + i\sin\theta = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

Amazingly enough, this series is the Taylor series for e^x with $i\theta$ substituted for x. Therefore, we have shown that

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

Solutions for Section 10.3

Exercises

1. We'll use

$$\begin{aligned}\sqrt{1+y} = (1+y)^{\frac{1}{2}} &= 1 + \left(\frac{1}{2}\right)y + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\frac{y^2}{2!} \\ &\quad + \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\frac{y^3}{3!} + \cdots \\ &= 1 + \frac{y}{2} - \frac{y^2}{8} + \frac{y^3}{16} - \cdots.\end{aligned}$$

Substitute $y = -2x$.

$$\begin{aligned}\sqrt{1-2x} &= 1 + \frac{(-2x)}{2} - \frac{(-2x)^2}{8} + \frac{(-2x)^3}{16} - \cdots \\ &= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} - \cdots\end{aligned}$$

2. Substitute $x = \theta^2$ into series for $\cos x$:

$$\cos(\theta^2) = 1 - \frac{(\theta^2)^2}{2!} + \frac{(\theta^2)^4}{4!} - \frac{(\theta^2)^6}{6!} + \cdots$$
$$= 1 - \frac{\theta^4}{2!} + \frac{\theta^8}{4!} - \frac{\theta^{12}}{6!} + \cdots.$$

3. Substitute $y = -x$ into $e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$. We get

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \cdots$$
$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots.$$

4.

$$\frac{t}{1+t} = t(1+t)^{-1} = t\left(1 + (-1)t + \frac{(-1)(-2)}{2!}t^2 + \frac{(-1)(-2)(-3)}{3!}t^3 + \cdots\right)$$
$$= t - t^2 + t^3 - t^4 + \cdots$$

5. Substituting $x = -2y$ into $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ gives

$$\ln(1-2y) = (-2y) - \frac{(-2y)^2}{2} + \frac{(-2y)^3}{3} - \frac{(-2y)^4}{4} + \cdots$$
$$= -2y - 2y^2 - \frac{8}{3}y^3 - 4y^4 - \cdots.$$

6. Since $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots$, integrating gives

$$\arcsin x = c + x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots.$$

Since $\arcsin 0 = 0$, $c = 0$.

7. Substituting $x = -z^2$ into $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$ gives

$$\frac{1}{\sqrt{1-z^2}} = 1 - \frac{(-z^2)}{2} + \frac{3(-z^2)^2}{8} - \frac{5(-z^2)^3}{16} + \cdots$$
$$= 1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \cdots.$$

8.

$$\phi^3\cos(\phi^2) = \phi^3\left(1 - \frac{(\phi^2)^2}{2!} + \frac{(\phi^2)^4}{4!} - \frac{(\phi^2)^6}{6!} + \cdots\right)$$
$$= \phi^3 - \frac{\phi^7}{2!} + \frac{\phi^{11}}{4!} - \frac{\phi^{15}}{6!} + \cdots$$

9.

$$\frac{z}{e^{z^2}} = ze^{-z^2} = z\left(1 + (-z^2) + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!} + \cdots\right)$$
$$= z - z^3 + \frac{z^5}{2!} - \frac{z^7}{3!} + \cdots$$

10.

$$\sqrt{(1+t)}\sin t = \left(1+\frac{t}{2}-\frac{t^2}{8}+\frac{t^3}{16}-\cdots\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\cdots\right)$$

Multiplying and collecting terms yields

$$\begin{aligned}\sqrt{(1+t)}\sin t &= t+\frac{t^2}{2}-\left(\frac{t^3}{3!}+\frac{t^3}{8}\right)+\left(\frac{t^4}{16}-\frac{t^4}{12}\right)+\cdots\\ &= t+\frac{1}{2}t^2-\frac{7}{24}t^3-\frac{1}{48}t^4+\cdots.\end{aligned}$$

11.

$$e^t\cos t = \left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right)$$

Multiplying out and collecting terms gives

$$\begin{aligned}e^t\cos t &= 1+t+\left(\frac{t^2}{2!}-\frac{t^2}{2!}\right)+\left(\frac{t^3}{3!}-\frac{t^3}{2!}\right)+\left(\frac{t^4}{4!}+\frac{t^4}{4!}-\frac{t^4}{(2!)^2}\right)+\cdots\\ &= 1+t-\frac{t^3}{3}-\frac{t^4}{6}+\cdots.\end{aligned}$$

12. Substituting the series for $\sin\theta = \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots$ into

$$\sqrt{1+y} = 1+\frac{1}{2}y-\frac{1}{8}y^2+\frac{1}{16}y^3-\cdots$$

gives

$$\begin{aligned}\sqrt{1+\sin\theta} &= 1+\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)-\frac{1}{8}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)^2\\ &\quad+\frac{1}{16}\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots\right)^3-\cdots\\ &= 1+\theta-\frac{\theta^2}{8}+\left(\frac{\theta^3}{16}-\frac{\theta^3}{3!}\right)+\cdots\\ &= 1+\theta-\frac{1}{8}\theta^2-\frac{5}{48}\theta^3+\cdots\end{aligned}$$

13. Multiplying out gives $(1+x)^3 = 1+3x+3x^2+x^3$. Since this polynomial equals the original function for all x, it must be the Taylor series. The general term is $0\cdot x^n$ for $n\geq 4$.

14. Substituting t^2 into the series for $\sin x$ gives

$$\begin{aligned}\sin(t^2) &= t^2-\frac{(t^2)^3}{3!}+\frac{(t^2)^5}{5!}+\cdots+\frac{(-1)^k(t^2)^{2k+1}}{(2k+1)!}+\cdots\\ &= t^2-\frac{t^6}{3!}+\frac{t^{10}}{5!}+\cdots+\frac{(-1)^k t^{4k+2}}{(2k+1)!}+\cdots\end{aligned}$$

Therefore

$$\begin{aligned}t\sin(t^2)-t^3 &= \left(t^3-\frac{t^7}{3!}+\frac{t^{11}}{5!}+\cdots+\frac{(-1)^k t^{4k+3}}{(2k+1)!}+\cdots\right)-t^3\\ &= -\frac{t^7}{3!}+\frac{t^{11}}{5!}+\cdots+\frac{(-1)^k t^{4k+3}}{(2k+1)!}+\cdots \quad \text{for } k\geq 1.\end{aligned}$$

15. Using the Binomial theorem:

$$\begin{aligned}
&\frac{1}{\sqrt{1-x}}\\
&= (1-x)^{-1/2}\\
&= 1 + \left(-\frac{1}{2}\right)(-x) + \frac{(-1/2)(-3/2)(-x)^2}{2!} + \cdots + \frac{(-1/2)(-3/2)\cdots(-\frac{1}{2}-n+1)(-x)^n}{n!} + \cdots \text{ for } n \geq 1.
\end{aligned}$$

Substituting y^2 for x:

$$\begin{aligned}
\frac{1}{\sqrt{1-y^2}} &= (1-y^2)^{-1/2}\\
&= 1 + \frac{1}{2}y^2 + \frac{3}{8}y^4 + \cdots + \frac{(1/2)(3/2)\cdots(\frac{1}{2}+n-1)y^{2n}}{n!} + \cdots \text{ for } n \geq 1.
\end{aligned}$$

16.

$$\begin{aligned}
\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} &= \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}\\
&= \frac{1}{2}\left(1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 + \cdots\right)
\end{aligned}$$

17.

$$\begin{aligned}
\frac{a}{\sqrt{a^2+x^2}} = \frac{a}{a(1+\frac{x^2}{a^2})^{\frac{1}{2}}} &= \left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}}\\
&= 1 + \left(-\frac{1}{2}\right)\frac{x^2}{a^2} + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x^2}{a^2}\right)^2\\
&\quad + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x^2}{a^2}\right)^3 + \cdots\\
&= 1 - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{3}{8}\left(\frac{x}{a}\right)^4 - \frac{5}{16}\left(\frac{x}{a}\right)^6 + \cdots
\end{aligned}$$

Problems

18. **(a)** Writing

$$f(x) = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

and using the Binomial expansion, we have

$$f(x) \approx P_2(x) = b\left(1 - \frac{1}{2}\frac{x^2}{a^2}\right) = b - \frac{bx^2}{2a^2}.$$

(b) A graph of the upper half the ellipse is shown in Figure 10.7. Since the graph has a horizontal tangent at $x = 0$, the coefficient of x is 0.

(c) The parabola is

$$y = b - \frac{bx^2}{2a^2}.$$

Its x-intercepts are $x = \pm\sqrt{2}a$.

(d) The graphs of

$$y = f(x) = 2\sqrt{1 - \frac{x^2}{9}} \quad \text{and} \quad y = 2 - \frac{x^2}{9}$$

are shown in Figure 10.8. The maximum difference occurs at $x = 0.1$ or $x = -0.1$, so

$$\text{Maximum error} = 2 - \frac{(0.1)^2}{9} - 2\sqrt{1 - \frac{(0.1)^2}{9}} \approx 3 \cdot 10^{-7}.$$

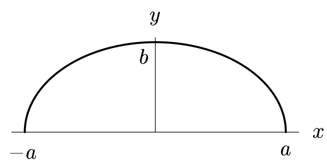


Figure 10.7: Graph of $y = b\sqrt{1 - x^2/a^2}$

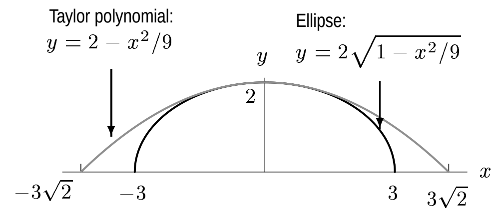


Figure 10.8

19. The Taylor expansion about $\theta = 0$ for $\sin\theta$ is

$$\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

So

$$1 + \sin\theta = 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

The Taylor expansion about $\theta = 0$ for $\cos\theta$ is

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots.$$

The Taylor expansion for $\dfrac{1}{1+\theta}$ about $\theta = 0$ is

$$\frac{1}{1+\theta} = 1 - \theta + \theta^2 - \theta^3 + \theta^4 - \cdots.$$

So, substituting $-\theta^2$ for θ:

$$\begin{aligned}\frac{1}{1-\theta^2} &= 1 - (-\theta^2) + (-\theta^2)^2 - (-\theta^2)^3 + (-\theta^2)^4 + \cdots \\ &= 1 + \theta^2 + \theta^4 + \theta^6 + \theta^8 + \cdots.\end{aligned}$$

For small θ, we can neglect the terms above quadratic in these expansions, giving:

$$\begin{aligned}1 + \sin\theta &\approx 1 + \theta \\ \cos\theta &\approx 1 - \frac{\theta^2}{2} \\ \frac{1}{1-\theta^2} &\approx 1 + \theta^2.\end{aligned}$$

For all $\theta \neq 0$, we have

$$1 - \frac{\theta^2}{2} < 1 + \theta^2.$$

Also, since $\theta^2 < \theta$ for $0 < \theta < 1$, we have

$$1 - \frac{\theta^2}{2} < 1 + \theta^2 < 1 + \theta.$$

So, for small positive θ, we have

$$\cos\theta < \frac{1}{1-\theta^2} < 1 + \sin\theta.$$

20. From the series for $\ln(1+y)$,

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots,$$

we get

$$\ln(1+y^2) = y^2 - \frac{y^4}{2} + \frac{y^6}{3} - \frac{y^8}{4} + \cdots$$

The Taylor series for $\sin y$ is

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$$

So

$$\sin y^2 = y^2 - \frac{y^6}{3!} + \frac{y^{10}}{5!} - \frac{y^{14}}{7!} + \cdots$$

The Taylor series for $\cos y$ is

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots$$

So

$$1 - \cos y = \frac{y^2}{2!} - \frac{y^4}{4!} + \frac{y^6}{6!} + \cdots$$

Near $y = 0$, we can drop terms beyond the fourth degree in each expression:

$$\begin{aligned} \ln(1+y^2) &\approx y^2 - \frac{y^4}{2} \\ \sin y^2 &\approx y^2 \\ 1 - \cos y &\approx \frac{y^2}{2!} - \frac{y^4}{4!}. \end{aligned}$$

(Note: These functions are all even, so what holds for negative y will hold for positive y.)
Clearly $1 - \cos y$ is smallest, because the y^2 term has a factor of $\frac{1}{2}$. Thus, for small y,

$$\frac{y^2}{2!} - \frac{y^4}{4!} < y^2 - \frac{y^4}{2} < y^2$$

so

$$1 - \cos y < \ln(1+y^2) < \sin(y^2).$$

21. The Taylor series about 0 for $y = \dfrac{1}{1-x^2}$ is

$$y = 1 + x^2 + x^4 + x^6 + \cdots.$$

The series for $y = (1+x)^{1/4}$ is, using the binomial expansion,

$$y = 1 + \frac{1}{4}x + \frac{1}{4}\left(-\frac{3}{4}\right)\frac{x^2}{2!} + \frac{1}{4}\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)\frac{x^3}{3!} + \cdots.$$

The series for $y = \sqrt{1+\dfrac{x}{2}} = (1+\dfrac{x}{2})^{1/2}$ is, again using the binomial expansion,

$$y = 1 + \frac{1}{2}\cdot\frac{x}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\cdot\frac{x^2}{8} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{x^3}{48} + \cdots.$$

Similarly for $y = \dfrac{1}{\sqrt{1-x}} = (1-x)^{-(1/2)}$,

$$y = 1 + \left(-\frac{1}{2}\right)(-x) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{x^2}{2!} + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdot\frac{-x^3}{3!} + \cdots.$$

Near 0, let's truncate these series after their x^2 terms:

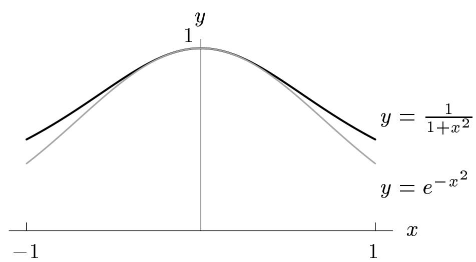
$$\frac{1}{1-x^2} \approx 1+x^2,$$
$$(1+x)^{1/4} \approx 1+\frac{1}{4}x-\frac{3}{32}x^2,$$
$$\sqrt{1+\frac{x}{2}} \approx 1+\frac{1}{4}x-\frac{1}{32}x^2,$$
$$\frac{1}{\sqrt{1-x}} \approx 1+\frac{1}{2}x+\frac{3}{8}x^2.$$

Thus $\frac{1}{1-x^2}$ looks like a parabola opening upward near the origin, with y-axis as the axis of symmetry, so (a) = I.

Now $\frac{1}{\sqrt{1-x}}$ has the largest positive slope ($\frac{1}{2}$), and is concave up (because the coefficient of x^2 is positive). So (d) = II.

The last two both have positive slope ($\frac{1}{4}$) and are concave down. Since $(1+x)^{\frac{1}{4}}$ has the smallest second derivative (i.e., the most negative coefficient of x^2), (b) = IV and therefore (c) = III.

22.



(a)

$$e^{-x^2} = 1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$$
$$\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots$$

Notice that the first two terms are the same in both series.

(b) $\dfrac{1}{1+x^2}$ is greater.

(c) Even, because the only terms involved are of even degree.

(d) The coefficients for e^{-x^2} become extremely small for higher powers of x, and we can "counteract" the effect of these powers for large values of x. The series for $\frac{1}{1+x^2}$ has no such coefficients.

23. (a) The Taylor approximation to $f(x) = \cosh x$ about $x = 0$ is of the form

$$\cosh x \approx \cosh(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots + \frac{f^{(n)}(0)x^n}{n!}.$$

We have the following results:

$$f(x) = \cosh x \quad \text{so } f(0) = 1,$$
$$f'(x) = \sinh x \quad \text{so } f'(0) = 0,$$
$$f''(x) = \frac{d}{dx}(\sinh x) = \cosh x \quad \text{so } f''(0) = 1,$$
$$f'''(x) = \sinh x \quad \text{so } f'''(0) = 0.$$

The derivatives continue to alternate between $\cosh x$ and $\sinh x$, so their values at 0 continue to alternate between 0 and 1. Therefore

$$\cosh x \approx 1 + 0\cdot x + 1\cdot\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} + 1\cdot\frac{x^4}{4!} + \cdots,$$

so the degree 8 Taylor approximation is given by

$$\cosh x \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}.$$

(b) We use the polynomial obtained from part (a) to estimate $\cosh 1$,

$$\cosh 1 \approx 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} = 1.543080357.$$

Compared to the actual value of $\cosh 1 = 1.543080635\ldots$, the error is less than 10^{-6}.

(c) Since $\dfrac{d}{dx}(\cosh x) = \sinh x$, we have

$$\begin{aligned}\sinh x &\approx \frac{d}{dx}\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}\right)\\ &= \frac{2x}{2!} + \frac{4x^3}{4!} + \frac{6x^5}{6!} + \frac{8x^7}{8!}\\ &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}.\end{aligned}$$

24. (a)

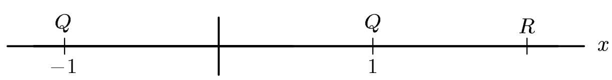
$$\begin{aligned}f(x) &= (1+ax)(1+bx)^{-1} = (1+ax)\left(1 - bx + (bx)^2 - (bx)^3 + \cdots\right)\\ &= 1 + (a-b)x + (b^2 - ab)x^2 + \cdots\end{aligned}$$

(b) $e^x = 1 + x + \frac{x^2}{2} + \cdots$
Equating coefficients:

$$\begin{aligned}a - b &= 1,\\ b^2 - ab &= \frac{1}{2}.\end{aligned}$$

Solving gives $a = \frac{1}{2}, b = -\frac{1}{2}$.

25.



$$\begin{aligned}E &= kQ\left(\frac{1}{(R-1)^2} - \frac{1}{(R+1)^2}\right)\\ &= \frac{kQ}{R^2}\left(\frac{1}{(1-\frac{1}{R})^2} - \frac{1}{(1+\frac{1}{R})^2}\right)\end{aligned}$$

Since $|\frac{1}{R}| < 1$, we can expand the two terms using the binomial expansion:

$$\begin{aligned}\frac{1}{(1-\frac{1}{R})^2} &= \left(1 - \frac{1}{R}\right)^{-2}\\ &= 1 - 2\left(-\frac{1}{R}\right) + (-2)(-3)\frac{(-\frac{1}{R})^2}{2!} + (-2)(-3)(-4)\frac{(-\frac{1}{R})^3}{3!} + \cdots\\ \frac{1}{(1+\frac{1}{R})^2} &= \left(1 + \frac{1}{R}\right)^{-2}\\ &= 1 - 2\left(\frac{1}{R}\right) + (-2)(-3)\frac{(\frac{1}{R})^2}{2!} + (-2)(-3)(-4)\frac{(\frac{1}{R})^3}{3!} + \cdots\end{aligned}$$

Substituting, we get:

$$E = \frac{kQ}{R^2}\left[1 + \frac{2}{R} + \frac{3}{R^2} + \frac{4}{R^3} + \cdots - \left(1 - \frac{2}{R} + \frac{3}{R^2} - \frac{4}{R^3} + \cdots\right)\right] \approx \frac{kQ}{R^2}\left(\frac{4}{R} + \frac{8}{R^3}\right),$$

using only the first two non-zero terms.

