Week 2 Assignment

Mathematical Models in Biology *An Introduction*

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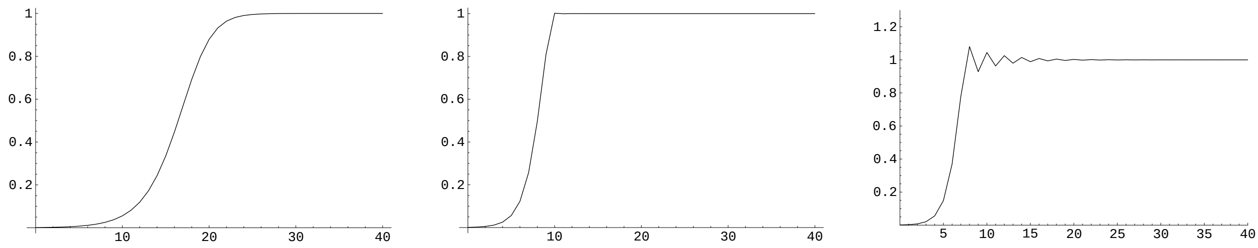
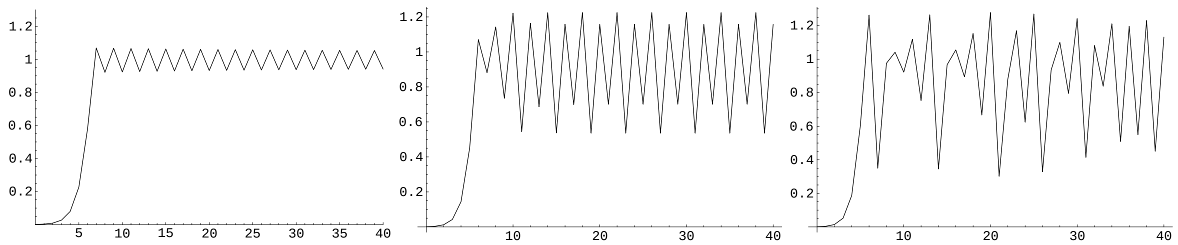
Ch 1.3 - Ex(4, 5, 6(a,c), 7(a), 8, 11)
Ch 1.4 - Ex(1 ,2 ,3)

1.3.4 Increasing r in steps of .25 we can get a good idea of what's happening as we change r. The first graph is at $r = .5$. It displays the approach to equilibrium without oscillations. The consecutive graphs up until $r = 1.25$ all display this behavior. We see at $r = 1.25$ that the graph goes just above equilibrium, almost a hint of what's to come. Then as r increases above 1.25 we se this overshoot become more and more severe resulting in oscillations before reaching equilibrium. Then at $r = 2$ the system begins to oscillate continuously never reaching equilibrium with "'two cycle behavior"'. When $r = 2.5$ the system has 4 cycle behavior and when $r = 2.75$ the system is in a chaotic state.



(a) r = .5 (b) r = 1.25 (c) 1.25 < r < 2.0



(d) r = 2 (e) r = 2.5 (f) r = 2.75

1.3.5 a)

$$\begin{aligned} \Delta N &= rN(1-N) \\ N^* = 0,1 &\Rightarrow \Delta N = 0 \\ \Delta N = 0 &\Rightarrow Stability \end{aligned}$$

1.3.6 a)

$$\begin{aligned} P_{t+1} &= 1.3P_t - .02P_t^2 \\ \Delta P &= P_{t+1} - P_t \\ P_{t+1} - P_t &= 1.3P_t - .02P_t^2 - P_t \;=\; .3P_t - .02P_t^2 \\ &= .02P_t(15 - P_t) \\ \Delta P &= 0 \Rightarrow P^* = 0,15 \end{aligned}$$

c)

$$\begin{aligned} \Delta P &= .2P(1 - P/20) \\ \Delta P &= 0 \Rightarrow P^* = 0,20 \end{aligned}$$

1.3.7 a) Before answering the question through linearization, if we think about it we are modeling life and the one thing that living being want to do is grow and spread. So if their are enough beings to sustain reproduction than the carrying capacity will be the sable point. In these models we could add a parameter that would alow us to control the number needed to sustain reproduction but in this model their is no such term so any displacement away from zero will increase the number to carrying capacity, making 0 population an unstable point.

