Mathematical Models in Biology *An Introduction*
by Elizebeth Allman and John A. Rhodes

2.3.1. The model does behave as expected, showing slow exponential growth in both classes, with decaying oscillations superposed.

2.3.4. a. In zeroing out the first row, no new ungerminated seeds are added to the population. Since the $(2,1)$ entry has been replaced with zero, no ungerminated seeds progress into the class of sexually immature plants. This eliminates the class of ungerminated seeds from the population. (One reason for considering this model would be to understand the effect of ungerminated seeds on the population dynamics, by imagining what would happen in their absence.)
b. The dominant eigenvalues of the model in the text is 1.1694 and the dominant eigenvalue of the altered matrix is 1.1336. This means that both models predict exponential growth, though the growth rate for the model with no ungerminated seeds is slightly slower. If the ungerminated seed entry of the dominant eigenvectors is discarded, there is also little difference in the stable stage vector for the two models.
c. The ungerminated seeds might be gathered by animals and spread throughout a region, possibly germinating in a later year and spreading the plant species. Also, if the plants have a bad year (due to factors not included in the model, such as drought, extreme cold, fire, etc.) and many fail to survive, the ungerminated seeds still remain in the area despite the temporary adverse growing conditions. If they then germinate at a later date, this may help the population recover. Even though they have little effect on the 'normal year' population dynamics, the ungerminated seeds may well be important.

2.3.5. a. The model should produce slow exponential growth. One way to see this is to notice that after one time step 40% of the first class survives to reproduce and 30% remain in the first class. Of the 30%, the model indicates that 40%, or $(.3)(.4) = 12\%$ will survive to reach the reproduction stage after a second time step. This means that at least $.4 + .12 = 52\%$ of the first class will survive to reproduce. Since on average, each adult produces two offspring, we should expect at least $(.52)2 = 1.04 > 1$ offspring produced by individual members of the first class on average. Thus, the population will grow slowly. In fact, the growth rate should be a little larger than 1.04, since $(.3)^2(.4) = .036 = 3.6\%$ of the first class progress into the second stage after three time steps and then reproduce. Similarly, for four, five, ... time steps. Clearly, the situation is somewhat complicated and an eigenvalue analysis can help us understand the growth trend more easily.
b. The eigenvalue 1.0569 is dominant with eigenvector $(.9353, .3540)$. The other eigenvalue is $-.7569$ with corresponding eigenvector $(-.8841, .4672)$.
c. The intrinsic growth rate is 1.0569, a number a little bit bigger than 1.04 as anticipated by (a). The stable stage distribution is $(2.6423, 1)$.
d. Using eigenvectors calculated by MATLAB, $(5,5) = 9.0100(.9353, .3540) + 3.8757(-.8841, .4672)$.
e. $\mathbf{x}_t = 9.0100(1.0569)^t(.9353, .3540) + 3.8757(-.7569)^t(-.8841, .4672)$.

2.3.7. The dominant eigenvalue is .6791 so the coyote population will decline rather rapidly. The stable stage distribution is $(2.2636, 1, 7.5877)$.

2.3.9. a. The transition matrix $P = \begin{pmatrix} 0 & 5 \\ 1/6 & 1/4 \end{pmatrix}$ is for an Usher model.
b. The dominant eigenvalue is 1.0464 with eigenvector $(.9788, .2048)$. The other eigenvalue is $-.7964$ with eigenvector $(-.9876, .1573)$.
c. The intrinsic growth rate is 1.0464 and the population will grow. The stable stage distribution is $(4.7783, 1)$.

2.4.1. a. A: $\lambda_1 = 1$ and $\lambda_2 = .6$; B: $\lambda_1 = -1$ and $\lambda_2 = 5$; C: $\lambda_1 = -3$ and $\lambda_2 = 2$
b. A: $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (1, -1)$; B: $\mathbf{v}_1 = (-2, 1)$, $\mathbf{v}_2 = (1, 1)$; C: $\mathbf{v}_1 = (-3, 2)$, $\mathbf{v}_2 = (1, 1)$

2.4.4. a. $\lambda_1 = \lambda_2 = 2$, $\mathbf{v}_1 = (1, 0)$, and $\mathbf{v}_2 = (0, 1)$
b. $\lambda_1 = \lambda_2 = 2$ and $\mathbf{v}_1 = (1, 0)$. However, it is impossible to find a second eigenvector, since $B - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the only solutions to $(B - 2I)\mathbf{x} = \mathbf{0}$ are $(c, 0) = c\mathbf{v}_1$. (The 1 in the $(1, 2)$ entry forces the second entry of any nonzero eigenvector to be zero.)