26. Using the binomial expansion we have

$$\begin{aligned}\sqrt{a^2+x^2} &= a\left(1+\frac{x^2}{a^2}\right)^{1/2} \\ &= a\left(1+\frac{1}{2}\frac{x^2}{a^2}+\frac{(1/2)(-1/2)}{2!}\frac{x^4}{a^4}+\frac{(1/2)(-1/2)(-3/2)}{3!}\frac{x^6}{a^6}+\cdots\right) \\ &= a\left(1+\frac{1}{2}\frac{x^2}{a^2}-\frac{1}{8}\frac{x^4}{a^4}+\frac{1}{16}\frac{x^6}{a^6}+\cdots\right).\end{aligned}$$

Similarly, we have

$$\sqrt{a^2-x^2} = a\left(1-\frac{1}{2}\frac{x^2}{a^2}-\frac{1}{8}\frac{x^4}{a^4}-\frac{1}{16}\frac{x^6}{a^6}-\cdots\right).$$

Combining gives

$$z=\sqrt{a^2+x^2}-\sqrt{a^2-x^2}=a\left(2\cdot\frac{1}{2}\frac{x^2}{a^2}+2\cdot\frac{1}{16}\frac{x^6}{a^6}+\cdots\right)=\frac{x^2}{a}+\frac{1}{8}\frac{x^6}{a^5}+\cdots.$$

27. This time we are interested in how a function behaves at large values in its domain. Therefore, we don't want to expand $V=2\pi\sigma(\sqrt{R^2+a^2}-R)$ about $R=0$. We want to find a variable which becomes small as R gets large. Since $R>a$, it is helpful to write

$$V=R2\pi\sigma\left(\sqrt{1+\frac{a^2}{R^2}}-1\right).$$

We can now expand a series in terms of $(\frac{a}{R})^2$. This may seem strange, but suspend your disbelief. The Taylor series for $\sqrt{1+\frac{a^2}{R^2}}$ is

$$1+\frac{1}{2}\frac{a^2}{R^2}+\frac{(1/2)(-1/2)}{2}\left(\frac{a^2}{R^2}\right)^2+\cdots$$

So $V=R2\pi\sigma\left(1+\frac{1}{2}\frac{a^2}{R^2}-\frac{1}{8}\left(\frac{a^2}{R^2}\right)^2+\cdots-1\right)$. For large R, we can drop the $-\frac{1}{8}\frac{a^4}{R^4}$ term and terms of higher order, so

$$V\approx\frac{\pi\sigma a^2}{R}.$$

Notice that what we really did by expanding around $(\frac{a}{R})^2=0$ was expanding around $R=\infty$. We then get a series that converges for large R.

28. **(a)** If $\phi=0$,

$$\text{left side}=b(1+1+1)=3b\approx 0$$

so the equation is almost satisfied and there could be a solution near $\phi=0$.

(b) We have

$$\begin{aligned}\sin\phi &= \phi-\frac{\phi^3}{3!}+\frac{\phi^5}{5!}-\cdots \\ \cos\phi &= 1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}-\cdots\end{aligned}$$

So

$$\cos^2\phi=\left(1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}-\cdots\right)\left(1-\frac{\phi^2}{2!}+\frac{\phi^4}{4!}-\cdots\right).$$

Neglecting terms of order ϕ^2 and higher, we get

$$\begin{aligned}\sin\phi &\approx \phi \\ \cos\phi &\approx 1 \\ \cos^2\phi &\approx 1.\end{aligned}$$

So $\phi+b(1+1+1)\approx 0$, whence $\phi\approx -3b$.

29. (a) Factoring the expression for $t_1 - t_2$, we get

$$\begin{aligned}\Delta t = t_1 - t_2 &= \frac{2l_2}{c(1-v^2/c^2)} - \frac{2l_1}{c\sqrt{1-v^2/c^2}} - \frac{2l_2}{c\sqrt{1-v^2/c^2}} + \frac{2l_1}{c(1-v^2/c^2)} \\ &= \frac{2(l_1+l_2)}{c(1-v^2/c^2)} - \frac{2(l_1+l_2)}{c\sqrt{1-v^2/c^2}} \\ &= \frac{2(l_1+l_2)}{c}\left(\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}}\right).\end{aligned}$$

Expanding the two terms within the parentheses in terms of v^2/c^2 gives

$$\begin{aligned}\left(1-\frac{v^2}{c^2}\right)^{-1} &= 1 + \frac{v^2}{c^2} + \frac{(-1)(-2)}{2!}\left(\frac{-v^2}{c^2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{-v^2}{c^2}\right)^3 + \cdots \\ &= 1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots \\ \left(1-\frac{v^2}{c^2}\right)^{-1/2} &= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2!}\left(\frac{-v^2}{c^2}\right)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{3!}\left(\frac{-v^2}{c^2}\right)^3 + \cdots \\ &= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\end{aligned}$$

Thus, we have

$$\begin{aligned}\Delta t &= \frac{2(l_1+l_2)}{c}\left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6} + \cdots - 1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} - \frac{5}{16}\frac{v^6}{c^6} - \cdots\right) \\ &= \frac{2(l_1+l_2)}{c}\left(\frac{1}{2}\frac{v^2}{c^2} + \frac{5}{8}\frac{v^4}{c^4} + \frac{11}{16}\frac{v^6}{c^6} + \cdots\right) \\ \Delta t &\approx \frac{(l_1+l_2)}{c}\left(\frac{v^2}{c^2} + \frac{5}{4}\frac{v^4}{c^4}\right).\end{aligned}$$

(b) For small v. we can neglect all but the first nonzero term, so

$$\Delta t \approx \frac{(l_1+l_2)}{c}\cdot\frac{v^2}{c^2} = \frac{(l_1+l_2)}{c^3}v^2.$$

Thus, Δt is proportional to v^2 with constant of proportionality $(l_1+l_2)/c^3$.

30. (a) $\mu = \dfrac{mM}{m+M}$.

If $M >> m$, then the denominator $m + M \approx M$, so $\mu \approx \dfrac{mM}{M} = m$.

(b)

$$\mu = m\left(\frac{M}{m+M}\right) = m\left(\frac{\frac{1}{M}M}{\frac{m}{M}+\frac{M}{M}}\right) = m\left(\frac{1}{1+\frac{m}{M}}\right)$$

We can use the binomial expansion since $\frac{m}{M} < 1$.

$$\mu = m\left[1 - \frac{m}{M} + \left(\frac{m}{M}\right)^2 - \left(\frac{m}{M}\right)^3 + \cdots\right]$$

(c) If $m \approx \dfrac{1}{1836}M$, then $\frac{m}{M} \approx \frac{1}{1836} \approx 0.000545$.

So a first order approximation to μ would give $\mu = m(1 - 0.000545)$. The percentage difference from $\mu = m$ is -0.0545%.

31. (a) For $a/h < 1$, we have

$$\frac{1}{(a^2+h^2)^{1/2}} = \frac{1}{h(1+a^2/h^2)^{1/2}} = \frac{1}{h}\left(1 - \frac{1}{2}\frac{a^2}{h^2} + \frac{3}{8}\frac{a^4}{h^4} - \ldots\right).$$

Thus

$$F = \frac{2GMmh}{a^2}\left(\frac{1}{h} - \frac{1}{h}\left(1 - \frac{1}{2}\frac{a^2}{h^2} + \frac{3}{8}\frac{a^4}{h^4} - \dots\right)\right)$$
$$= \frac{2GMmh}{a^2 h}\left(1 - 1 + \frac{1}{2}\frac{a^2}{h^2} - \frac{3}{8}\frac{a^4}{h^4} - \dots\right)$$
$$= \frac{2GMm}{a^2}\frac{1}{2}\frac{a^2}{h^2}\left(1 - \frac{3}{4}\frac{a^2}{h^2}\dots\right) = \frac{GMm}{h^2}\left(1 - \frac{3}{4}\frac{a^2}{h^2} - \dots\right).$$

(b) Taking only the first nonzero term gives

$$F \approx \frac{GMm}{h^2}.$$

Notice that this approximation to F is independent of a.

(c) If $a/h = 0.02$, then $a^2/h^2 = 0.0004$, so

$$F \approx \frac{GMm}{h^2}(1 - \frac{3}{4}(0.0004)) = \frac{GMm}{h^2}(1 - 0.0003).$$

Thus, the approximations differ by 0.0003 = 0.03%.

32. (a) If h is much smaller than R, we can say that $(R+h) \approx R$, giving the approximation

$$F = \frac{mgR^2}{(R+h)^2} \approx \frac{mgR^2}{R^2} = mg.$$

(b)

$$F = \frac{mgR^2}{(R+h)^2} = \frac{mg}{(1+h/R)^2} = mg(1+h/R)^{-2}$$
$$= mg\left(1 + \frac{(-2)}{1!}\left(\frac{h}{R}\right) + \frac{(-2)(-3)}{2!}\left(\frac{h}{R}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{h}{R}\right)^3 + \cdots\right)$$
$$= mg\left(1 - \frac{2h}{R} + \frac{3h^2}{R^2} - \frac{4h^3}{R^3} + \cdots\right)$$

(c) The first order correction comes from term $-2h/R$. The approximation for F is then given by

$$F \approx mg\left(1 - \frac{2h}{R}\right).$$

If the first order correction alters the estimate for F by 10%, we have

$$\frac{2h}{R} = 0.10 \qquad \text{so} \quad h = 0.05R \approx 0.05(6400) = 320 \text{ km}.$$

The approximation $F \approx mg$ is good to within 10% — that is, up to about 300 km.

33. (a) We take the left-hand Riemann sum with the formula

$$\text{Left-hand sum} = (1 + 0.9608 + 0.8521 + 0.6977 + 0.5273)(0.2) = 0.8076.$$

Similarly,

$$\text{Right-hand sum} = (0.9608 + 0.8521 + 0.6977 + 0.5273 + 0.3679)(0.2) = 0.6812.$$

(b) Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$
$$e^{-x^2} \approx 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!}$$
$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}.$$

(c)

$$\int_0^1 e^{-x^2}dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right) dx$$
$$= \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right)\Bigg|_0^1 = 0.74286.$$

(d) We can improve the left and right sum values by averaging them to get 0.74439 or by increasing the number of subdivisions. We can improve on the estimate using the Taylor approximation by taking more terms.

34. (a) The Taylor series for $1/(1-x) = 1 + x + x^2 + x^3 + \ldots$, so

$$\frac{1}{0.98} = \frac{1}{1-0.02} = 1 + (0.02) + (0.02)^2 + (0.02)^3 + \ldots$$
$$= 1.020408\ldots$$

(b) Since $d/dx(1/(1-x)) = (1/(1-x))^2$, the Taylor series for $1/(1-x)^2$ is

$$\frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Thus

$$\frac{1}{(0.99)^2} = \frac{1}{(1-0.01)^2} = 1 + 2(0.01) + 3(0.0001) + 4(0.000001) + \cdots$$
$$= 1.0203040506\ldots$$

Solutions for Section 10.4

Exercises

1. Let $f(x) = (1-x)^{1/3}$, so $f(0.5) = (0.5)^{1/3}$. The error bound in the Taylor approximation of degree 3 for $f(0.5) = 0.5^{\frac{1}{3}}$ about $x = 0$ is:

$$|E_3| = |f(0.5) - P_3(0.5)| \le \frac{M \cdot |0.5 - 0|^4}{4!} = \frac{M(0.5)^4}{24},$$

where $|f^{(4)}(x)| \le M$ for $0 \le x \le 0.5$. Now, $f^{(4)}(x) = -\frac{80}{81}(1-x)^{-(11/3)}$. By looking at the graph of $(1-x)^{-(11/3)}$, we see that $|f^{(4)}(x)|$ is maximized for x between 0 and 0.5 when $x = 0.5$. Thus,

$$|f^{(4)}| \le \frac{80}{81}\left(\frac{1}{2}\right)^{-(11/3)} = \frac{80}{81} \cdot 2^{11/3},$$

so

$$|E_3| \le \frac{80 \cdot 2^{11/3} \cdot (0.5)^4}{81 \cdot 24} \approx 0.033.$$

2. Let $f(x) = \ln(1+x)$. The error bound in the Taylor approximation of degree 3 about $x = 0$ is:

$$|E_4| = |f(0.5) - P_3(0.5)| \le \frac{M \cdot |0.5 - 0|^4}{4!} = \frac{M(0.5)^4}{24},$$

where $|f^{(4)}(x)| \le M$ for $0 \le x \le 0.5$. Since $f^{(4)}(x) = \frac{3!}{(1+x)^4}$ and the denominator attains its minimum when $x = 0$, we have $|f^{(4)}(x)| \le 3!$, so

$$|E_4| \le \frac{3!\,(0.5)^4}{24} \approx 0.016.$$

3. Let $f(x) = (1+x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+x}}$. The error bound for the Taylor approximation of degree three for $f(2) = \frac{1}{\sqrt{3}}$ about $x = 0$ is:

$$|E_3| = |f(2) - P_3(2)| \le \frac{M \cdot |2-0|^4}{4!} = \frac{M \cdot 2^4}{24},$$

where $|f^{(4)}| \le M$ for $0 \le x \le 2$. Since $f^{(4)}(x) = \frac{105}{16}(1+x)^{-(9/2)}$, we see that if x is between 0 and 2, $|f^{(4)}x)| \le \frac{105}{16}$. Thus,

$$|E_3| \le \frac{105}{16} \cdot \frac{2^4}{24} = \frac{105}{24} = 4.375.$$

Again, this is not a very helpful bound on the error, but that is to be expected as the Taylor series does not converge at $x = 2$. (At $x = 2$, we are outside the interval of convergence.)

4. Let $f(x) = \tan x$. The error bound for the Taylor approximation of degree three for $f(1) = \tan 1$ about $x = 0$ is:

$$|E_3| = |f(1) - P_3(x)| \le \frac{M \cdot |1-0|^4}{4!} = \frac{M}{24}$$

where $|f^{(4)}(x)| \le M$ for $0 \le x \le 1$. Now, $f^{(4)}(x) = \frac{16 \sin x}{\cos^3 x} + \frac{24 \sin^3 x}{\cos^5 x}$. From a graph of $f^{(4)}(x)$, we see that $f^{(4)}(x)$ is increasing for x between 0 and 1. Thus,

$$|f^{(4)}(x)| \le |f^{(4)}(1)| \approx 396,$$

so

$$|E_3| \le \frac{396}{24} = 16.5.$$

This is not a very helpful error bound! The reason the error bound is so huge is that $x = 1$ is getting near the vertical asymptote of the tangent graph, and the fourth derivative is enormous there.

Problems

5. **(a)** The Taylor polynomial of degree 0 about $t = 0$ for $f(t) = e^t$ is simply $P_0(x) = 1$. Since $e^t \ge 1$ on $[0, 0.5]$, the approximation is an underestimate.

(b) Using the zero degree error bound, if $|f'(t)| \le M$ for $0 \le t \le 0.5$, then

$$|E_0| \le M \cdot |t| \le M(0.5).$$

Since $|f'(t)| = |e^t| = e^t$ is increasing on $[0, 0.5]$,

$$|f'(t)| \le e^{0.5} < \sqrt{4} = 2.$$

Therefore

$$|E_0| \le (2)(0.5) = 1.$$

(Note: By looking at a graph of $f(t)$ and its 0^{th} degree approximation, it is easy to see that the greatest error occurs when $t = 0.5$, and the error is $e^{0.5} - 1 \approx 0.65 < 1$. So our error bound works.)

6. **(a)** The second-degree Taylor polynomial for $f(t) = e^t$ is $P_2(t) = 1 + t + t^2/2$. Since the full expansion of $e^t = 1 + t + t^2/2 + t^3/6 + t^4/24 + \cdots$ is clearly larger than $P_2(t)$ for $t > 0$, $P_2(t)$ is an underestimate on $[0, 0.5]$.

(b) Using the second-degree error bound, if $|f^{(3)}(t)| \le M$ for $0 \le t \le 0.5$, then

$$|E_2| \le \frac{M}{3!} \cdot |t|^3 \le \frac{M(0.5)^3}{6}.$$

Since $|f^{(3)}(t)| = e^t$, and e^t is increasing on $[0, 0.5]$,

$$f^{(3)}(t) \le e^{0.5} < \sqrt{4} = 2.$$

So

$$|E_2| \le \frac{(2)(0.5)^3}{6} < 0.047.$$

7. (a) θ is the first degree approximation of $f(\theta) = \sin\theta$; it is also the second degree approximation, since the next term in the Taylor expansion is 0.
$P_1(\theta) = \theta$ is an overestimate for $0 < \theta \le 1$, and is an underestimate for $-1 \le \theta < 0$. (This can be seen easily from a graph.)

(b) Using the second degree error bound, if $|f^{(3)}(\theta)| \le M$ for $-1 \le \theta \le 1$, then

$$|E_2| \le \frac{M \cdot |\theta|^3}{3!} \le \frac{M}{6}.$$

For what value of M is $|f^{(3)}(\theta)| \le M$ for $-1 \le \theta \le 1$? Well, $|f^{(3)}(\theta)| = |-\cos\theta| \le 1$. So $|E_2| \le \frac{1}{6} = 0.17$.

8. (a) $\theta - \dfrac{\theta^3}{3!}$ is the third degree Taylor approximation of $f(\theta) = \sin\theta$; it is also the fourth degree approximation, since the next term in the Taylor expansion is 0.
$P_3(\theta)$ is an underestimate for $0 < \theta \le 1$, and is an overestimate for $-1 \le \theta < 0$. (This can be checked with a calculator.)

(b) Using the fourth degree error bound, if $|f^{(5)}(\theta)| \le M$ for $-1 \le \theta \le 1$, then

$$|E_4| \le \frac{M \cdot |\theta|^5}{5!} \le \frac{M}{120}.$$

For what value of M is $|f^{(5)}(\theta)| \le M$ for $-1 \le \theta \le 1$? Since $f^{(5)}(\theta) = \cos\theta$ and $|\cos\theta| \le 1$, we have

$$|E_4| \le \frac{1}{120} \le 0.0084.$$

9. (a) The vertical distance between the graph of $y = \cos x$ and $y = P_{10}(x)$ at $x = 6$ is no more than 4, so

$$|\text{Error in } P_{10}(6)| \le 4.$$

Since at $x = 6$ the $\cos x$ and $P_{20}(x)$ graphs are indistinguishable in this figure, the error must be less than the smallest division we can see, which is about 0.2 so,

$$|\text{Error in } P_{20}(6)| \le 0.2.$$

(b) The maximum error occurs at the ends of the interval, that is, at $x = -9, x = 9$. At $x = 9$, the graphs of $y = \cos x$ and $y = P_{20}(x)$ are no more than 1 apart, so

$$\left|\begin{array}{l}\text{Maximum error in } P_{20}(x)\\ \text{for } -9 \le x \le 9\end{array}\right| \le 1.$$

(c) We are looking for the largest x-interval on which the graphs of $y = \cos x$ and $y = P_{10}(x)$ are indistinguishable. This is hard to estimate accurately from the figure, though $-4 \le x \le 4$ certainly satisfies this condition.