First look at P* = 0, this makes our equation

$$\begin{aligned} P_{t+1} &= 1.3P_t - .02P_t^2 \\ \rightarrow\ 0 + p_{t+1} &= 1.3(0+p_t) - .02(0+p_t)^2 \\ p_{t+1} &= 1.3p_t - .02p_t^2 \end{aligned}$$

Now we say that the perturbation p_t is extremely small and the term $.02p_t^2 \approx 0$ so that $p_{t+1} \approx 1.3p_t$. This means that with each time step the perturbation is increased by a factor of 1.3 which will drive the population away from that point making $P^* = 0$ unstable.

Now looking at P* = 15 so our equation is

$$\begin{aligned} P_{t+1} &= 1.3P_t - .02P_t^2 \\ \rightarrow\ 15 + p_{t+1} &= 1.3(15+p_t) - .02(15+p_t)^2 \\ &= 19.5 + 1.3p_t - .02((15+p_t)(15+p_t)) \\ &= 19.5 + 1.3p_t - .02(225 + 30p_t + p_t^2 \\ 15 + p_{t+1} &= 15 + .7p_t + .02p_t^2 \\ p_{t+1} &= .7p_t + .02p_t^2 \end{aligned}$$

We can use the same argument of a small perturbation as before and end up with the perturbation being changed by a factor of .7 each time step which brings us back to that stability making $P^* = 15$ stable.

1.3.8

$$\begin{aligned} P_{t+1} &= P_t + rP_t(1-P_t) \\ \Delta P &= rP_t(1-P_t) \\ \Delta P &= 0 \ \Rightarrow\ P_t \ = \ 0,1 \end{aligned}$$

$$\begin{aligned} p_t &= p_t + rp_t(1-p_t) \\ p_t &\approx p_t(1+r) \\ |p_t(1+r)| &> 1 \ \ if \ |r| > 0 \end{aligned}$$

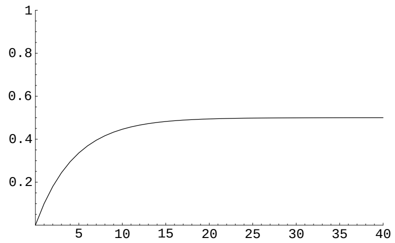
$$\begin{aligned} p_t + 1 &= p_t + 1 + r(p_t+1)(1-p_t+1) \\ &= p_t + 1 + r(-p_t - p_t^2) \\ &= p_t - rp_t - rP_t^2 + 1 \\ p_{t+1} &= p_t(1-r) - rp_t^2 \\ p_{t+1} &\approx p_t(1-r) \\ |p_t(1-r)| &< 1 \ \ if \ 0 < |r| < 1 \end{aligned}$$

1.3.11 a) The value for r should be $0 < r < 1$. The reason for this is that if for some reason you had no oxygen in your blood than the $(L - B)$ value would be just L and if $r < 1$ than you could have more oxygen getting to your blood than your lungs could physically hold. Also if $r < 0$ than your lungs would be taking oxygen out of your blood even when there is no oxygen there to be taken out.

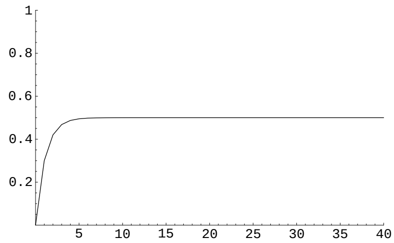
b) For this question we just ave to solve the equation given to us for L and substitute it into the ΔB equation.

$$\begin{aligned} L + B &= K \\ L &= K - B \\ \Delta B &= r(L - B) \\ \Delta B &= r(K - 2B) \end{aligned}$$

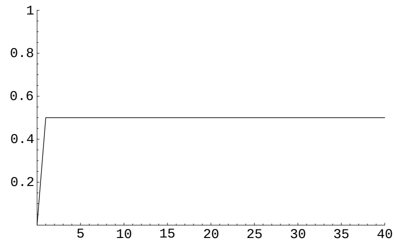
c) When $0 < r \leq .5$, r has the effect of increasing the rate of how fast B reaches equilibrium. When $.5 < r < 1$ we still reach equilibrium but the high growth factor causes us to over shoot the equilibrium and then drop back down below equilibrium in an oscillation that eventually reaches and equilibrium. The closer you get to 1 the longer it takes to reach equilibrium. If found that for every order of magnitude closer to 1 I brought r, I had to add an order of magnitude more steps to se the graph reach equilibrium. Then when you get to $r = 1$ the B oscillates forever never equalizing, and $1 < r$ oscillates out of control and blows up.



