Non-Linear Dynamics Homework Solutions
Week 2

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Please email me at smachr09@evergreen.edu with any questions or concerns reguarding these solutions.

For the excercises from section 3.2, we sketch all qualitatively different vector fields that arise as r is varied, show that transcritical bifurcations occur at critical values of r to be found and sketch the bifurcation diagram of x^* vs r.

3.2.3

$$\dot{x} = x - rx(1-x)$$

First we locate the fixed points of our system as a function of r. To do this we factor our expression for $\dot{x}$.

$$\begin{aligned} \dot{x} &= x - rx(1-x) \\ &= x(1 - r(1+x)) \end{aligned}$$

If $r = 0$, then $\dot{x} = x$ and $x^* = 0$ is the only fixed point and is unstable. Otherwise, $r \neq 0$, and so $\dot{x}$ has zeros at $x^* = 0$ and $x^* = (r-1)/r$ (Note that we need $r \neq 0$ to divide by r). Since $\dot{x}$ is a quadratic polynomial when $r \neq 0$, its graph takes the form of a parabola which intersect the x axis at the fixed points we found above. Since r is the leading term of the polynomial $\dot{x}$, the sign of r determines whether the parabola points up or down. Peicing together this information we obtain the vector fields below.

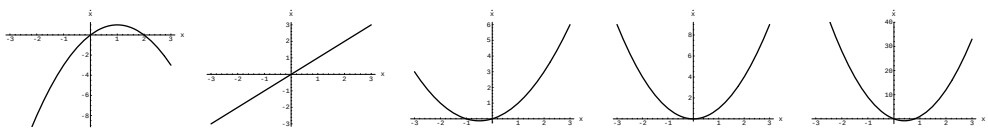

Figure 1: Qualitatively Different Vector Fields

To see that the fixed point point $x^* = 0$ switches stability at a critical value (ie that a transcritical bifurcation occurs), we could simply look at the diagrams above, or we could see

that if we let $\dot{x} = f(x)$, then $f'(x) = 2rx + (1 - r)$. Evaluated at the fixed point in question, this reduces to $f'(x) = (1 - r)$. Thus we see that for $r < -1$, $f'(x) < 0$ indicating that the fixed point is stable. For $r > -1$, $f'(x) > 0$ indicating that the fixed point is unstable. Hence, we have a transcritical bifurcation at $r_c = 1$.

We give Figure 2 for a 3-D plot of $\dot{x}$ vs r and x and the bifurcation diagram.

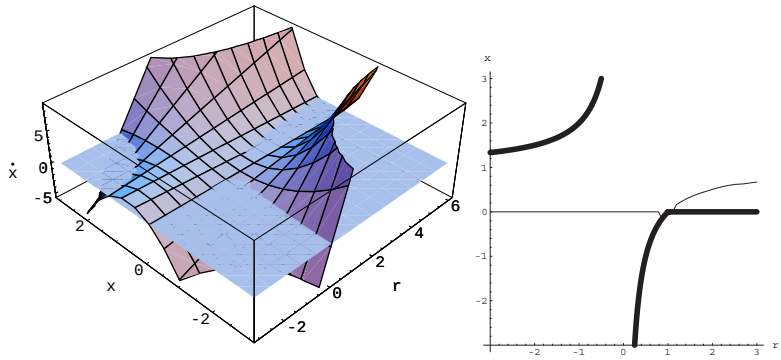

Figure 2: 3-D plot of $\dot{x}$ along with Bifurcation Diagram

3.2.4

$$\dot{x} = x(r - e^x)$$

As with 3.2.3, we start looking for the fixed points of the system. The point $x^* = 0$ will always be fixed. We can also have $\dot{x} = 0$ if $r - e^x = 0$, which occurs iff $\ln r = x$, so $x^* = \ln r$ is our other fixed point. Note that $\ln r$ is undefined if $r < 0$. When r is such that $0 < r < 1$, $\ln r < 0$. When $r = 1$ $\ln r = 0$ and when $r > 1$ $\ln r > 0$. Each of these regions corresponds to a qualitatively different vector field, each of which we plot in Figure 3.

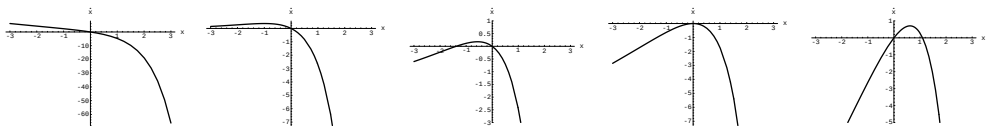

Figure 3: Qualitatively Different Vector Fields

Again, we could resort to a graphical analysis to convince us that a transcritical bifurcation occurs at $r_c = 1$, but as with 3.2.3, we shall present a more rigorous arguement. First, we find that if $\dot{x} = f(x)$, the $f'(x) = r - (x + 1)e^x$, which simplifies to $f'(x) = r - 1$ when $x = 0$. This is positive for $r > 1$ and negative for $r < 1$ indicating that stability of the fixed point switches at $r_c = 1$.