10. The maximum possible error for the n^{th} degree Taylor polynomial about $x = 0$ approximating $\cos x$ is $|E_n| \le \frac{M\cdot|x-0|^{n+1}}{(n+1)!}$, where $|\cos^{(n+1)} x| \le M$ for $0 \le x \le 1$. Now the derivatives of $\cos x$ are simply $\cos x, \sin x, -\cos x$, and $-\sin x$. The largest magnitude these ever take is 1, so $|\cos^{(n+1)}(x)| \le 1$, and thus $|E_n| \le \frac{|x|^{n+1}}{(n+1)!} \le \frac{1}{(n+1)!}$. The same argument works for $\sin x$.

11. By the results of Problem 10, if we approximate $\cos 1$ using the n^{th} degree polynomial, the error is at most $\frac{1}{(n+1)!}$. For the answer to be correct to four decimal places, the error must be less than 0.00005. Thus, the first n such that $\frac{1}{(n+1)!} < 0.00005$ will work. In particular, when $n = 7$, $\frac{1}{8!} = \frac{1}{40370} < 0.00005$, so the 7^{th} degree Taylor polynomial will give the desired result. For six decimal places, we need $\frac{1}{(n+1)!} < 0.0000005$. Since $n = 9$ works, the 9^{th} degree Taylor polynomial is sufficient.

12. (a)

Table 10.1
$E_1 = \sin x - x$

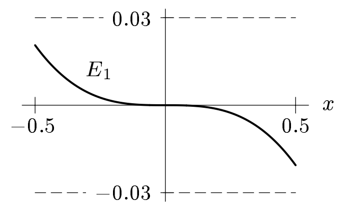
x	$\sin x$	E
-0.5	-0.4794	0.0206
-0.4	-0.3894	0.0106
-0.3	-0.2955	0.0045
-0.2	-0.1987	0.0013
-0.1	-0.0998	0.0002

Table 10.2
$E_1 = \sin x - x$

x	$\sin x$	E
0	0	0
0.1	0.0998	-0.0002
0.2	0.1987	-0.0013
0.3	0.2955	-0.0045
0.4	0.3894	-0.0106
0.5	0.4794	-0.0206

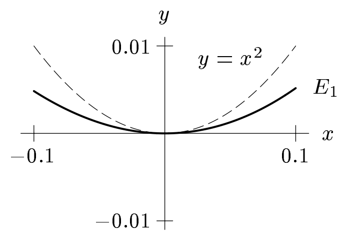
(b) See answer to part (a) above.

(c)



The fact that the graph of E_1 lies between the horizontal lines at ± 0.03 shows that $|E_1| < 0.03$ for $-0.5 \leq x \leq 0.5$.

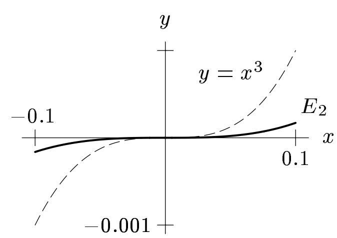
13. **(a)**



The graph of E_1 looks like a parabola. Since the graph of E_1 is sandwiched between the graph of $y = x^2$ and the x axis, we have

$$|E_1| \leq x^2 \quad \text{for} \quad |x| \leq 0.1.$$

(b)



The graph of E_2 looks like a cubic, sandwiched between the graph of $y = x^3$ and the x axis, so

$$|E_2| \leq x^3 \quad \text{for} \quad |x| \leq 0.1.$$

(c) Using the Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we see that

$$E_1 = e^x - (1 + x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Thus for small x, the $x^2/2!$ term dominates, so

$$E_1 \approx \frac{x^2}{2!},$$

and so E_1 is approximately a quadratic.

Similarly

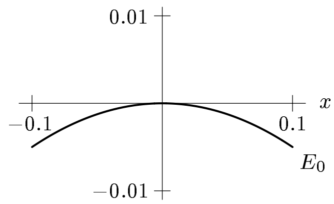
$$E_2 = e^x - (1 + x + \frac{x^2}{2}) = \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Thus for small x, the $x^3/3!$ term dominates, so

$$E_2 \approx \frac{x^3}{3!}$$

and so E_2 is approximately a cubic.

14.



The graph of E_0 looks like a parabola, and the graph shows

$$|E_0| < 0.01 \quad \text{for} \quad |x| \le 0.1.$$

(In fact $|E_0| < 0.005$ on this interval.) Since

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$
$$E_0 = \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

So, for small x,

$$E_0 \approx -\frac{x^2}{2},$$

and therefore the graph of E_0 is parabolic.

15. Since $f(x) = e^x$, the $(n+1)^{\text{st}}$ derivative $f^{(n+1)}(x)$ is also e^x, no matter what n is. Now fix a number x and let $M = e^x$, then $|f^{(n+1)}(t)| \le e^t \le e^x$ on the interval $0 \le t \le x$. (This works for $x \ge 0$; if $x < 0$ then we can take $M = 1$.) The important observation is that for any x the *same* number M bounds all the higher derivatives $f^{(n+1)}(x)$.

By the error bound formula, we now have

$$|E_n(x)| = |e^x - P_n(x)| \le \frac{M|x|^{n+1}}{(n+1)!} \quad \text{for every } n.$$

To show that the errors go to zero, we must show that for a fixed x and a fixed number M,

$$\frac{M}{(n+1)!}|x|^{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Since M is fixed, we need only show that

$$\frac{1}{(n+1)!}|x|^{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

This was shown in the text on page 456. Therefore, the Taylor series $1 + x + x^2/2! + \cdots$ does converge to e^x.

16.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Write the error in approximating $\sin x$ by the Taylor polynomial of degree $n = 2k+1$ as E_n so that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots (-1)^k \frac{x^{2k+1}}{(2k+1)!} + E_n.$$

(Notice that $(-1)^k = 1$ if k is even and $(-1)^k = -1$ if k is odd.) We want to show that if x is fixed, $E_n \to 0$ as $k \to \infty$. Since $f(x) = \sin x$, all the derivatives of $f(x)$ are $\pm \sin x$ or $\pm \cos x$, so we have for all n and all x

$$|f^{(n+1)}(x)| \le 1.$$

Using the bound on the error given in the text on page 456, we see that

$$|E_n| \le \frac{1}{(2k+2)!}|x|^{2k+2}.$$

By the argument in the text on page 456, we know that for all x,

$$\frac{|x|^{2k+2}}{(2k+2)!} = \frac{|x|^{n+1}}{(n+1)!} \to 0 \quad \text{as} \quad n = 2k+1 \to \infty.$$

Thus the Taylor series for $\sin x$ does converge to $\sin x$ for every x.

Solutions for Section 10.5

Exercises

1. No, a Fourier series has terms of the form $\cos nx$, not $\cos^n x$.

2. Not a Fourier series because terms are not of the form $\sin nx$.

3. Yes. Terms are of the form $\sin nx$ and $\cos nx$.

4. Yes. This is a Fourier series where the $\cos nx$ terms all have coefficients of zero.

5.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -1\,dx + \int_{0}^{\pi} 1\,dx\right] = 0$$

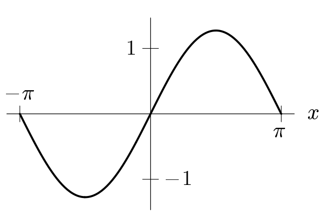
$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x\,dx + \int_{0}^{\pi}\cos x\,dx\right] \\ &= \frac{1}{\pi}\left[-\sin x\Big|_{-\pi}^{0} + \sin x\Big|_{0}^{\pi}\right] = 0. \end{aligned}$$

Similarly, a_2 and a_3 are both 0.
(In fact, notice $f(x)\cos nx$ is an odd function, so $\int_{-\pi}^{\pi} f(x)\cos nx = 0$.)

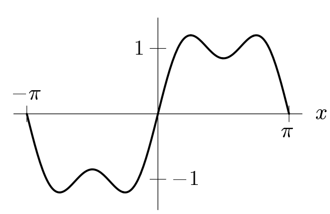
$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin x\,dx + \int_{0}^{\pi}\sin x\,dx\right] \\ &= \frac{1}{\pi}\left[\cos x\Big|_{-\pi}^{0} + (-\cos x)\Big|_{0}^{\pi}\right] = \frac{4}{\pi}. \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin 2x\,dx + \int_{0}^{\pi}\sin 2x\,dx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{2}\cos 2x\Big|_{-\pi}^{0} + (-\frac{1}{2}\cos 2x)\Big|_{0}^{\pi}\right] = 0. \\ b_3 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 3x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin 3x\,dx + \int_{0}^{\pi}\sin 3x\,dx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{3}\cos 3x\Big|_{-\pi}^{0} + (-\frac{1}{3}\cos 3x)\Big|_{0}^{\pi}\right] = \frac{4}{3\pi}. \end{aligned}$$

Thus, $F_1(x) = F_2(x) = \frac{4}{\pi}\sin x$ and $F_3(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$.



$F_1(x) = F_2(x) = \frac{4}{\pi}\sin x$



$F_3(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x$

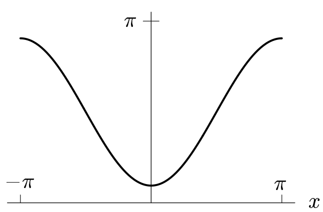
6. First,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right] = \frac{1}{2\pi}\left[-\frac{x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi}\right] = \frac{\pi}{2}.$$

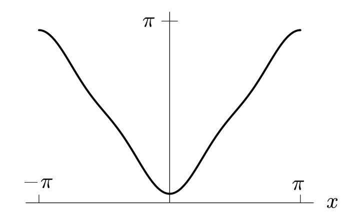
To find the a_i's, we use the integral table. For $n \geq 1$,

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx\right] \\ &= \frac{1}{\pi}\left[\left(-\frac{x}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right)\Big|_{-\pi}^{0} \right. \\ &\qquad \left. + \left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\Big|_{0}^{\pi}\right] \\ &= \frac{1}{\pi}\left(-\frac{1}{n^2} + \frac{1}{n^2}\cos(-n\pi) + \frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right) \\ &= \frac{2}{\pi n^2}(\cos n\pi - 1) \end{aligned}$$

Thus, $a_1 = -\frac{4}{\pi}, a_2 = 0$, and $a_3 = -\frac{4}{9\pi}$. To find the b_i's, note that $f(x)$ is even, so for $n \geq 1$, $f(x)\sin(nx)$ is odd. Thus, $\int_{-\pi}^{\pi} f(x)\sin(nx) = 0$, so all the b_i's are 0. $F_1 = F_2 = \frac{\pi}{2} - \frac{4}{\pi}\cos x, F_3 = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$.



$F_1(x) = F_2(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x$



$F_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$

7. The energy of the function $f(x)$ is

$$\begin{aligned} E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{3\pi}x^3\Big|_{-\pi}^{\pi} \\ &= \frac{1}{3\pi}(\pi^3 - (-\pi^3)) = \frac{2\pi^3}{3\pi} = \frac{2}{3}\pi^2 = 6.57974. \end{aligned}$$

From Problem 6, we know all the b_i's are 0 and $a_0 = \frac{\pi}{2}$, $a_1 = -\frac{4}{\pi}$, $a_2 = 0$, $a_3 = -\frac{4}{9\pi}$. Therefore the energy in the constant term and first three harmonics is

$$\begin{aligned} A_0^2 + A_1^2 + A_2^2 + A_3^2 &= 2a_0^2 + a_1^2 + a_2^2 + a_3^2 \\ &= 2\left(\frac{\pi^2}{4}\right) + \frac{16}{\pi^2} + 0 + \frac{16}{81\pi^2} = 6.57596 \end{aligned}$$

which means that they contain $\dfrac{6.57596}{6.57974} = 0.99942 \approx 99.942\%$ of the total energy.

8. First, we find a_0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{2\pi}\left(\frac{x^3}{3}\Big|_{-\pi}^{\pi}\right) = \frac{\pi^2}{3}$$

To find $a_n, n \geq 1$, we use the integral table (III-15 and III-16).

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx &= \frac{1}{\pi}\left[\frac{x^2}{n}\sin(nx) + \frac{2x}{n^2}\cos(nx) - \frac{2}{n^3}\sin(nx)\right]\Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos(n\pi) + \frac{2\pi}{n^2}\cos(-n\pi)\right] \\ &= \frac{4}{n^2}\cos(n\pi) \end{aligned}$$

Again, $\cos(n\pi) = (-1)^n$ for all integers n, so $a_n = (-1)^n \frac{4}{n^2}$. Note that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx.$$

x^2 is an even function, and $\sin nx$ is odd, so $x^2 \sin nx$ is odd. Thus $\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0$, and $b_n = 0$ for all n. We deduce that the n^{th} Fourier polynomial for f (where $n \geq 1$) is

$$F_n(x) = \frac{\pi^2}{3} + \sum_{i=1}^{n} (-1)^i \frac{4}{i^2}\cos(ix).$$

In particular, we have the graphs in Figure 10.9.

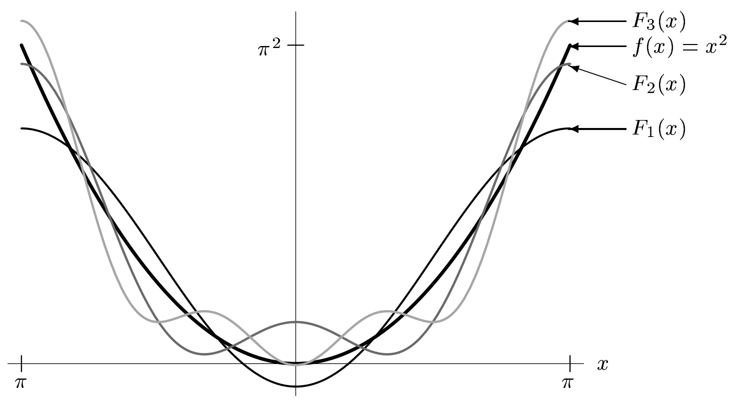


Figure 10.9

9.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} x\,dx = \frac{\pi}{4}$$

As in Problem 10, we use the integral table (III-15 and III-16) to find formulas for a_n and b_n.

$$\begin{aligned} a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx &= \frac{1}{\pi}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\Bigg|_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right) \\ &= \frac{1}{n^2\pi}\Bigg(\cos(n\pi) - 1\Bigg). \end{aligned}$$

Note that since $\cos(n\pi) = (-1)^n$, $a_n = 0$ if n is even and $a_n = -\frac{2}{n^2\pi}$ if n is odd.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx \\ &= \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Bigg|_0^{\pi} \\ &= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi)\right) \\ &= -\frac{1}{n}\cos(n\pi) \\ &= \frac{1}{n}(-1)^{n+1} \quad \text{if } n \geq 1 \end{aligned}$$

We have that the n^{th} Fourier polynomial for h (for $n \geq 1$) is

$$H_n(x) = \frac{\pi}{4} + \sum_{i=1}^{n} \left(\frac{1}{i^2\pi}\left(\cos(i\pi) - 1\right) \cdot \cos(ix) + \frac{(-1)^{i+1}\sin(ix)}{i} \right).$$

This can also be written as

$$H_n(x) = \frac{\pi}{4} + \sum_{i=1}^{n} \frac{(-1)^{i+1}\sin(ix)}{i} + \sum_{i=1}^{\left[\frac{n}{2}\right]} \frac{-2}{(2i-1)^2\pi}\cos((2i-1)x)$$

where $\left[\frac{n}{2}\right]$ denotes the biggest integer smaller than or equal to $\frac{n}{2}$. In particular, we have the graphs in Figure 10.10.

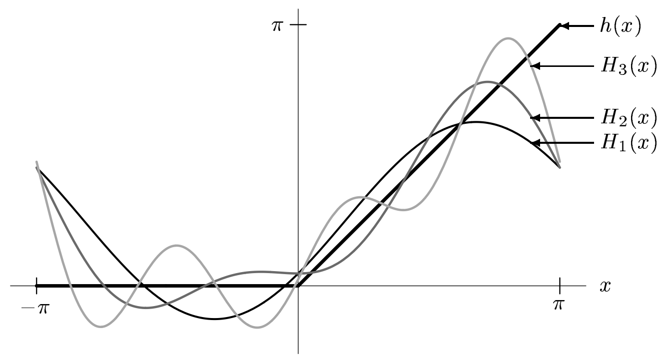


Figure 10.10

10. To find the n^{th} Fourier polynomial, we must come up with a general formula for a_n and b_n. First, we find a_0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\bigg|_{-\pi}^{\pi}\right] = 0$$

Now we use the integral table (III-15 and III-16) to find a_n and b_n for $n \geq 1$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left(\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right)\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\cos(-n\pi)\right) = 0 \end{aligned}$$

(Note that since $x\cos nx$ is odd, we could have deduced that $\int_{-\pi}^{\pi} x\cos nx = 0$.)

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) - \frac{\pi}{n}\cos(-n\pi)\right) \\ &= -\frac{2}{n}\cos(n\pi) \end{aligned}$$

Notice that $\cos(n\pi) = (-1)^n$ for all integers n, so $b_n = (-1)^{n+1}(\frac{2}{n})$.
Thus the n^{th} Fourier polynomial for g is

$$G_n(x) = \sum_{i=1}^{n}(-1)^{i+1}\frac{2}{i}\sin(ix).$$

In particular, we have the graphs in Figure 10.11.