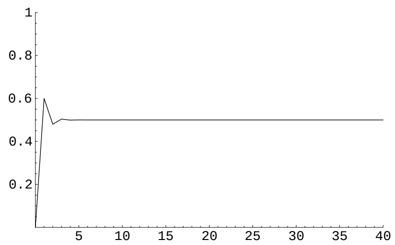
(g) r = .1



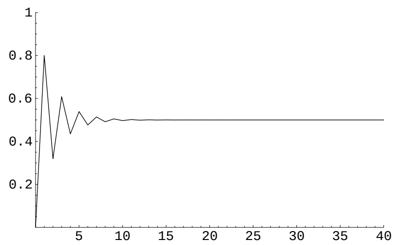
(h) r = .3



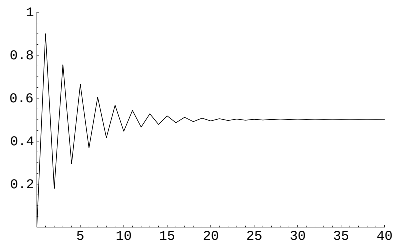
(i) r = .5



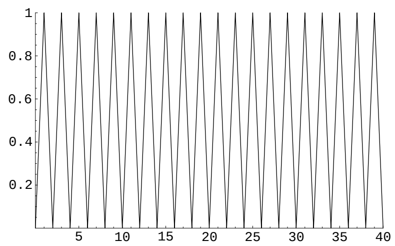
(j) r = .6



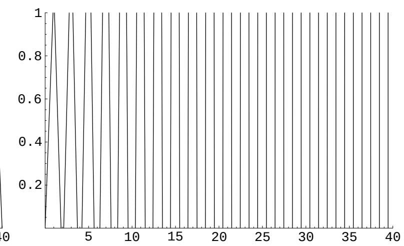
(k) r = .8



(l) r = .9



(m) r = 1.0



(n) r = 1.1

d) Setting $\Delta B = 0$ and then solving for B gives us B^* which is $K/2$. This agrees with what we saw in the previous graphs which reached equilibrium at $B = .5$. Intuitively why does this make sense? If we look at the way we arrived at K it does. K is a constant that says, in a single breath the sum of oxygen concentration in the lungs and oxygen concentration in the blood is a constant concentration.

$$\begin{aligned} \Delta B &= r(K - 2B) \\ 0 &= r(K - 2B) \\ r2B &= rK \\ 2B &= k \\ B^* &= \frac{K}{2} \end{aligned}$$

e)

$$\begin{aligned} B &= B^* + b \\ \Delta B &= r(K - 2(B^* + b)) \\ &= r(K - K - 2b) \\ &= r(-2b) \end{aligned}$$

f)

$$\begin{aligned} \Delta B &= r(-2b) \\ &= b_{t+1} - b_t \\ b_{t+1} &= b_t + r(-2b) \end{aligned}$$

$$\begin{aligned} \Delta B &= B_{t+1} - B_t \\ B_{t+1} &= B_t + r(K - 2B_t) \end{aligned}$$

g) Let the volume of the lungs be V_L and the volume of the blood stream be V_B. Then the total volume of oxygen would be $K = LV_L + BV_B$

1.4.1 a) For a particular value of P_t, if the relative growth rate is larger than 1, then the population will *increase* over the next time interval, whereas if it is smaller than 1, the population will *decrease*.

b) Negative relative growth rates don't make sense because if you had a relative growth rate that was negative your population would be going either from negative to positive or vice versa, which makes no sense. You could have a relative growth rate of zero though, it would just mean that your population was instantly decimated.