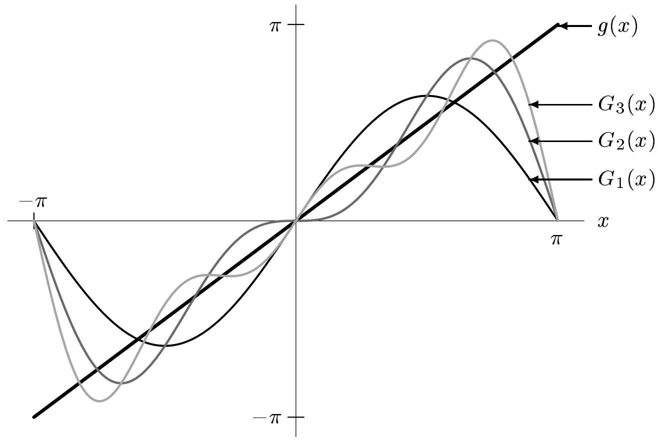
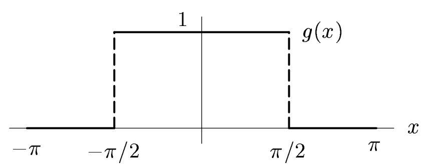


Figure 10.11

Problems

11. (a) The graph of $g(x)$ is



First find the Fourier coefficients: a_0 is the average value of g on $[-\pi, \pi]$ so from the graph, it is clear that

$$a_0 = \frac{1}{2\pi}(\pi \times 1) = \frac{1}{2},$$

or analytically,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{1}{2\pi}x\bigg|_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\ &= \frac{1}{2\pi}(\pi) = \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos kx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos kx\,dx = \frac{1}{k\pi}\sin kx\bigg|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{k\pi}\left(\sin\frac{k\pi}{2} - \sin\left(-\frac{k\pi}{2}\right)\right) = \frac{1}{k\pi}\left(2\sin\frac{k\pi}{2}\right), \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin kx\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sin kx\,dx = -\frac{1}{k\pi}\cos kx\bigg|_{-\pi/2}^{\pi/2} \\ &= -\frac{1}{k\pi}\left(\cos\frac{k\pi}{2} - \cos\left(-\frac{k\pi}{2}\right)\right). = -\frac{1}{k\pi}(0) = 0 \end{aligned}$$

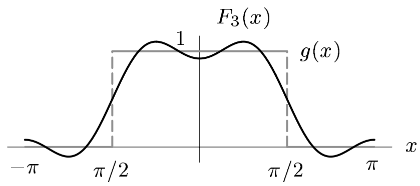
So,

$$a_1 = \frac{1}{\pi}\left(2\sin\frac{\pi}{2}\right) = \frac{2}{\pi},$$

$$a_2 = \frac{1}{2\pi}\left(2\sin\frac{2\pi}{2}\right) = 0,$$
$$a_3 = \frac{1}{3\pi}\left(2\sin\frac{3\pi}{2}\right) = -\frac{2}{3\pi},$$

which gives

$$F_3(x) = \frac{1}{2} + \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x.$$



(b) There are cosines instead of sines (but the energy spectrum remains the same).

12. We have $f(x) = x$, $0 \le x < 1$. Let $t = 2\pi x - \pi$. Notice that as x varies from 0 to 1, t varies from $-\pi$ to π. Thus if we rewrite the function in terms of t, we can find the Fourier series in terms of t in the usual way. To do this, let $g(t) = f(x) = x = \frac{t+\pi}{2\pi}$ on $-\pi \le t < \pi$. We now find the fourth degree Fourier polynomial for g.

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{t+\pi}{2\pi}\,dt = \frac{1}{(2\pi)^2}\left(\frac{t^2}{2} + \pi t\right)\Bigg|_{-\pi}^{\pi} = \frac{1}{2}$$

Notice, a_0 is the average value of both f and g. For $n \ge 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t+\pi}{2\pi}\cos(nt)dt = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}(t\cos(nt) + \pi\cos(nt))dt \\ &= \frac{1}{2\pi^2}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt) + \frac{\pi}{n}\sin(nt)\right]\Bigg|_{-\pi}^{\pi} \\ &= 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{t+\pi}{2\pi}\sin(nt)\,dt = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}(t\sin(nt) + \pi\sin(nt))\,dt \\ &= \frac{1}{2\pi^2}\left[-\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt) - \frac{\pi}{n}\cos(nt)\right]\Bigg|_{-\pi}^{\pi} \\ &= \frac{1}{2\pi^2}(-\frac{4\pi}{n}\cos(\pi n)) = -\frac{2}{\pi n}\cos(\pi n) = \frac{2}{\pi n}(-1)^{n+1}. \end{aligned}$$

We get the integrals for a_n and b_n using the integral table (formulas III-15 and III-16).

Thus, the Fourier polynomial of degree 4 for g is:

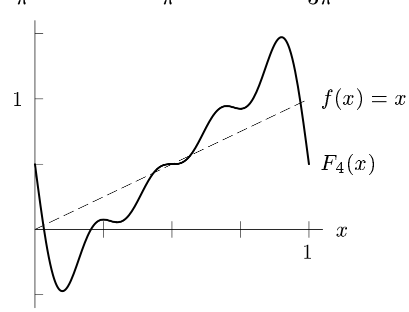
$$G_4(t) = \frac{1}{2} + \frac{2}{\pi}\sin t - \frac{1}{\pi}\sin 2t + \frac{2}{3\pi}\sin 3t - \frac{1}{2\pi}\sin 4t.$$

Now, since $g(t) = f(x)$, the Fourier polynomial of degree 4 for f can be found by replacing t in terms of x again. Thus,

$$F_4(x) = \frac{1}{2} + \frac{2}{\pi}\sin(2\pi x - \pi) - \frac{1}{\pi}\sin(4\pi x - 2\pi) + \frac{2}{3\pi}\sin(6\pi x - 3\pi) - \frac{1}{2\pi}\sin(8\pi x - 4\pi).$$

Now, using the fact that $\sin(x - \pi) = -\sin x$ and $\sin(x - 2\pi) = \sin x$, etc., we have:

$$F_4(x) = \frac{1}{2} - \frac{2}{\pi}\sin(2\pi x) - \frac{1}{\pi}\sin(4\pi x) - \frac{2}{3\pi}\sin(6\pi x) - \frac{1}{2\pi}\sin(8\pi x).$$

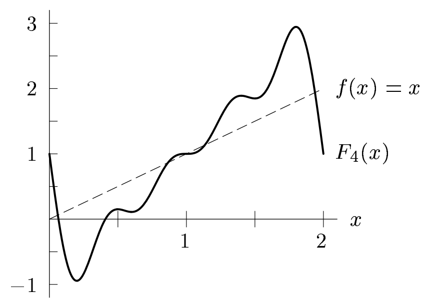


13. Since the period is 2, we make the substitution $t = \pi x - \pi$. Thus, $x = \frac{t+\pi}{\pi}$. We find the Fourier coefficients. Notice that all of the integrals are the same as in Problem 12 except for an extra factor of 2. Thus, $a_0 = 1$, $a_n = 0$, and $b_n = \frac{4}{\pi n}(-1)^{n+1}$, so:

$$G_4(t) = 1 + \frac{4}{\pi}\sin t - \frac{2}{\pi}\sin 2t + \frac{4}{3\pi}\sin 3t - \frac{1}{\pi}\sin 4t.$$

Again, we substitute back in to get a Fourier polynomial in terms of x:

$$\begin{aligned} F_4(x) &= 1 + \frac{4}{\pi}\sin(\pi x - \pi) - \frac{2}{\pi}\sin(2\pi x - 2\pi) \\ &\quad + \frac{4}{3\pi}\sin(3\pi x - 3\pi) - \frac{1}{\pi}\sin(4\pi x - 4\pi) \\ &= 1 - \frac{4}{\pi}\sin(\pi x) - \frac{2}{\pi}\sin(2\pi x) - \frac{4}{3\pi}\sin(3\pi x) - \frac{1}{\pi}\sin(4\pi x). \end{aligned}$$



Notice in this case, the terms in our series are $\sin(n\pi x)$, not $\sin(2\pi n x)$, as in Problem 12. In general, the terms will be $\sin(n\frac{2\pi}{b}x)$, where b is the period.

14. The signal received on earth is in the form of a periodic function $h(t)$, which can be expanded in a Fourier series

$$\begin{aligned} h(t) &= a_0 + a_1\cos t + a_2\cos 2t + a_3\cos 3t + \cdots \\ &\quad + b_1\sin t + b_2\sin 2t + b_3\sin 3t + \cdots \end{aligned}$$

If the periodic noise consists of *only* the second and higher harmonics of the Fourier series, then the original signal contributed the fundamental harmonic plus the constant term, i.e.,

$$\underbrace{a_0}_{\text{constant term}} + \underbrace{a_1\cos t + b_1\sin t}_{\text{fundamental harmonic}} = \underbrace{A\cos t}_{\text{original signal}}.$$

In order to find A, we need to find a_0, a_1, and b_1. Looking at the graph of $h(t)$, we see

$$\begin{aligned} a_0 = \text{average value of } h(t) &= \frac{1}{2\pi}(\text{Area above the } x\text{-axis} - \text{Area below the } x\text{-axis}) \\ &= \frac{1}{2\pi}\left[80\left(\frac{\pi}{2}\right) - \left(50\left(\frac{\pi}{4}\right) + 30\left(\frac{\pi}{4}\right) + 30\left(\frac{\pi}{4}\right) + 50\left(\frac{\pi}{4}\right)\right)\right] \\ &= \frac{1}{2\pi}\left[80\left(\frac{\pi}{2}\right) - 80\left(\frac{\pi}{2}\right)\right] = \frac{1}{2\pi}\cdot 0, = 0 \end{aligned}$$

$$\begin{aligned} a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\cos t\,dt \\ &= \frac{1}{\pi}\left[\int_{-\pi}^{-3\pi/4} -50\cos t\,dt + \int_{-3\pi/4}^{-\pi/2} 0\cos t\,dt + \int_{-\pi/2}^{-\pi/4} -30\cos t\,dt \right. \\ &\quad \left. + \int_{-\pi/4}^{\pi/4} 80\cos t\,dt + \int_{\pi/4}^{\pi/2} -30\cos t\,dt + \int_{\pi/2}^{3\pi/4} 0\cos t\,dt + \int_{3\pi/4}^{\pi} -50\cos t\,dt\right] \\ &= \frac{1}{\pi}\left[-50\sin t\Big|_{-\pi}^{-3\pi/4} - 30\sin t\Big|_{-\pi/2}^{-\pi/4}\right. \end{aligned}$$

$$
\begin{aligned}
& \left. +80\sin t\Big|_{-\pi/4}^{\pi/4} - 30\sin t\Big|_{\pi/4}^{\pi/2} - 50\sin t\Big|_{3\pi/4}^{\pi}\right] \\
&= \frac{1}{\pi}\left[-50\left(-\frac{\sqrt{2}}{2}-0\right) - 30\left(-\frac{\sqrt{2}}{2}-(-1)\right) + 80\left(\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)\right)\right. \\
&\quad \left. -30\left(1-\frac{\sqrt{2}}{2}\right) - 50\left(0-\frac{\sqrt{2}}{2}\right)\right] \\
&= \frac{1}{\pi}[25\sqrt{2}+15\sqrt{2}-30+40\sqrt{2}+40\sqrt{2}-30+15\sqrt{2}+25\sqrt{2}] \\
&= \frac{1}{\pi}[160\sqrt{2}-60] = 52.93,
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\sin t\,dt \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{-3\pi/4} -50\sin t\,dt + \int_{-3\pi/4}^{-\pi/2} 0\sin t\,dt + \int_{-\pi/2}^{-\pi/4} -30\sin t\,dt\right. \\
&\quad \left. + \int_{-\pi/4}^{\pi/4} 80\sin t\,dt + \int_{\pi/4}^{\pi/2} -30\sin t\,dt + \int_{\pi/2}^{3\pi/4} 0\sin t\,dt + \int_{3\pi/4}^{\pi} -50\sin t\,dt\right] \\
&= \frac{1}{\pi}\left[50\cos t\Big|_{-\pi}^{-3\pi/4} + 30\cos t\Big|_{-\pi/2}^{-\pi/4} - 80\cos t\Big|_{-\pi/4}^{\pi/4} + 30\cos t\Big|_{\pi/4}^{\pi/2} + 50\cos t\Big|_{3\pi/4}^{\pi}\right] \\
&= \frac{1}{\pi}\left[50\left(-\frac{\sqrt{2}}{2}-(-1)\right) + 30\left(\frac{\sqrt{2}}{2}-0\right) - 80\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right)\right. \\
&\quad \left. +30\left(0-\frac{\sqrt{2}}{2}\right) + 50\left(-1-(-\frac{\sqrt{2}}{2})\right)\right] \\
&= \frac{1}{\pi}\left[-25\sqrt{2}+50+15\sqrt{2}-0-15\sqrt{2}-50+25\sqrt{2}\right] = \frac{1}{\pi}(0) = 0.
\end{aligned}
$$

Also, we could have just noted that $b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} h(t)\sin t\,dt = 0$ because $h(t)\sin t$ is an odd function.

Substituting in, we get

$$a_0 + a_1\cos t + b_1\sin t = 0 + 52.93\cos t + 0 = A\cos t.$$

So $A = 52.93$.

15. The energy spectrum of the flute shows that the first two harmonics have equal energies and contribute the most energy by far. The higher harmonics contribute relatively little energy. In contrast, the energy spectrum of the bassoon shows the comparative weakness of the first two harmonics to the third harmonic which is the strongest component.

16. Let $f(x) = a_k\cos kx + b_k\sin kx$. Then the energy of f is given by

$$
\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx &= \frac{1}{\pi}\int_{-\pi}^{\pi}(a_k\cos kx + b_k\sin kx)^2\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(a_k^2\cos^2 kx - 2a_kb_k\cos kx\sin kx + b_k^2\sin^2 kx)\,dx \\
&= \frac{1}{\pi}\left[a_k^2\int_{-\pi}^{\pi}\cos^2 kx\,dx - 2a_kb_k\int_{-\pi}^{\pi}\cos kx\sin kx\,dx + b_k^2\int_{-\pi}^{\pi}\sin^2 kx\,dx\right] \\
&= \frac{1}{\pi}\left[a_k^2\pi - 2a_kb_k\cdot 0 + b_k^2\pi\right] = a_k^2 + b_k^2.
\end{aligned}
$$

17. Since each square in the graph has area $(\frac{\pi}{4})\cdot(0.2)$,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\cdot(\frac{\pi}{4})\cdot(0.2)\text{ [Number of squares under graph above } x\text{-axis}$$
$$\text{– Number of squares above graph below } x \text{ axis]}$$
$$\approx \frac{1}{2\pi}\cdot(\frac{\pi}{4})\cdot(0.2)\cdot[13+11-14] = 0.25.$$

Approximate the Fourier coefficients using Riemann sums.

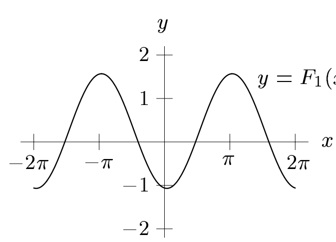
$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx \\
&\approx \frac{1}{\pi}\left[f(-\pi)\cos(-\pi) + f\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right) + f(0)\cos(0) + f\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)\right]\cdot\frac{\pi}{2} \\
&= \frac{1}{\pi}\left[(0.92)(-1) + (1)(0) + (-1.7)(1) + (0.7)(0)\right]\cdot\frac{\pi}{2} \\
&= -1.31
\end{aligned}$$

Similarly for b_1:

$$\begin{aligned}
b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx \\
&\approx \frac{1}{\pi}\left[f(-\pi)\sin(-\pi) + f\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + f(0)\sin(0) + f\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)\right]\cdot\frac{\pi}{2} \\
&= \frac{1}{\pi}\left[(0.92)(0) + (1)(-1) + (-1.7)(0) + (0.7)(1)\right]\cdot\frac{\pi}{2} \\
&= -0.15.
\end{aligned}$$

So our first Fourier approximation is

$$F_1(x) = 0.25 - 1.31\cos x - 0.15\sin x.$$



Similarly for a_2:

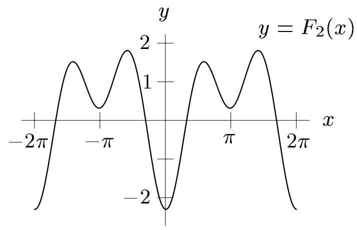
$$\begin{aligned}
a_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx \\
&\approx \frac{1}{\pi}\left[f(-\pi)\cos(-2\pi) + f\left(-\frac{\pi}{2}\right)\cos(-\pi) + f(0)\cos(0) + f\left(\frac{\pi}{2}\right)\cos(-\pi)\right]\cdot\frac{\pi}{2} \\
&= \frac{1}{\pi}\left[(0.92)(1) + (1)(-1) + (-1.7)(1) + (0.7)(-1)\right]\cdot\frac{\pi}{2} \\
&= -1.24
\end{aligned}$$

Similarly for b_2:

$$\begin{aligned}
b_2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin 2x\,dx \\
&\approx \frac{1}{\pi}\left[f(-\pi)\sin(-2\pi) + f\left(-\frac{\pi}{2}\right)\sin(-\pi) + f(0)\sin(0) + f\left(\frac{\pi}{2}\right)\sin(-\pi)\right]\cdot\frac{\pi}{2} \\
&= \frac{1}{\pi}\left[(0.92)(0) + (1)(0) + (-1.7)(0) + (0.7)(0)\right]\cdot\frac{\pi}{2} \\
&= 0.
\end{aligned}$$

So our second Fourier approximation is

$$F_2(x) = 0.25 - 1.31\cos x - 0.15\sin x - 1.24\cos 2x.$$



As you can see from comparing our graphs of F_1 and F_2 to the original, our estimates of the Fourier coefficients are not very accurate.

There are other methods of estimating the Fourier coefficients such as taking other Riemann sums, using Simpson's rule, and using the trapezoid rule. With each method, the greater the number of subdivisions, the more accurate the estimates of the Fourier coefficients.

The actual function graphed in the problem was

$$y = \frac{1}{4} - 1.3\cos x - \frac{\sin(\frac{3}{5})}{\pi}\sin x - \frac{2}{\pi}\cos 2x - \frac{\cos 1}{3\pi}\sin 2x$$

$$= 0.25 - 1.3\cos x - 0.18\sin x - 0.63\cos 2x - 0.057\sin 2x.$$

18. The Fourier series for f is

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx.$$

Pick any positive integer m. Then multiply through by $\sin mx$, to get

$$f(x)\sin mx = a_0 \sin mx + \sum_{k=1}^{\infty} a_k \cos kx \sin mx + \sum_{k=1}^{\infty} b_k \sin kx \sin mx.$$

Now, integrate term-by-term on the interval $[-\pi, \pi]$ to get

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left(a_0\sin mx + \sum_{k=1}^{\infty} a_k\cos kx\sin mx + \sum_{k=1}^{\infty} b_k\sin kx\sin mx\right)dx$$
$$= a_0\int_{-\pi}^{\pi}\sin mx\,dx + \sum_{k=1}^{\infty}\left(a_k\int_{-\pi}^{\pi}\cos kx\sin mx\,dx\right)$$
$$+ \sum_{k=1}^{\infty}\left(b_k\int_{-\pi}^{\pi}\sin kx\sin mx\,dx\right).$$

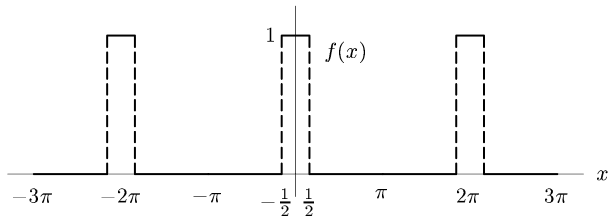
Since m is a positive integer, we know that the first term of the above expression is zero (because $\int_{-\pi}^{\pi}\sin mx\,dx = 0$). Since $\int_{-\pi}^{\pi}\cos kx\sin mx\,dx = 0$, we know that everything in the first infinite sum is zero. Since $\int_{-\pi}^{\pi}\sin kx\sin mx\,dx = 0$ where $k \neq m$, the second infinite sum reduces down to the case where $k = m$ so

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_m\int_{-\pi}^{\pi}\sin mx\sin mx\,dx = b_m\pi.$$

Divide by π to get

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

19. (a)



The energy of the pulse train f is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{\pi}\int_{-1/2}^{1/2} 1^2\,dx = \frac{1}{\pi}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \frac{1}{\pi}.$$

Next, find the Fourier coefficients:

$$a_0 = \text{average value of } f \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{ Area}) = \frac{1}{2\pi}(1) = \frac{1}{2\pi},$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1/2}^{1/2}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-1/2}^{1/2}$$
$$= \frac{1}{k\pi}\left(\sin\left(\frac{k}{2}\right) - \sin\left(-\frac{k}{2}\right)\right) = \frac{1}{k\pi}\left(2\sin\left(\frac{k}{2}\right)\right),$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1/2}^{1/2}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-1/2}^{1/2}$$
$$= -\frac{1}{k\pi}\left(\cos\left(\frac{k}{2}\right) - \cos\left(-\frac{k}{2}\right)\right) = \frac{1}{k\pi}(0) = 0.$$

The energy of f contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{2\pi}\right)^2 = \frac{1}{2\pi^2}$$

which is

$$\frac{A_0^2}{E} = \frac{1/2\pi^2}{1/\pi} = \frac{1}{2\pi} \approx 0.159155 = 15.9155\% \quad \text{of the total.}$$

The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} = \frac{\left(\frac{2\sin\frac{1}{2}}{\pi}\right)^2}{\frac{1}{\pi}} \approx 0.292653.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 0.159155 + 0.292653 = 0.451808\%.$$

(b) The formula for the energy of the k^{th} harmonic is

$$A_k^2 = a_k^2 + b_k^2 = \left(\frac{2\sin\frac{k}{2}}{k\pi}\right)^2 + 0^2 = \frac{4\sin^2\frac{k}{2}}{k^2\pi^2}.$$

By graphing it as a continuous function for $k \geq 1$, we see its overall behavior as k gets larger. See Figure 10.12. The energy spectrum for the first five terms is graphed below as well in Figure 10.13.

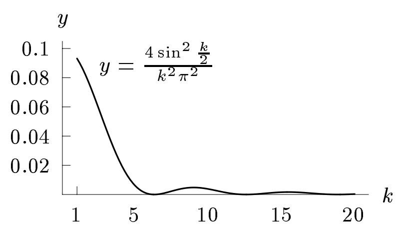


Figure 10.12

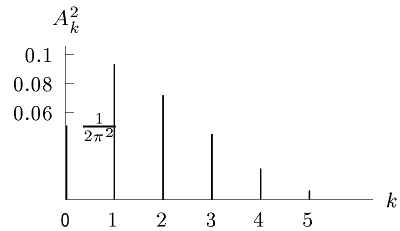
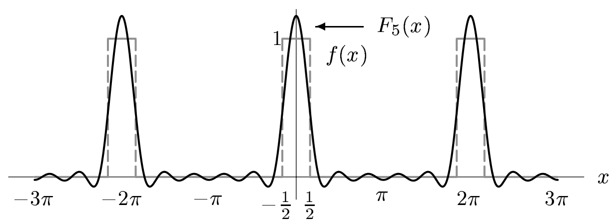


Figure 10.13

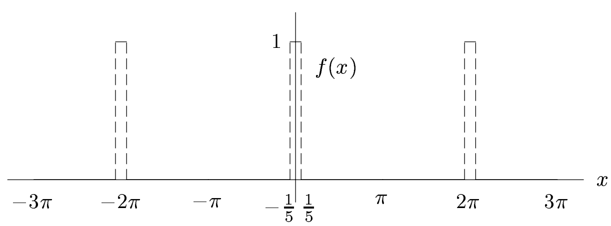
(c) The constant term and the first five harmonics are needed to capture 90% of the energy of f. This was determined by adding the fractions of energy of f contained in each harmonic until the sum reached at least 90% of the total energy of f:

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} + \frac{A_2^2}{E} + \frac{A_3^2}{E} + \frac{A_4^2}{E} + \frac{A_5^2}{E} \approx 90.1995\%.$$

(d) $F_5(x) = \frac{1}{2\pi} + \frac{2\sin(\frac{1}{2})}{\pi}\cos x + \frac{\sin 1}{\pi}\cos 2x + \frac{2\sin(\frac{3}{2})}{3\pi}\cos 3x + \frac{\sin 2}{2\pi}\cos 4x + \frac{2\sin(\frac{5}{2})}{5\pi}\cos 5x$



20. (a)



The energy of the pulse train f is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{\pi}\int_{-1/5}^{1/5} 1^2\,dx = \frac{1}{\pi}\left(\frac{1}{5} - \left(-\frac{1}{5}\right)\right) = \frac{2}{5\pi}.$$

Next, find the Fourier coefficients:

$$a_0 = \text{average value of } f \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{ Area}) = \frac{1}{2\pi}\left(\frac{2}{5}\right) = \frac{1}{5\pi},$$

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1/5}^{1/5}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-1/5}^{1/5} \\ &= \frac{1}{k\pi}\left(\sin\left(\frac{k}{5}\right) - \sin\left(-\frac{k}{5}\right)\right) = \frac{1}{k\pi}\left(2\sin\left(\frac{k}{5}\right)\right), \end{aligned}$$

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1/5}^{1/5}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-1/5}^{1/5} \\ &= -\frac{1}{k\pi}\left(\cos\left(\frac{k}{5}\right) - \cos\left(-\frac{k}{5}\right)\right) = \frac{1}{k\pi}(0) = 0. \end{aligned}$$

The energy of f contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{5\pi}\right)^2 = \frac{2}{25\pi^2}$$

which is

$$\frac{A_0^2}{E} = \frac{2/25\pi^2}{2/5\pi} = \frac{1}{5\pi} \approx 0.063662 = 6.3662\% \quad \text{of the total.}$$

The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} = \frac{\left(\frac{2\sin\frac{1}{5}}{\pi}\right)^2}{\frac{2}{5\pi}} \approx 0.12563.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 0.06366 + 0.12563 = 0.18929 = 18.929\%.$$

(b) The formula for the energy of the k^{th} harmonic is

$$A_k^2 = a_k^2 + b_k^2 = \left(\frac{2\sin\frac{k}{5}}{k\pi}\right)^2 + 0^2 = \frac{4\sin^2\frac{k}{5}}{k^2\pi^2}.$$

By graphing this formula as a continuous function for $k \geq 1$, we see its overall behavior as k gets larger in Figure 10.14. The energy spectrum for the first five terms is shown in Figure 10.15.

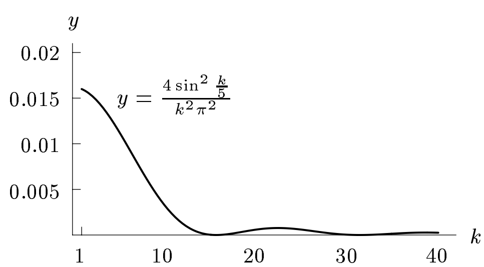


Figure 10.14

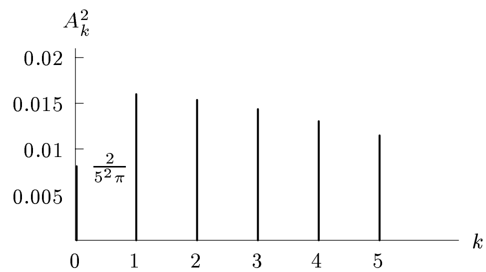


Figure 10.15

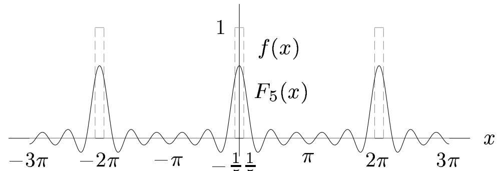
(c) The constant term and the first five harmonics contain

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} + \frac{A_2^2}{E} + \frac{A_3^2}{E} + \frac{A_4^2}{E} + \frac{A_5^2}{E} \approx 61.5255\%$$

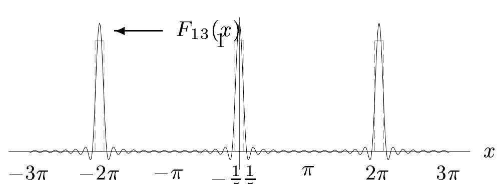
of the total energy of f.

(d) The fifth Fourier approximation to f is

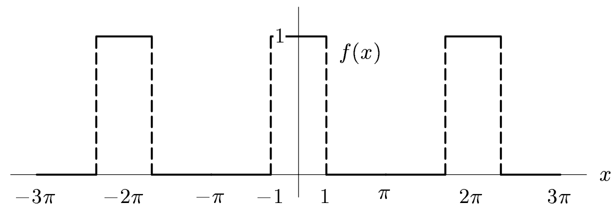
$$F_5(x) = \frac{1}{5\pi} + \frac{2\sin(\frac{1}{5})}{\pi}\cos x + \frac{\sin(\frac{2}{5})}{\pi}\cos 2x + \frac{2\sin(\frac{3}{5})}{3\pi}\cos 3x + \frac{\sin(\frac{4}{5})}{2\pi}\cos 4x + \frac{2\sin 1}{5\pi}\cos 5x.$$



For comparison, below is the thirteenth Fourier approximation to f.



21. (a)



The energy of the pulse train f is

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-1}^{1} 1^2 = \frac{1}{\pi}(1-(-1)) = \frac{2}{\pi}.$$

Next, find the Fourier coefficients:

$$a_0 = \text{average value of } f \text{ on } [-\pi,\pi] = \frac{1}{2\pi}(\text{ Area}) = \frac{1}{2\pi}(2) = \frac{1}{\pi},$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx = \frac{1}{\pi}\int_{-1}^{1}\cos kx\,dx = \frac{1}{k\pi}\sin kx\Big|_{-1}^{1}$$
$$= \frac{1}{k\pi}(\sin k - \sin(-k)) = \frac{1}{k\pi}(2\sin k),$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx = \frac{1}{\pi}\int_{-1}^{1}\sin kx\,dx = -\frac{1}{k\pi}\cos kx\Big|_{-1}^{1}$$
$$= -\frac{1}{k\pi}(\cos k - \cos(-k)) = \frac{1}{k\pi}(0) = 0.$$

The energy of f contained in the constant term is

$$A_0^2 = 2a_0^2 = 2\left(\frac{1}{\pi}\right)^2 = \frac{2}{\pi^2}$$

which is

$$\frac{A_0^2}{E} = \frac{2/\pi^2}{2/\pi} = \frac{1}{\pi} \approx 0.3183 = 31.83\% \quad \text{of the total.}$$

The fraction of energy contained in the first harmonic is

$$\frac{A_1^2}{E} = \frac{a_1^2}{E} = \frac{\left(\frac{2\sin 1}{\pi}\right)^2}{\frac{2}{\pi}} \approx 0.4508 = 45.08\%.$$

The fraction of energy contained in both the constant term and the first harmonic together is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} \approx 0.7691 = 76.91\%.$$

(b) The fraction of energy contained in the second harmonic is

$$\frac{A_2^2}{E} = \frac{a_2^2}{E} = \frac{\left(\frac{\sin 2}{\pi}\right)^2}{\frac{2}{\pi}} \approx 0.1316 = 13.16\%$$

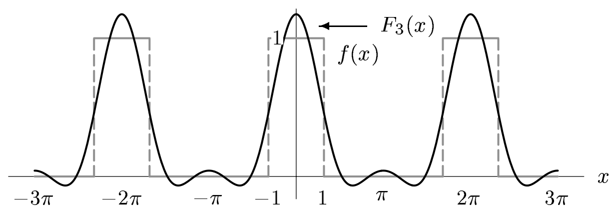
so the fraction of energy contained in the constant term and first two harmonics is

$$\frac{A_0^2}{E} + \frac{A_1^2}{E} + \frac{A_2^2}{E} \approx 0.7691 + 0.1316 = 0.9007 = 90.07\%.$$

Therefore, the constant term and the first two harmonics are needed to capture 90% of the energy of f.

(c)

$$F_3(x) = \frac{1}{\pi} + \frac{2\sin 1}{\pi}\cos x + \frac{\sin 2}{\pi}\cos 2x + \frac{2\sin 3}{3\pi}\cos 3x$$



22. As c gets closer and closer to 0, the energy of the pulse train will also approach 0, since

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-c/2}^{c/2}1^2\,dx = \frac{1}{\pi}\left(\frac{c}{2} - \left(-\frac{c}{2}\right)\right) = \frac{c}{\pi}.$$

The energy spectrum shows the *relative* distribution of the energy of f among its harmonics. The fraction of energy carried by each harmonic gets smaller as c gets closer to 0, as shown by comparing the k^{th} terms of the Fourier series for pulse trains with $c = 2, 1, 0.4$. For instance, notice that the *fraction* or *percentage* of energy carried by the constant term gets smaller as c gets smaller; the same is true for the energy carried by the first harmonic.

If each harmonic contributes less energy, then more harmonics are needed to capture a fixed percentage of energy. For example, if $c = 2$, only the constant term and the first two harmonics are needed to capture 90% of the total energy of that pulse train. If $c = 1$, the constant term and the first five harmonics are needed to get 90% of the energy of that pulse train. If $c = 0.4$, the constant term and the first thirteen harmonics are needed to get 90% of the energy of that pulse train. This means that more harmonics, or more terms in the series, are needed to get an accurate approximation. Compare the graphs of the fifth and thirteenth Fourier approximations of f in Problem 20.

23. By formula II-11 of the integral table,

$$\int_{-\pi}^{\pi}\cos kx\cos mx\,dx = \frac{1}{m^2-k^2}\left(m\cos(kx)\sin(mx) - k\sin(kx)\cos(mx)\right)\Big|_{-\pi}^{\pi}.$$

Again, since $\sin(n\pi) = 0$ for any integer n, it is easy to see that this expression is simply 0.

24. We make the substitution $u = mx$, $dx = \frac{1}{m}du$. Then

$$\int_{-\pi}^{\pi}\cos^2 mx\,dx = \frac{1}{m}\int_{u=-m\pi}^{u=m\pi}\cos^2 u\,du.$$

By Formula IV-18 of the integral table, this equals

$$\frac{1}{m}\left[\frac{1}{2}\cos u\sin u\right]\Bigg|_{-m\pi}^{m\pi} + \frac{1}{m}\frac{1}{2}\int_{-m\pi}^{m\pi}1\,du = 0 + \frac{1}{2m}u\Bigg|_{-m\pi}^{m\pi} = \frac{1}{2m}u\Bigg|_{-m\pi}^{m\pi}$$
$$= \frac{1}{2m}(2m\pi) = \pi.$$

25. The easiest way to do this is to use Problem 24.

$$\int_{-\pi}^{\pi}\sin^2 mx\,dx = \int_{-\pi}^{\pi}(1-\cos^2 mx)\,dx = \int_{-\pi}^{\pi}dx - \int_{-\pi}^{\pi}\cos^2 mx\,dx$$
$$= 2\pi - \pi \quad \text{using Problem 24}$$
$$= \pi.$$

26. By formula II-12 of the integral table,

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin kx \cos mx \, dx \\
&= \frac{1}{m^2-k^2}\left(m\sin(kx)\sin(mx)+k\cos(kx)\cos(mx)\right)\Big|_{-\pi}^{\pi} \\
&= \frac{1}{m^2-k^2}\Big[m\sin(k\pi)\sin(m\pi)+k\cos(k\pi)\cos(m\pi) \\
&\qquad -m\sin(-k\pi)\sin(-m\pi)-k\cos(-k\pi)\cos(-m\pi)\Big].
\end{aligned}$$

Since k and m are positive integers, $\sin(k\pi)=\sin(m\pi)=\sin(-k\pi)=\sin(-m\pi)=0$. Also, $\cos(k\pi)=\cos(-k\pi)$ since $\cos x$ is even. Thus this expression reduces to 0. [Note: since $\sin kx \cos mx$ is odd, so $\int_{-\pi}^{\pi}\sin kx\cos mx\,dx$ must be 0.]

27. Using formula II-10 in the integral table,

$$\int_{-\pi}^{\pi} \sin kx \sin mx \, dx = \frac{1}{m^2-k^2}\Big[k\cos(kx)\sin(mx)-m\sin(kx)\cos(mx)\Big]\Big|_{-\pi}^{\pi}.$$

Again, since $\sin(n\pi)=0$ for all integers n, this expression reduces to 0.

28. **(a)** To show that $g(t)$ is periodic with period 2π, we calculate

$$g(t+2\pi)=f\left(\frac{b(t+2\pi)}{2\pi}\right)=f\left(\frac{bt}{2\pi}+b\right)=f\left(\frac{bt}{2\pi}\right)=g(t).$$

Since $g(t+2\pi)=g(t)$ for all t, we know that $g(t)$ is periodic with period 2π. In addition

$$g\left(\frac{2\pi x}{b}\right)=f\left(\frac{b(2\pi x/b)}{2\pi}\right)=f(x).$$

(b) We make the change of variable $t=2\pi x/b$, $dt=(2\pi/b)dx$ in the usual formulas for the Fourier coefficients of $g(t)$, as follows:

$$a_0=\frac{1}{2\pi}\int_{t=-\pi}^{\pi} g(t)dt=\frac{1}{2\pi}\int_{x=-b/2}^{b/2} g\left(\frac{2\pi x}{b}\right)\frac{2\pi}{b}dx=\frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\,dx$$

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{t=-\pi}^{\pi} g(t)\cos(kt)\,dt=\frac{1}{\pi}\int_{x=-b/2}^{b/2} g\left(\frac{2\pi x}{b}\right)\cos\left(\frac{2\pi kx}{b}\right)\frac{2\pi}{b}\,dx \\
&= \frac{2}{b}\int_{-b/2}^{b/2} f(x)\cos\left(\frac{2\pi kx}{b}\right)\,dx \\
b_k &= \frac{1}{\pi}\int_{t=-\pi}^{\pi} g(t)\sin(kt)\,dt=\frac{1}{\pi}\int_{x=-b/2}^{b/2} g\left(\frac{2\pi x}{b}\right)\sin\left(\frac{2\pi kx}{b}\right)\frac{2\pi}{b}\,dx \\
&= \frac{2}{b}\int_{-b/2}^{b/2} f(x)\sin\left(\frac{2\pi kx}{b}\right)\,dx
\end{aligned}$$

(c) By part (a), the Fourier series for $f(x)$ can be obtained by substituting $t=2\pi x/b$ into the Fourier series for $g(t)$ which was found in part (b).