c) Geometric models have constant relative growth rates so

$$\frac{P_{t+1}}{P_t} = \lambda$$

Logistic models have the general form of

$$P_{t+1} = (1 + r)P_t - \frac{rP_t^2}{k}$$

which would give them an expression for the relative growth rate like

$$\frac{P_{t+1}}{P_t} = (1 + r) - \frac{rP_t}{k}$$

For the discrete logistic model we have

$$P_{t+1} = Pe^{r(1-P_t/k)}$$

which would give us an expression of the form

$$\frac{P_{t+1}}{P_t} = e^{r(1-P_t/k)}$$

The last model is

$$P_{t+1} = \frac{\lambda P_t}{(1 + aP_t)^\beta}$$

which will look like

$$\frac{P_{t+1}}{P_t} = \frac{\lambda}{(1 + aP_t)^\beta}$$

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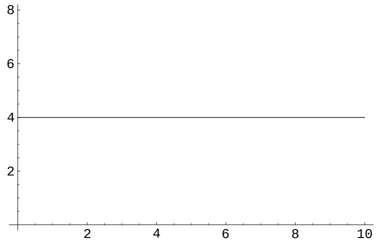
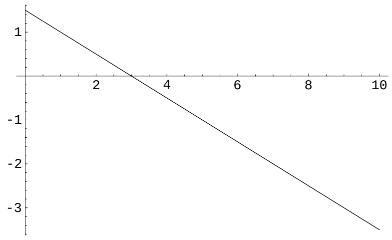
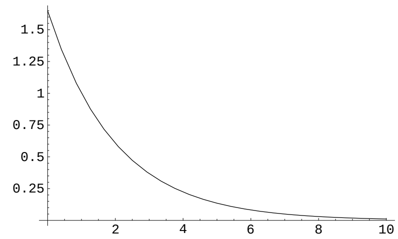


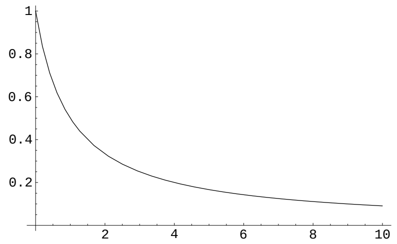
(o) geometric



(p) logistic



(q) discrete logistic

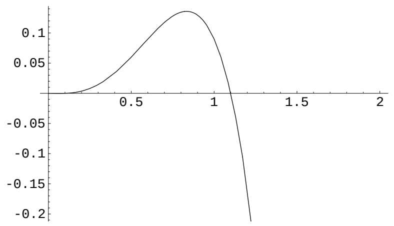


(r) last model in section 1.4.1

d)

1.4.2 This effect is directly correlated to the P term being multiplied by everything. It has the effect of causing an unstable point at $P = 0$ which is important in a model because if there is no population to begin with that their will never be a population. It also says in a way that the reproduction rate is directly related to the amount of population, the lower the population the lower the chance of finding a mate.

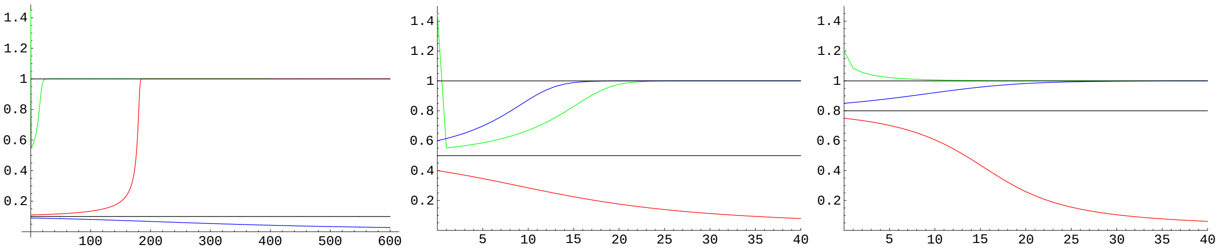
1.4.3 a) The range $0 < L < K$ is just any range of numbers grater than zero and less than K. $\Delta P/P < 0$ when $0 < P < L$ or $P > K$ is saying that the Population must be grater than L in order to sustains reproduction but if P is grater than K than the population is over carrying capacity. So L is the minimum population you must have and K is your carrying capacity. A possible graph of $\Delta P/P$ vs P would be



(s) $\Delta P/P$ vs P, for $K = 1.1$ and $L = .1$

b) The reason that $\Delta P/P = P(K - P)(P - L)$ has what we are looking for is because if $P > K$ or $P < L$ then the per capita growth rate is negative, and if $P = 0$ than everything goes to zero. We have three stable points here, $P^* = K,\ L,\ 0$. $P = L$ is unstable, and $P = K, 0$ is stable. This is a better model than the logistic model because their is a minimum population parameter which you would want to consider in a population model.

c) The graphs below definitely have the behavior we could expect, three equilibrium points at $P^* = K, L, 0$ with $P^* = K, 0$ stable and $P^* = L$ unstable. The green line represents the behavior if the initial population is greater than the carrying capacity, the red line is if the initial population is below the minimum population, and blue is if the initial population is between the carrying capacity and minimum population.



(t) $\Delta P/P$ vs P, for $K = 1.1$ and $L = .1$ (u) $\Delta P/P$ vs P, for $K = 1.1$ and $L = .1$ (v) $\Delta P/P$ vs P, for $K = 1.1$ and $L = .1$

d) We need a model where per capita growth rate is always bigger than -1