Solutions for Chapter 10 Review

Exercises

1. $e^x \approx 1 + e(x-1) + \frac{e}{2}(x-1)^2$

2. $\ln x \approx \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2$

3. $\sin x \approx -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x + \frac{\pi}{4}\right) + \frac{1}{2\sqrt{2}}\left(x + \frac{\pi}{4}\right)^2$

4. Differentiating $f(x) = \tan x$, we get $f'(x) = 1/\cos^2 x$, $f''(x) = 2\sin x/\cos^3 x$.

Since $\tan(\pi/4) = 1$, $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, we have $f(\pi/4) = 1$, $f'(\pi/4) = 1/(1/\sqrt{2})^2 = 2$, $f''(\pi/4) = \frac{2(1/\sqrt{2})}{(1/\sqrt{2})^3} = 4$, so

$$\begin{aligned}\tan x &\approx f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + \frac{f''(\frac{\pi}{4})}{2!}\left(x - \frac{\pi}{4}\right)^2\\ &= 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2.\end{aligned}$$

5. $f'(x) = 3x^2 + 14x - 5$, $f''(x) = 6x + 14$, $f'''(x) = 6$. The Taylor polynomial about $x = 1$ is

$$\begin{aligned}P_3(x) &= 4 + \frac{12}{1!}(x-1) + \frac{20}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3\\ &= 4 + 12(x-1) + 10(x-1)^2 + (x-1)^3.\end{aligned}$$

Notice that if you multiply out and collect terms in $P_3(x)$, you will get $f(x)$ back.

6.

$$\begin{aligned}\theta^2\cos\theta^2 &= \theta^2\left(1 - \frac{(\theta^2)^2}{2!} + \frac{(\theta^2)^4}{4!} - \frac{(\theta^2)^6}{6!} + \cdots\right)\\ &= \theta^2 - \frac{\theta^6}{2!} + \frac{\theta^{10}}{4!} - \frac{\theta^{14}}{6!} + \cdots\end{aligned}$$

7. Substituting $y = t^2$ in $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots$ gives

$$\sin t^2 = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots$$

8.

$$\begin{aligned}\frac{1}{\sqrt{4-x}} &= \frac{1}{2\sqrt{1-\frac{x}{2}}} = \frac{1}{2}\left(1 - \frac{x}{2}\right)^{-\frac{1}{2}}\\ &= \frac{1}{2}\left(1 - \left(-\frac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^2\right.\\ &\quad\left. - \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{2}\right)^3 + \cdots\right)\\ &= \frac{1}{2} + \frac{1}{8}x + \frac{3}{64}x^2 + \frac{5}{256}x^3 + \cdots\end{aligned}$$

9. Substituting $y = -4z^2$ into $\frac{1}{1+y} = 1 - y + y^2 - y^3 + \cdots$ gives

$$\frac{1}{1-4z^2} = 1 + 4z^2 + 16z^4 + 64z^6 + \cdots$$

10.

$$\frac{a}{a+b} = \frac{a}{a(1+\frac{b}{a})} = \left(1+\frac{b}{a}\right)^{-1} = 1 - \frac{b}{a} + \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^3 + \cdots$$

11.

$$\begin{aligned}\sqrt{R-r} &= \sqrt{R}\left(1-\frac{r}{R}\right)^{\frac{1}{2}} \\ &= \sqrt{R}\left(1+\frac{1}{2}\left(-\frac{r}{R}\right)+\frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{r}{R}\right)^2\right. \\ &\quad \left. +\frac{1}{3!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{r}{R}\right)^3+\cdots\right) \\ &= \sqrt{R}\left(1-\frac{1}{2}\frac{r}{R}-\frac{1}{8}\frac{r^2}{R^2}-\frac{1}{16}\frac{r^3}{R^3}-\cdots\right)\end{aligned}$$

Problems

12. (a) Factoring out $7(1.02)^3$ and using the formula for the sum of a finite geometric series with $a = 7(1.02)^3$ and $r = 1/1.02$, we see

$$\begin{aligned}\text{Sum} &= 7(1.02)^3 + 7(1.02)^2 + 7(1.02) + 7 + \frac{7}{(1.02)} + \frac{7}{(1.02)^3} + \cdots + \frac{7}{(1.02)^{100}} \\ &= 7(1.02)^3\left(1+\frac{1}{(1.02)}+\frac{1}{(1.02)^2}+\cdots+\frac{1}{(1.02)^{103}}\right) \\ &= 7(1.02)^3\frac{\left(1-\frac{1}{(1.02)^{104}}\right)}{1-\frac{1}{1.02}} \\ &= 7(1.02)^3\left(\frac{(1.02)^{104}-1}{(1.02)^{104}}\frac{1.02}{0.02}\right) \\ &= \frac{7(1.02^{104}-1)}{0.02(1.02)^{100}}.\end{aligned}$$

(b) Using the Taylor expansion for e^x with $x = (0.1)^2$, we see

$$\begin{aligned}\text{Sum} &= 7 + 7(0.1)^2 + \frac{7(0.1)^4}{2!} + \frac{7(0.1)^6}{3!} + \cdots \\ &= 7\left(1+(0.1)^2+\frac{(0.1)^4}{2!}+\frac{(0.1)^6}{3!}+\cdots\right) \\ &= 7e^{(0.1)^2} \\ &= 7e^{0.01}.\end{aligned}$$

13. Infinite geometric series with $a = 1$, $x = -1/3$, so

$$\text{Sum} = \frac{1}{1-(-1/3)} = \frac{3}{4}.$$

14. Finite geometric series which can be rewritten as

$$8\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{13}}\right) = 8\left(\frac{1-1/2^{14}}{1-1/2}\right) = 16\left(1-\frac{1}{2^{14}}\right).$$

15. This is the series for e^x with $x = -2$ substituted. Thus

$$1 - 2 + \frac{4}{2!} - \frac{8}{3!} + \frac{16}{4!} + \cdots = 1 + (-2) + \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!} + \cdots = e^{-2}.$$

16. This is the series for $\sin x$ with $x = 2$ substituted. Thus

$$2 - \frac{8}{3!} + \frac{32}{5!} - \frac{128}{7!} + \cdots = 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots = \sin 2.$$

17. Factoring out a 3, we see

$$3\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots\right) = 3e^1 = 3e.$$

18. Factoring out a 0.1, we see

$$0.1\left(0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \frac{(0.1)^7}{7!} + \cdots\right) = 0.1\sin(0.1).$$

19. The second degree Taylor polynomial for $f(x)$ around $x = 3$ is

$$\begin{aligned} f(x) &\approx f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 \\ &= 1 + 5(x-3) - \frac{10}{2!}(x-3)^2 = 1 + 5(x-3) - 5(x-3)^2. \end{aligned}$$

Substituting $x = 3.1$, we get

$$f(3.1) \approx 1 + 5(3.1-3) - 5(3.1-3)^2 = 1 + 5(0.1) - 5(0.01) = 1.45.$$

20. Write out series expansions about $x = 0$, and compare the first few terms:

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\ 1 - \cos x &= 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\ e^x - 1 &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ \arctan x &= \int \frac{dx}{1+x^2} = \int (1 - x^2 + x^4 - \cdots)\,dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots \qquad \text{(note that the arbitrary constant is 0)} \\ x\sqrt{1-x} &= x(1-x)^{1/2} = x\left(1 - \frac{1}{2}x + \frac{(1/2)(-1/2)}{2}x^2 + \cdots\right) \\ &= x - \frac{x^2}{2} + \frac{x^3}{8} + \cdots \end{aligned}$$

So, considering just the first term or two (since we are interested in small x)

$$1 - \cos x < x\sqrt{1-x} < \ln(1+x) < \arctan x < \sin x < x < e^x - 1.$$

21. The graph in Figure 10.16 suggests that the Taylor polynomials converge to $f(x) = \dfrac{1}{1+x}$ on the interval $(-1, 1)$. The Taylor expansion is

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots,$$

so the ratio test gives

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|(-1)^{n+1}x^{n+1}|}{|(-1)^n x^n|} = |x|.$$

Thus, the series converges if $|x| < 1$; that is $-1 < x < 1$.

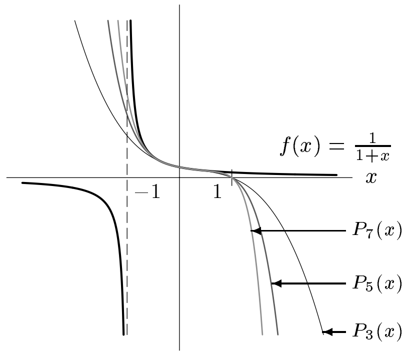


Figure 10.16

22. First we use the Taylor series expansion for $\ln(1+t)$,

$$\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots$$

to find the Taylor series expansion of $\ln(1+x+x^2)$ by putting $t = x + x^2$. We get

$$\ln(1+x+x^2) = x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \cdots.$$

Next we use the Taylor series for $\sin x$ to get

$$\sin^2 x = (\sin x)^2 = (x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots)^2 = x^2 - \frac{1}{3}x^4 + \cdots.$$

Finally,

$$\frac{\ln(1+x+x^2) - x}{\sin^2 x} = \frac{\frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 + \cdots}{x^2 - \frac{1}{3}x^4 + \cdots} \to \frac{1}{2}, \quad \text{as} \quad x \to 0.$$

23. **(a)** The series for $\frac{\sin 2\theta}{\theta}$ is

$$\frac{\sin 2\theta}{\theta} = \frac{1}{\theta}\left(2\theta - \frac{(2\theta)^3}{3!} + \frac{(2\theta)^5}{5!} - \cdots\right) = 2 - \frac{4\theta^2}{3} + \frac{4\theta^4}{15} - \cdots$$

so $\lim_{\theta\to 0} \frac{\sin 2\theta}{\theta} = 2$.

(b) Near $\theta = 0$, we make the approximation

$$\frac{\sin 2\theta}{\theta} \approx 2 - \frac{4}{3}\theta^2$$

so the parabola is $y = 2 - \frac{4}{3}\theta^2$.

24. **(a)** $f(t) = te^t$.
Use the Taylor expansion for e^t :

$$f(t) = t\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)$$
$$= t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \cdots$$

(b)

$$\int_0^x f(t)\,dt = \int_0^x te^t\,dt = \int_0^x \left(t + t^2 + \frac{t^3}{2!} + \frac{t^4}{3!} + \cdots\right)\,dt$$
$$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4\cdot 2!} + \frac{t^5}{5\cdot 3!} + \cdots \Big|_0^x$$
$$= \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4\cdot 2!} + \frac{x^5}{5\cdot 3!} + \cdots$$

(c) Substitute $x = 1$:

$$\int_0^1 te^t\,dt = \frac{1}{2} + \frac{1}{3} + \frac{1}{4\cdot 2!} + \frac{1}{5\cdot 3!} + \cdots$$

In the integral above, to integrate by parts, let $u = t,\ dv = e^t\,dt$, so $du = dt$, $v = e^t$.

$$\int_0^1 te^t\,dt = te^t\Big|_0^1 - \int_0^1 e^t\,dt = e - (e-1) = 1$$

Hence

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4\cdot 2!} + \frac{1}{5\cdot 3!} + \cdots = 1.$$

25. **(a)** Since $\sqrt{4-x^2} = 2\sqrt{1-x^2/4}$, we use the Binomial expansion

$$\begin{aligned}\sqrt{4-x^2} &\approx 2\left(1 + \frac{1}{2}\left(-\frac{x^2}{4}\right) + \frac{1}{2!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{x^2}{4}\right)^2\right)\\ &= 2\left(1 - \frac{x^2}{8} - \frac{x^4}{128}\right) = 2 - \frac{x^2}{4} - \frac{x^4}{64}.\end{aligned}$$

(b) Substituting the Taylor series in the integral gives

$$\int_0^1 \sqrt{4-x^2}\,dx \approx \int_0^1 \left(2 - \frac{x^2}{4} - \frac{x^4}{64}\right)\,dx = 2x - \frac{x^3}{12} - \frac{x^5}{320}\Big|_0^1 = 1.9135.$$

(c) Since $x = 2\sin t$, we have $dx = 2\cos t\,dt$; in addition $t = 0$ when $x = 0$ and $t = \pi/6$ when $x = 1$. Thus

$$\begin{aligned}\int_0^1 \sqrt{4-x^2}\,dx &= \int_0^{\pi/6} \sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt\\ &= \int_0^{\pi/6} 2\cdot 2\sqrt{1-\sin^2 t}\cos t\,dt = 4\int_0^{\pi/6} \cos^2 t\,dt.\end{aligned}$$

Using the table of integrals, we find

$$4\int_0^{\pi/6} \cos^2 t\,dt = 4\cdot\frac{1}{2}(\cos t\sin t + t)\Big|_0^{\pi/6} = 2\left(\cos\frac{\pi}{6}\sin\frac{\pi}{6} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}.$$

(d) Using a calculator, $(\sqrt{3}/3) + (\pi/3) = 1.9132$, so the answers to parts (b) and (c) agree to three decimal places.

26. **(a)** Since $\int(1-x^2)^{-1/2}dx = \arcsin x$, we use the Taylor series for $(1-x^2)^{-1/2}$ to find the Taylor series for $\arcsin x$:

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots$$

so

$$\arcsin x = \int(1-x^2)^{-1/2}dx = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots$$

(b) From Example 4 in Section 10.3, we know

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

so that

$$\frac{\arctan x}{\arcsin x} = \frac{x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots}{x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots} \to 1, \quad \text{as} \quad x \to 0.$$

27. (a) The Taylor polynomial of degree 2 is

$$V(x) \approx V(0) + V'(0)x + \frac{V''(0)}{2}x^2.$$

Since $x = 0$ is a minimum, $V'(0) = 0$ and $V''(0) > 0$. We can not say anything about the sign or value of $V(0)$. Thus

$$V(x) \approx V(0) + \frac{V''(0)}{2}x^2.$$

(b) Differentiating gives an approximation to $V'(x)$ at points near the origin

$$V'(x) \approx V''(0)x.$$

Thus, the force on the particle is approximated by $-V''(0)x$.

$$\text{Force} = -V'(x) \approx -V''(0)x.$$

Since $V''(0) > 0$, the force is approximately proportional to x with negative proportionality constant, $-V''(0)$. This means that when x is positive, the force is negative, which means pointing toward the origin. When x is negative, the force is positive, which means pointing toward the origin. Thus, the force always points toward the origin.

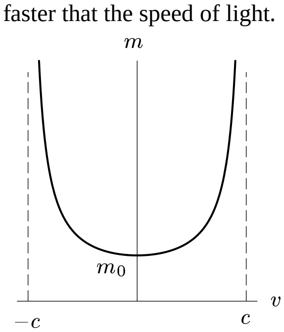
Physical principles tell us that the particle is at equilibrium at the minimum potential. The direction of the force toward the origin supports this, as the force is tending to restore the particle to the origin.

28. (a) Since the expression under the square root sign, $1 - \frac{v^2}{c^2}$ must be positive in order to give a real value of m, we have

$$\begin{aligned} 1 - \frac{v^2}{c^2} &> 0 \\ \frac{v^2}{c^2} &< 1 \\ v^2 &< c^2, \\ \text{so} \quad -c < v &< c. \end{aligned}$$

In other words, the object can never travel faster that the speed of light.

(b)



(c) Notice that $m = m_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. If we substitute $u = -\frac{v^2}{c^2}$, we get $m = m_0(1+u)^{-1/2}$ and we can use the binomial expansion to get:

$$\begin{aligned} m &= m_0\left(1 - \frac{1}{2}u + \frac{(-1/2)(-3/2)}{2!}u^2 + \cdots\right) \\ &= m_0\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right). \end{aligned}$$

(d) We would expect this series to converge only for values of the original function that exist, namely when $|v| < c$.

29. (a) To find when V takes on its minimum values, set $\frac{dV}{dr} = 0$. So

$$\begin{aligned} -V_0 \frac{d}{dr}\left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right) &= 0 \\ -V_0\left(-12r_0^6 r^{-7} + 12r_0^{12} r^{-13}\right) &= 0 \\ 12r_0^6 r^{-7} &= 12r_0^{12} r^{-13} \\ r_0^6 &= r^6 \\ r &= r_0. \end{aligned}$$

Rewriting $V'(r)$ as $\dfrac{12r_0^6 V_0}{r_7}\left(1 - \left(\dfrac{r_0}{r}\right)^6\right)$, we see that $V'(r) > 0$ for $r > r_0$ and $V'(r) < 0$ for $r < r_0$. Thus, $V = -V_0(2(1)^6 - (1)^{12}) = -V_0$ is a minimum.
(Note: We discard the negative root $-r_0$ since the distance r must be positive.)

(b)

$$\begin{aligned} V(r) &= -V_0\left(2\left(\frac{r_0}{r}\right)^6 - \left(\frac{r_0}{r}\right)^{12}\right) & V(r_0) &= -V_0 \\ V'(r) &= -V_0(-12r_0^6 r^{-7} + 12r_0^{12} r^{-13}) & V'(r_0) &= 0 \\ V''(r) &= -V_0(84r_0^6 r^{-8} - 156r_0^{12} r^{-14}) & V''(r_0) &= 72V_0 r_0^{-2} \end{aligned}$$

The Taylor series is thus:

$$V(r) = -V_0 + 72V_0 r_0^{-2} \cdot (r - r_0)^2 \cdot \frac{1}{2} + \cdots$$

(c) The difference between V and its minimum value $-V_0$ is

$$V - (-V_0) = 36V_0 \frac{(r - r_0)^2}{r_0^2} + \cdots$$

which is approximately proportional to $(r - r_0)^2$ since terms containing higher powers of $(r - r_0)$ have relatively small values for r near r_0.

(d) From part (a) we know that $dV/dr = 0$ when $r = r_0$, hence $F = 0$ when $r = r_0$. Since, if we discard powers of $(r - r_0)$ higher than the second,

$$V(r) \approx -V_0\left(1 - 36\frac{(r - r_0)^2}{r_0^2}\right)$$

giving

$$F = -\frac{dV}{dr} \approx 72 \cdot \frac{r - r_0}{r_0^2}(-V_0) = -72V_0 \frac{r - r_0}{r_o^2}.$$

So F is approximately proportional to $(r - r_0)$.

30. (a) $F = \frac{GM}{R^2} + \frac{Gm}{(R+r)^2}$

(b) $F = \frac{GM}{R^2} + \frac{Gm}{R^2}\frac{1}{(1+\frac{r}{R})^2}$

Since $\frac{r}{R} < 1$, use the binomial expansion:

$$\frac{1}{(1+\frac{r}{R})^2} = \left(1 + \frac{r}{R}\right)^{-2} = 1 - 2\left(\frac{r}{R}\right) + (-2)(-3)\frac{(\frac{r}{R})^2}{2!} + \cdots$$

$$F = \frac{GM}{R^2} + \frac{Gm}{R^2}\left[1 - 2\left(\frac{r}{R}\right) + 3\left(\frac{r}{R}\right)^2 - \cdots\right].$$

(c) Discarding higher power terms, we get

$$\begin{aligned} F &\approx \frac{GM}{R^2} + \frac{Gm}{R^2} - \frac{2Gmr}{R^3} \\ &= \frac{G(M+m)}{R^2} - \frac{2Gmr}{R^3}. \end{aligned}$$

Looking at the expression, we see that the term $\frac{G(M+m)}{R^2}$ is the field strength at a distance R from a single particle of mass $M + m$. The correction term, $-\frac{2Gmr}{R^3}$, is negative because the field strength exerted by a particle of mass $(M + m)$ at a distance R would clearly be larger than the field strength at P in the question.

31. Since expanding $f(x+h)$ and $g(x+h)$ in Taylor series gives

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \dots,$$
$$g(x+h) = g(x) + g'(x)h + \frac{f''(x)}{2!}h^2 + \dots,$$

we substitute to get

$$\begin{aligned}
&\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \frac{(f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \dots)(g(x) + g'(x)h + \frac{1}{2}g''(x)h^2 + \dots) - f(x)g(x)}{h} \\
&= \frac{f(x)g(x) + (f'(x)g(x) + f(x)g'(x))h + \text{ Terms in } h^2 \text{ and higher powers } - f(x)g(x)}{h} \\
&= \frac{h(f'(x)g(x) + f(x)g'(x) + \text{ Terms in } h \text{ and higher powers})}{h} \\
&= f'(x)g(x) + f(x)g'(x) + \text{ Terms in } h \text{ and higher powers.}
\end{aligned}$$

Thus, taking the limit as $h \to 0$, we get

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

32. Expanding $f(y+k)$ and $g(x+h)$ in Taylor series gives

$$f(y+k) = f(y) + f'(y)k + \frac{f''(y)}{2!}k^2 + \cdots,$$
$$g(x+h) = g(x) + g'(x)h + \frac{g''(x)}{2!}h^2 + \cdots.$$

Now let $y = g(x)$ and $y + k = g(x+h)$. Then $k = g(x+h) - g(x)$ so

$$k = g'(x)h + \frac{g''(x)}{2!}h^2 + \cdots.$$

Substituting $g(x+h) = y + k$ and $y = g(x)$ in the series for $f(y+k)$ gives

$$f(g(x+h)) = f(g(x)) + f'(g(x))k + \frac{f''(g(x))}{2!}k^2 + \cdots.$$

Now, substituting for k, we get

$$\begin{aligned}
f(g(x+h)) &= f(g(x)) + f'(g(x)) \cdot (g'(x)h + \frac{g''(x)}{2!}h^2 + \cdots) + \frac{f''(g(x))}{2!}(g'(x)h + \dots)^2 + \cdots \\
&= f(g(x)) + (f'(g(x))) \cdot g'(x)h + \text{Terms in } h^2 \text{ and higher powers.}
\end{aligned}$$

So, substituting for $f(g(x+h))$ and dividing by h, we get

$$\frac{f(g(x+h)) - f(g(x))}{h} = f'(g(x)) \cdot g'(x) + \text{ Terms in } h \text{ and higher powers,}$$

and thus, taking the limit as $h \to 0$,

$$\begin{aligned}
\frac{d}{dx}f(g(x)) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= f'(g(x)) \cdot g'(x).
\end{aligned}$$

33. **(a)** Notice $g'(0) = 0$ because g has a critical point at $x = 0$. So, for $n \geq 2$,

$$g(x) \approx P_n(x) = g(0) + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots + \frac{g^{(n)}(0)}{n!}x^n.$$

(b) The Second Derivative test says that if $g''(0) > 0$, then 0 is a local minimum and if $g''(0) < 0$, 0 is a local maximum.

(c) Let $n = 2$. Then $P_2(x) = g(0) + \frac{g''(0)}{2!}x^2$. So, for x near 0,

$$g(x) - g(0) \approx \frac{g''(0)}{2!}x^2.$$

If $g''(0) > 0$, then $g(x) - g(0) \geq 0$, as long as x stays near 0. In other words, there exists a small interval around $x = 0$ such that for any x in this interval $g(x) \geq g(0)$. So $g(0)$ is a local minimum.
The case when $g''(0) < 0$ is treated similarly; then $g(0)$ is a local maximum.

34. The situation is more complicated. Let's first consider the case when $g'''(0) \neq 0$. To be specific let $g'''(0) > 0$. Then

$$g(x) \approx P_3(x) = g(0) + \frac{g'''(0)}{3!}x^3.$$

So, $g(x) - g(0) \approx \frac{g'''(0)}{3!}x^3$. (Notice that $\frac{g'''(0)}{3!} > 0$ is a constant.) Now, no matter how small an open interval I around $x = 0$ is, there are always some x_1 and x_2 in I such that $x_1 < 0$ and $x_2 > 0$, which means that $\frac{g'''(0)}{3!}x_1^3 < 0$ and $\frac{g'''(0)}{3!}x_2^3 > 0$, i.e. $g(x_1) - g(0) < 0$ and $g(x_2) - g(0) > 0$. Thus, $g(0)$ is neither a local minimum nor a local maximum. (If $g'''(0) < 0$, the same conclusion still holds. Try it! The reasoning is similar.)

Now let's consider the case when $g'''(0) = 0$. If $g^{(4)}(0) > 0$, then by the fourth degree Taylor polynomial approximation to g at $x = 0$, we have

$$g(x) - g(0) \approx \frac{g^{(4)}(0)}{4!}x^4 > 0$$

for x in a small open interval around $x = 0$. So $g(0)$ is a local minimum. (If $g^{(4)}(0) < 0$, then $g(0)$ is a local maximum.)

In general, suppose that $g^{(k)}(0) \neq 0$, $k \geq 2$, and all the derivatives of g with order less than k are 0. In this case g looks like cx^k near $x = 0$, which determines its behavior there. Then $g(0)$ is neither a local minimum nor a local maximum if k is odd. For k even, $g(0)$ is a local minimum if $g^{(k)}(0) > 0$, and $g(0)$ is a local maximum if $g^{(k)}(0) < 0$.

35. Let us begin by finding the Fourier coefficients for $f(x)$. Since f is odd, $\int_{-\pi}^{\pi} f(x)\,dx = 0$ and $\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$. Thus $a_i = 0$ for all $i \geq 0$. On the other hand,

$$\begin{aligned} b_i = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx &= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin(nx)\,dx + \int_{0}^{\pi}\sin(nx)\,dx\right] \\ &= \frac{1}{\pi}\left[\frac{1}{n}\cos(nx)\Big|_{-\pi}^{0} - \frac{1}{n}\cos(nx)\Big|_{0}^{\pi}\right] \\ &= \frac{1}{n\pi}\left[\cos 0 - \cos(-n\pi) - \cos(n\pi) + \cos 0\right] \\ &= \frac{2}{n\pi}\left(1 - \cos(n\pi)\right). \end{aligned}$$

Since $\cos(n\pi) = (-1)^n$, this is 0 if n is even, and $\frac{4}{n\pi}$ if n is odd. Thus the n^{th} Fourier polynomial (where n is odd) is

$$F_n(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \cdots + \frac{4}{n\pi}\sin(nx).$$

As $n \to \infty$, the n^{th} Fourier polynomial must approach $f(x)$ on the interval $(-\pi, \pi)$, except at the point $x = 0$ (where f is not continuous). In particular, if $x = \frac{\pi}{2}$,

$$\begin{aligned} F_n(1) &= \frac{4}{\pi}\sin\frac{\pi}{2} + \frac{4}{3\pi}\sin\frac{3\pi}{2} + \frac{4}{5\pi}\sin\frac{5\pi}{2} + \frac{4}{7\pi}\sin\frac{7\pi}{2} + \cdots + \frac{4}{n\pi}\sin\frac{n\pi}{2} \\ &= \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{2n+1}\frac{1}{2n+1}\right). \end{aligned}$$

But $F_n(1)$ approaches $f(\frac{\pi}{2}) = 1$ as $n \to \infty$, so

$$\frac{\pi}{4}F_n(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{2n+1}\frac{1}{2n+1} \to \frac{\pi}{4}\cdot 1 = \frac{\pi}{4}.$$

36. Let $t = 2\pi x - \pi$. Then, $g(t) = f(x) = e^{2\pi x} = e^{t+\pi}$. Notice that as x varies from 0 to 1, t varies from $-\pi$ to π. Thus, we can find the Fourier coefficients for $g(t)$:

$$a_o = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{t+\pi}dt = \frac{1}{2\pi}e^{t+\pi}\Big|_{-\pi}^{\pi} = \frac{e^{2\pi}-1}{2\pi},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{t+\pi}cos(nt)dt = \frac{e^{\pi}}{\pi}\int_{-\pi}^{\pi} e^t\cos(nt)dt.$$

Using the integral table, Formula II-8, yields:

$$\begin{aligned}
&= \frac{e^{\pi}}{\pi}\frac{1}{n^2+1}e^t(\cos(nt) + n\sin(nt))\Big|_{-\pi}^{\pi} \\
&= \frac{e^{\pi}}{\pi}\frac{1}{n^2+1}(e^{\pi} - e^{-\pi})(\cos(n\pi)) \\
&= \frac{(e^{2\pi}-1)}{\pi}\frac{(-1)^n}{n^2+1}
\end{aligned}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{t+\pi}\sin(nt)dt = \frac{e^{\pi}}{\pi}\int_{-\pi}^{\pi} e^t\sin(nt)dt.$$

Again, using the integral table, Formula II-9, yields:

$$\begin{aligned}
&= \frac{e^{\pi}}{\pi}\frac{1}{n^2+1}e^t(\sin(nt) - n\cos(nt))\Big|_{-\pi}^{\pi} \\
&= -\frac{e^{\pi}}{\pi}\frac{n}{n^2+1}(e^{\pi} - e^{-\pi})\cos(n\pi) \\
&= \frac{(e^{2\pi}-1)}{\pi}\frac{(-1)^{n+1}n}{n^2+1}.
\end{aligned}$$

Thus, after factoring a bit, we get:

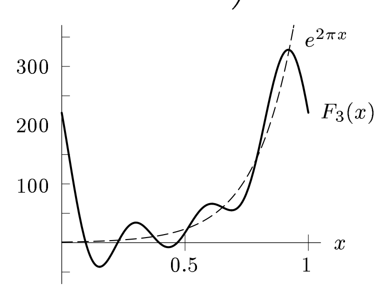
$$G_3(t) = \frac{e^{2\pi}-1}{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{5}\cos 2t - \frac{2}{5}\sin 2t - \frac{1}{10}\cos 3t + \frac{3}{10}\sin 3t\right).$$

Now, we substitute x back in for t:

$$\begin{aligned}
F_3(x) = \frac{e^{2\pi}-1}{\pi}(&\frac{1}{2} - \frac{1}{2}\cos(2\pi x - \pi) + \frac{1}{2}\sin(2\pi x - \pi) + \frac{1}{5}\cos(4\pi x - 2\pi) \\
&-\frac{2}{5}\sin(4\pi x - 2\pi) - \frac{1}{10}\cos(6\pi x - 3\pi) + \frac{3}{10}\sin(6\pi x - 3\pi)).
\end{aligned}$$

Recalling that $\cos(x-\pi) = -\cos x, \sin(x-\pi) = -\sin x, \cos(x-2\pi) = \cos x$, and $\sin(x-2\pi) = \sin x$, we have:

$$\begin{aligned}
F_3(x) = \frac{e^{2\pi}-1}{\pi}\Bigg(&\frac{1}{2} + \frac{1}{2}\cos 2\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{5}\cos 4\pi x - \frac{2}{5}\sin 4\pi x \\
&+ \frac{1}{10}\cos 6\pi x - \frac{3}{10}\sin 6\pi x\Bigg).
\end{aligned}$$



37. **(a)** Expand $f(x)$ into its Fourier series:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots + a_k \cos kx + \cdots \\ + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots + b_k \sin kx + \cdots$$

Then differentiate term-by-term:

$$f'(x) = -a_1 \sin x - 2a_2 \sin 2x - 3a_3 \sin 3x - \cdots - ka_k \sin kx - \cdots \\ +b_1 \cos x + 2b_2 \cos 2x + 3b_3 \cos 3x + \cdots + kb_k \cos kx + \cdots$$

Regroup terms:

$$f'(x) = +b_1 \cos x + 2b_2 \cos 2x + 3b_3 \cos 3x + \cdots + kb_k \cos kx + \cdots \\ -a_1 \sin x - 2a_2 \sin 2x - 3a_3 \sin 3x - \cdots - ka_k \sin kx - \cdots$$

which forms a Fourier series for the derivative $f'(x)$. The Fourier coefficient of $\cos kx$ is kb_k and the Fourier coefficient of $\sin kx$ is $-ka_k$. Note that there is no constant term as you would expect from the formula ka_k with $k = 0$. Note also that if the k^{th} harmonic f is absent, so is that of f'.

(b) If the amplitude of the k^{th} harmonic of f is

$$A_k = \sqrt{a_k^2 + b_k^2}, \quad k \geq 1,$$

then the amplitude of the k^{th} harmonic of f' is

$$\sqrt{(kb_k)^2 + (-ka_k)^2} = \sqrt{k^2(b_k^2 + a_k^2)} = k\sqrt{a_k^2 + b_k^2} = kA_k.$$

(c) The energy of the k^{th} harmonic of f' is k^2 times the energy of the k^{th} harmonic of f.

38. Let r_k and s_k be the Fourier coefficients of $Af + Bg$. Then

$$\begin{aligned} r_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[Af(x) + Bg(x) \right] dx \\ &= A \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \right] + B \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\, dx \right] \\ &= Aa_0 + Bc_0. \end{aligned}$$

Similarly,

$$\begin{aligned} r_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[Af(x) + Bg(x) \right] \cos(kx)\, dx \\ &= A \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \right] + B \left[\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(kx)\, dx \right] \\ &= Aa_k + Bc_k. \end{aligned}$$

And finally,

$$\begin{aligned} s_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[Af(x) + Bg(x) \right] \sin(kx)\, dx \\ &= A \left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx \right] + B \left[\frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(kx)\, dx \right] \\ &= Ac_k + Bd_k. \end{aligned}$$

39. Since $g(x) = f(x + c)$, we have that $[g(x)]^2 = [f(x + c)]^2$, so g^2 is f^2 shifted horizontally by c. Since f has period 2π, so does f^2 and g^2. If you think of the definite integral as an area, then because of the periodicity, integrals of f^2 over any interval of length 2π have the same value. So

$$\text{Energy of } f = \int_{-\pi}^{\pi} (f(x))^2\, dx = \int_{-\pi+c}^{\pi+c} (f(x))^2\, dx.$$

Now we know that

$$\text{Energy of } g = \frac{1}{\pi}\int_{-\pi}^{\pi}(g(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x+c))^2\,dx.$$

Using the substitution $t = x + c$, we see that the two energies are equal.

CAS Challenge Problems

40. (a) The Taylor polynomials of degree 10 are

$$\text{For } \sin^2 x, \qquad P_{10}(x) = x^2 - \frac{x^4}{3} + \frac{2\,x^6}{45} - \frac{x^8}{315} + \frac{2\,x^{10}}{14175}$$
$$\text{For } \cos^2 x, \qquad Q_{10}(x) = 1 - x^2 + \frac{x^4}{3} - \frac{2\,x^6}{45} + \frac{x^8}{315} - \frac{2\,x^{10}}{14175}$$

(b) The coefficients in $P_{10}(x)$ are the negatives of the corresponding coefficients of $Q_{10}(x)$. The constant term of $P_{10}(x)$ is 0 and the constant term of $Q_{10}(x)$ is 1. Thus, $P_{10}(x)$ and $Q_{10}(x)$ satisfy

$$Q_{10}(x) = 1 - P_{10}(x).$$

This makes sense because $\cos^2 x$ and $\sin^2 x$ satisfy the identity

$$\cos^2 x = 1 - \sin^2 x.$$

41. (a) The Taylor polynomials of degree 7 are

$$\text{For } \sin x, \qquad P_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}$$
$$\text{For } \sin x\cos x, \qquad Q_7(x) = x - \frac{2\,x^3}{3} + \frac{2\,x^5}{15} - \frac{4\,x^7}{315}$$

(b) The coefficient of x^3 in $Q_7(x)$ is $-2/3$, and the coefficient of x^3 in $P_7(x)$ is $-1/6$, so the ratio is

$$\frac{-2/3}{-1/6} = 4.$$

The corresponding ratios for x^5 and x^7 are

$$\frac{2/15}{1/120} = 16 \quad \text{and} \quad \frac{-4/315}{-1/5040} = 64.$$

(c) It appears that the ratio is always a power of 2. For x^3, it is $4 = 2^2$; for x^5, it is $16 = 2^4$; for x^7, it is $64 = 2^6$. This suggests that in general, for the coefficient of x^n, it is 2^{n-1}.

(d) From the identity $\sin(2x) = 2\sin x\cos x$, we expect that $P_7(2x) = 2Q_7(x)$. So, if a_n is the coefficient of x^n in $P_7(x)$, and if b_n is the coefficient of x^n in $Q_7(x)$, then, since the x^n terms $P_7(2x)$ and $2Q_7(x)$ must be equal, we have

$$a_n(2x)^n = 2b_n x^n.$$

Dividing both sides by x^n and combining the powers of 2, this gives the pattern we observed. For $a_n \neq 0$,

$$\frac{b_n}{a_n} = 2^{n-1}.$$

42. **(a)** For $f(x) = x^2$ we have $f'(x) = 2x$ so the tangent line is

$$\begin{aligned} y &= f(2) + f'(2)(x-2) = 4 + 4(x-2) \\ y &= 4x - 4. \end{aligned}$$

For $g(x) = x^3 - 4x^2 + 8x - 7$, we have $g'(x) = 3x^2 - 8x + 8$, so the tangent line is

$$\begin{aligned} y &= g(1) + g'(1)(x-1) = -2 + 3(x-1) \\ y &= 3x - 5. \end{aligned}$$

For $h(x) = 2x^3 + 4x^2 - 3x + 7$, we have $h'(x) = 6x^2 + 8x - 3$. So the tangent line is

$$\begin{aligned} y &= h(-1) + h'(-1)(x+1) = 12 - 5(x+1) \\ y &= -5x + 7. \end{aligned}$$

(b) Division by a CAS or by hand gives

$$\begin{aligned} \frac{f(x)}{(x-2)^2} &= \frac{x^2}{(x-2)^2} = 1 + \frac{4x-4}{(x-2)^2} \quad \text{so} \quad r(x) = 4x - 4, \\ \frac{g(x)}{(x-1)^2} &= \frac{x^3 - 4x^2 + 8x - 7}{(x-1)^2} = x - 2 + \frac{3x-5}{(x-1)^2} \quad \text{so} \quad r(x) = 3x - 5, \\ \frac{h(x)}{(x+1)^2} &= \frac{2x^3 + 4x^2 - 3x + 7}{(x+1)^2} = 2x + \frac{-5x+7}{(x+1)^2} = \quad \text{so} \quad r(x) = -5x + 7. \end{aligned}$$

(c) In each of these three cases, $y = r(x)$ is the equation of the tangent line. We conjecture that this is true in general.

(d) The Taylor expansion of a function $p(x)$ is

$$p(x) = p(a) + p'(a)(x-a) + \frac{p''(a)}{2!}(x-a)^2 + \frac{p'''(a)}{3!}(x-a)^3 + \cdots$$

Now divide $p(x)$ by $(x-a)^2$. On the right-hand side, all terms from $p''(a)(x-a)^2/2!$ onward contain a power of $(x-a)^2$ and divide exactly by $(x-a)^2$ to give a polynomial $q(x)$, say. So the remainder is $r(x) = p(a) + p'(a)(x-a)$, the tangent line.

43. **(a)** The Taylor polynomial is

$$P_{10}(x) = 1 + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160}$$

(b) All the terms have even degree. A polynomial with only terms of even degree is an even function. This suggests that f might be an even function.

(c) To show that f is even, we must show that $f(-x) = f(x)$.

$$\begin{aligned} f(-x) &= \frac{-x}{e^{-x}-1} + \frac{-x}{2} = \frac{x}{1 - \frac{1}{e^x}} - \frac{x}{2} = \frac{xe^x}{e^x - 1} - \frac{x}{2} \\ &= \frac{xe^x - \frac{1}{2}x(e^x - 1)}{e^x - 1} \\ &= \frac{xe^x - \frac{1}{2}xe^x + \frac{1}{2}x}{e^x - 1} = \frac{\frac{1}{2}xe^x + \frac{1}{2}x}{e^x - 1} = \frac{\frac{1}{2}x(e^x - 1) + x}{e^x - 1} \\ &= \frac{1}{2}x + \frac{x}{e^x - 1} = \frac{x}{e^x - 1} + \frac{x}{2} = f(x) \end{aligned}$$

44. **(a)** The Taylor polynomial is

$$P_{11}(x) = \frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320}.$$

(b) Evaluating, we get

$$P_{11}(1) = \frac{1^3}{3} - \frac{1^7}{42} + \frac{1^{11}}{1320} = 0.310281$$
$$S(1) = \int_0^1 \sin(t^2)\,dt = 0.310268.$$

We need to take about 6 decimal places in the answer as this allows us to see the error. (The values of $P_{11}(1)$ and $S(1)$ start to differ in the fifth decimal place.) Thus, the percentage error is $(0.310281 - 0.310268)/0.310268 = 0.000013/0.310268 = 0.000042 = 0.0042\%$. On the other hand,

$$P_{11}(2) = \frac{2^3}{3} - \frac{2^7}{42} + \frac{2^{11}}{1320} = 1.17056$$
$$S(2) = \int_0^2 \sin(t^2)\,dt = 0.804776.$$

The percentage error in this case is $(1.17056 - 0.804776)/0.804776 = 0.365784/0.804776 = 0.454517$, or about 45%.

CHECK YOUR UNDERSTANDING

1. False. For example, both $f(x) = x^2$ and $g(x) = x^2 + x^3$ have $P_2(x) = x^2$.
2. False. The approximation $\sin\theta \approx \theta - \theta^3/3!$ holds for θ in radians, not degrees.
3. False. $P_2(x) = f(5) + f'(5)(x-5) + (f''(5)/2)(x-5)^2 = e^5 + e^5(x-5) + (e^5/2)(x-5)^2$.
4. False. Since -1 is the coefficient of x^2 in $P_2(x)$, we know that $f''(0)/2! = -1$, so $f''(0) < 0$, which implies that f is concave down near $x = 0$.
5. False. The Taylor series for $\sin x$ about $x = \pi$ is calculated by taking derivatives and using the formula

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots.$$

The series for $\sin x$ about $x = \pi$ turns out to be

$$-(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \cdots.$$

6. True. Since f is even, $f(-x) = f(x)$ for all x. Taking the derivative of both sides of this equation, we get $f'(-x)(-1) = f'(x)$, which at $x = 0$ gives $-f'(0) = f'(0)$, so $f'(0) = 0$. Taking the derivative again gives $f''(-x) = f''(x)$, i.e., f'' is even. Using the same reasoning again, we get that $f'''(0) = 0$, and, continuing in this way, we get $f^{(n)}(0) = 0$ for all odd n. Thus, for all odd n, the coefficient of x^n in the Taylor series is $f^{(n)}(0)/n! = 0$, so all the terms with odd exponent are zero.
7. True. Since the Taylor series for $\cos x$ has only even powers, multiplying by x^3 gives only odd powers.
8. True. The coefficient of x^7 is $-8/7!$, so

$$\frac{f^{(7)}(0)}{7!} = \frac{-8}{7!}$$

giving $f^{(7)}(0) = -8$.

9. False. The derivative of $f(x)g(x)$ is not $f'(x)g'(x)$. If this statement were true, the Taylor series for $(\cos x)(\sin x)$ would have all zero terms.
10. True. Since the derivative of a sum is the sum of the derivatives, Taylor series add.
11. False. For example the quadratic approximation to $\cos x$ for x near 0 is $1 - x^2/2$, whereas the linear approximation is the constant function 1. Although the quadratic approximation is better near 0, for large values of x it takes large negative values, whereas the linear approximation stays equal to 1. Since $\cos x$ oscillates between 1 and -1, the linear approximation is better than the quadratic for large x (although it is not very good).

12. False. The Taylor series converges on its interval of convergence, whereas f may be defined outside this interval.

For example, the series

$$1 + x + x^2 + x^3 + \cdots \quad \text{converges to } \frac{1}{1-x} \text{ for } |x| < 1.$$

But $1/(1-x)$ is defined for $|x| > 1$.

13. True. For large x, the graph of $P_{10}(x)$ looks like the graph of its highest powered term, $x^{10}/10!$. But e^x grows faster than any power, so e^x gets further and further away from $x^{10}/10! \approx P_{10}(x)$.

14. False. For example, if $a = 0$ and $f(x) = \cos x$, then $P_1(x) = 1$, and $P_1(x)$ touches $\cos x$ at $x = 0, 2\pi, 4\pi, \ldots$.

15. False. If f is itself a polynomial of degree n then it is equal to its n^{th} Taylor polynomial.

16. True. By Theorem 10.1, $|E_n(x)| < 10|x|^{n+1}/(n+1)!$. Since $\lim_{n\to\infty} |x|^{n+1}/(n+1)! = 0$, $E_n(x) \to 0$ as $n \to \infty$, so the Taylor series converges to $f(x)$ for all x.

True

17. True. Since f is even, $f(x)\sin(mx)$ is odd for any m, so

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x(mx)\,dx = 0.$$

18. False. Since $f(-1) = g(-1)$ the graphs of f and g intersect at $x = -1$. Since $f'(-1) < g'(-1)$, the slope of f is less than the slope of g at $x = -1$. Thus $f(x) > g(x)$ for all x sufficiently close to -1 on the left, and $f(x) < g(x)$ for all x sufficiently close to -1 on the right.

19. True. If

$$P_2(x) = \text{Quadratic approximation to } f = f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2}(x+1)^2$$
$$Q_2(x) = \text{Quadratic approximation to } g = g(-1) + g'(-1)(x+1) + \frac{g''(-1)}{2}(x+1)^2$$

then $P_2(x) - Q_2(x) = (f''(-1) - g''(-1))(x+1)^2/2 < 0$ for all $x \neq -1$. Thus $P_2(x) < Q_2(x)$ for all $x \neq -1$. This implies that for x sufficiently close to -1 (but not equal to -1), we have $f(x) < g(x)$.

20. True. We have

$$L_1(x) + L_2(x) = (f_1(0) + f_1'(0)x) + (f_2(0) + f_2'(0)x) = (f_1(0) + f_2(0)) + (f_1'(0) + f_2'(0))x.$$

The right hand side is the linear approximation to $f_1 + f_2$ near $x = 0$.

21. False. The quadratic approximation to $f_1(x)f_2(x)$ near $x = 0$ is

$$f_1(0)f_2(0) + (f_1'(0)f_2(0) + f_1(0)f_2'(0))x + \frac{f_1''(0)f_2(0) + 2f_1'(0)f_2'(0) + f_1(0)f_2''(0)}{2}x^2.$$

On the other hand, we have

$$L_1(x) = f_1(0) + f_1'(0)x, \quad L_2(x) = f_2(0) + f_2'(0)x,$$

so

$$L_1(x)L_2(x) = (f_1(0) + f_1'(0)x)(f_2(0) + f_2'(0)x) = f_1(0)f_2(0) + (f_1'(0)f_2(0) + f_2'(0)f_1(0))x + f_1'(0)f_2'(0)x^2.$$

The first two terms of the right side agree with the quadratic approximation to $f_1(x)f_2(x)$ near $x = 0$, but the term of degree 2 does not.

For example, the linear approximation to e^x is $1+x$, but the quadratic approximation to $(e^x)^2 = e^{2x}$ is $1+2x+2x^2$, not $(1+x)^2 = 1+2x+x^2$.

22. False. The Taylor series for f near $x = 0$ always converges at $x = 0$, since $\sum_{n=0}^{\infty} C_n x^n$ at $x = 0$ is just the constant C_0.

23. True. When $x = 1$,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}.$$

Since $f^{(n)}(0) \geq n!$, the terms of this series are all greater than 1. So the series cannot converge

24. False. For example, if $f^{(n)}(0) = n!$, then the Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n,$$

which converges at $x = 1/2$.

PROJECTS FOR CHAPTER TEN

1. (a) A calculator gives $4\tan^{-1}(1/5) - \tan^{-1}(1/239) = 0.7853981634$, which agrees with $\pi/4$ to ten decimal places. Notice that you cannot verify that Machin's formula is *exactly* true numerically (because any calculator has only a finite number of digits.) Showing that the formula is exactly true requires a theoretical argument.

(b) The Taylor polynomial of degree 5 approximating $\arctan x$ is

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5}.$$

Thus,

$$\begin{aligned}\pi &= 4\left(4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)\right)\\ &\approx 4\left(4\left(\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5\right) - \left(\frac{1}{239} - \frac{1}{3}\left(\frac{1}{239}\right)^3 + \frac{1}{5}\left(\frac{1}{239}\right)^5\right)\right)\\ &\approx 3.141621029.\end{aligned}$$

The true value is $\pi = 3.141592653\ldots$.

(c) Because the values of x, namely $x = 1/5$ and $x = 1/239$, are much smaller than 1, the terms in the series get smaller much faster.

(d) (i) If $A = \arctan(120/119)$ and $B = -\arctan(1/239)$, then

$$\tan A = \frac{120}{119} \quad \text{and} \quad \tan B = -\frac{1}{239}.$$

Substituting

$$\tan(A+B) = \frac{(120/119) + (-1/239)}{1 - (120/119)(-1/239)} = 1.$$

Thus

$$A + B = \arctan 1,$$

so

$$\arctan\left(\frac{120}{119}\right) - \arctan\left(\frac{1}{239}\right) = \arctan 1.$$

(ii) If $A = B = \arctan(1/5)$, then

$$\tan(A+B) = \frac{(1/5) + (1/5)}{1 - (1/5)(1/5)} = \frac{5}{12}.$$

Thus

$$A + B = \arctan\left(\frac{5}{12}\right),$$

so

$$2\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{5}{12}\right).$$

If $A = B = 2\arctan(1/5)$, then $\tan A = \tan B = 5/12$, so

$$\tan(A+B) = \frac{(5/12)+(5/12)}{1-(5/12)(5/12)} = \frac{120}{119}.$$

Thus

$$A + B = \arctan\left(\frac{120}{119}\right),$$

so

$$4\arctan\left(\frac{1}{5}\right) = \arctan\left(\frac{120}{119}\right).$$

(iii) Using the result of part (a) and substituting the results of part (b), we obtain

$$4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right) = \arctan 1 = \frac{\pi}{4}.$$

2. (a) (i) Using a Taylor series expansion, we have

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{3!}h^3 + \cdots.$$

So we have

$$\frac{f(x_0) - f(x_0 - h)}{h} - f'(x_0) \approx \frac{f''(x_0)}{2}h + \cdots.$$

This suggests the following bound for small h:

$$\left|\frac{f(x_0) - f(x_0 - h)}{h} - f'(x_0)\right| \le \frac{Mh}{2},$$

where $|f''(x)| \le M$ for $|x - x_0| < |h|$.

(ii) We use Taylor series expansions:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \cdots$$
$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{3!}h^3 + \cdots.$$

Subtracting gives

$$\begin{aligned} f(x_0 + h) - f(x_0 - h) &= 2f'(x_0)h + \frac{2f'''(x_0)}{3!}h^3 + \cdots \\ &= 2f'(x_0)h + \frac{1}{3}f'''(x_0)h^3 + \cdots. \end{aligned}$$

So

$$\frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) + \frac{f'''(x_0)}{6}h^2 + \cdots.$$

This suggests the following bound for small h:

$$\left|\frac{f(x_0 + h) - f(x_0 - h)}{2h} - f'(x_0)\right| \le \frac{Mh^2}{6},$$

where $|f'''(x)| \le M$ for $|x - x_0| < |h|$.

(iii) Expanding each term in the numerator is a Taylor series, we have

$$\begin{aligned}f(x_0+2h) &= f(x_0)+2f'(x_0)h+2f''(x_0)h^2+\frac{4}{3}f'''(x_0)h^3\\&\quad+\frac{2}{3}f^{(4)}(x_0)h^4+\frac{4}{15}f^{(5)}(x_0)h^5+\cdots\\f(x_0+h) &= f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2}h^2+\frac{f'''(x_0)}{3!}h^3\\&\quad+\frac{f^{(4)}(x_0)}{4!}h^4+\frac{f^{(5)}(x_0)}{5!}h^5+\cdots,\\f(x_0-h) &= f(x_0)-f'(x_0)h+\frac{f''(x_0)}{2}h^2-\frac{f'''(x_0)}{3!}h^3\\&\quad+\frac{f^{(4)}(x_0)}{4!}h^4-\frac{f^{(5)}(x_0)}{5!}h^5+\cdots,\\f(x_0-2h) &= f(x_0)-2f'(x_0)h+2f''(x_0)h^2-\frac{4}{3}f'''(x_0)h^3\\&\quad+\frac{2}{3}f^{(4)}(x_0)h^4-\frac{4}{15}f^{(5)}(x_0)h^5+\cdots.\end{aligned}$$

Combining the expansions in pairs, we have

$$\begin{aligned}8f(x_0+h)-8f(x_0-h) &= 16f'(x_0)h+\frac{8}{3}f'''(x_0)h^3+\frac{2}{15}f^{(5)}(x_0)h^5+\cdots\\f(x_0+2h)-f(x_0-2h) &= 4f'(x_0)h+\frac{8}{3}f'''(x_0)h^3+\frac{8}{15}f^{(5)}(x_0)h^5+\cdots.\end{aligned}$$

Thus,

$$-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h)=12f'(x_0)h-\frac{6}{15}f^{(5)}(x_0)h^5+\cdots$$

so

$$\frac{-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h)}{12h}=f'(x_0)-\frac{f^{(5)}(x_0)}{30}h^4+\cdots.$$

This suggests the following bound for small h,

$$\left|\frac{-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h)}{12h}-f'(x_0)\right|\le\frac{Mh^4}{30},$$

where $|f^{(5)}(x)|\le M$ for $|x-x_0|\le|h|$.

(b) (i)

h	$(f(x_0)-f(x_0-h))/h$	Error
10^{-1}	0.951626	4.837×10^{-2}
10^{-2}	0.995017	4.983×10^{-3}
10^{-3}	0.9995	4.998×10^{-4}
10^{-4}	0.99995	5×10^{-5}

The errors are roughly proportional to h, agreeing with part (a).

(ii)

h	$(f(x_0+h)-f(x_0-h))/(2h)$	Error
10^{-1}	1.00167	1.668×10^{-3}
10^{-2}	1.00001667	1.667×10^{-5}
10^{-3}	1.0000001667	1.667×10^{-7}
10^{-4}	1.000000001667	1.667×10^{-9}

The errors are roughly proportional to h^2, agreeing with part (a).

(iii)

h	$(-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h))/(12h)$	Error
10^{-1}	0.99999667	3.337×10^{-6}
10^{-2}	0.9999999999667	3.333×10^{-10}
10^{-3}	0.99999999999999667	3.333×10^{-14}
10^{-4}	0.999999999999999999667	3.333×10^{-18}

The errors are roughly proportional to h^4, agreeing with part (a). This is the most accurate formula.

(c) (i)

h	$(f(x_0)-f(x_0-h))/h$	Error
10^{-1}	1.0001×10^{6}	1.00×10^{10}
10^{-2}	1.0001×10^{7}	1.00×10^{10}
10^{-3}	1.0101×10^{8}	1.01×10^{10}
10^{-4}	1.11111×10^{9}	1.11×10^{10}
10^{-5}	Undefined	Undefined
10^{-6}	-1.11111×10^{10}	-1.11×10^{9}
10^{-7}	-1.0101×10^{10}	-1.01×10^{8}
10^{-8}	-1.001×10^{10}	-1.00×10^{7}
10^{-9}	-1.0001×10^{10}	-1.00×10^{6}

(ii)

h	$(f(x_0+h)-f(x_0-h))/(2h)$	Error
10^{-1}	1×10^{2}	1×10^{10}
10^{-2}	1×10^{4}	1×10^{10}
10^{-3}	1.0001×10^{6}	1.0001×10^{10}
10^{-4}	1.0101×10^{8}	1.0101×10^{10}
10^{-5}	Undefined	Undefined
10^{-6}	-1.0101×10^{10}	-1.01×10^{8}
10^{-7}	-1.0001×10^{10}	-1.00×10^{6}
10^{-8}	-1.000001×10^{10}	-1.00×10^{4}
10^{-9}	-1.00000001×10^{10}	-1.00×10^{2}

(iii)

h	$(-f(x_0+2h)+8f(x_0+h)-8f(x_0-h)+f(x_0-2h))/(12h)$	Error
10^{-1}	1.25×10^{2}	1.00×10^{10}
10^{-2}	1.25×10^{4}	1.00×10^{10}
10^{-3}	1.25013×10^{6}	1.00×10^{10}
10^{-4}	1.26326×10^{8}	1.01×10^{10}
10^{-5}	Undefined	Undefined
10^{-6}	-9.99579×10^{9}	4.21×10^{6}
10^{-7}	$-9.9999995998\times10^{10}$	4.00×10^{2}
10^{-8}	$-9.99999999996\times10^{10}$	4.00×10^{-2}
10^{-9}	$-9.999999999999996\times10^{10}$	4.00×10^{-6}

For relatively large values of h, these approximation formulas fail miserably. The main reason is that $f(x)=1/x$ changes very quickly at $x_0=10^{-5}$. In fact, $f(x)\to\pm\infty$ as $x\to0$. So we must use very small values for h when estimating a limit (involving f and $x_0=10^{-5}$) as $h\to0$. Here, $h>10^{-5}$ is too big, since the values of x_0-h cross over the discontinuity at $x=0$. For smaller values of h, that make sure we stay on the good side of the abyss, these formulas work quite well. Already by $h=10^{-6}$, formula (c) is the best approximation.