Figure 4 shows a 3-D plot of $\dot{x}$ as a function of r and x, as well as a bifurcation diagram of the system.

3.4.4 This problem is as those from Section 3.2, only you must show that a pitchfork bifurcation occurs (instead of a transcritical) and determine whether it is a supercritical or subcritical bifurcation.

$$\dot{x} = x + \frac{rx}{1 + x^2}$$

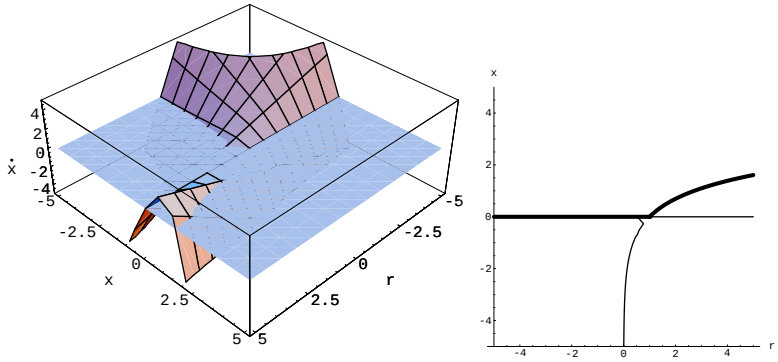

Figure 4: 3-D Plot of $\dot{x}$ and Bifurcation Diagram

Te begin we factor to get $\dot{x} = x(1 + r/(1 + x^2))$. Thus $\dot{x} = 0 \Longrightarrow x^* = 0$ or $1 + r/(1 + x^2) = 0$. The latter condition implies that $x^* = \sqrt{r - 1}$. For $r < 1$, the fixed points are not real, for $r = 1$ we get the same fixed point we always had, and for $r > 1$ we have two new fixed points. Thus we have a bifurcation occuring at $r_c = -1$ and what's more is that it must be some kind of pitchfork bifurcation. When we plot these solutions, it is clear that we have a subcritical pitchfork bifurcation. See Figures 5 and 6.

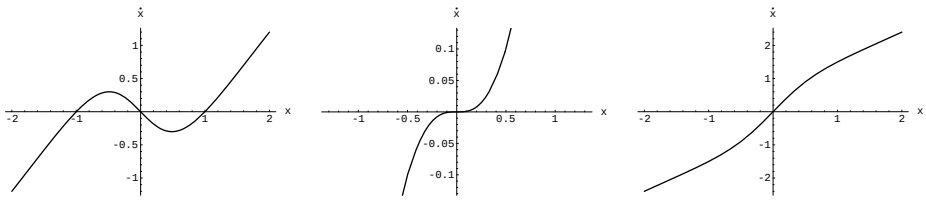

Figure 5: Qualitatively Different Vector Fields

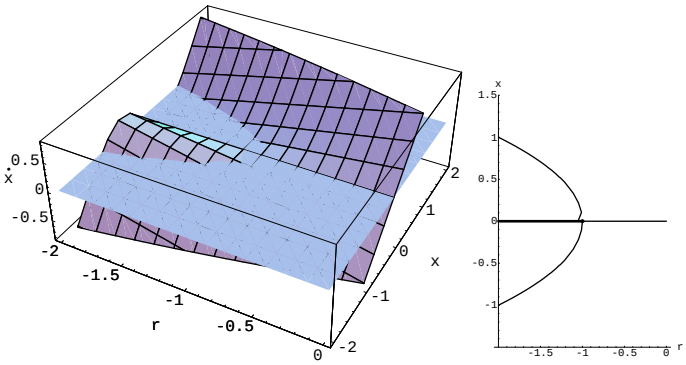

Figure 6: 3-D Plot of $\dot{x}$ and Bifurcation Diagram

For the remaining problems of this section we are to determine what kind(s) of bifurcation occurs, where they occur and sketch bifurcation diagrams of each systems fixed points.

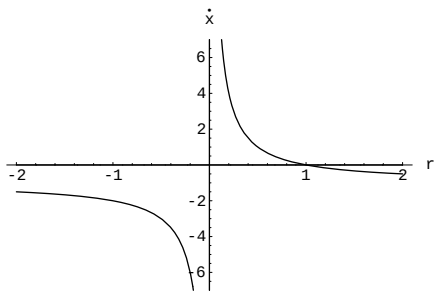

Figure 7: Plot of Fixed Points x^* vs. r

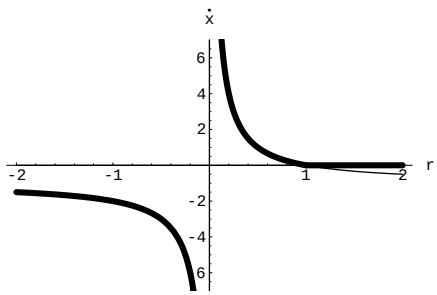

Figure 8: Bifurcation Diagram

3.4.6

$$\dot{x} = rx - \frac{x}{1+x}$$

Factoring this equation, it is easy to see that fixed points will lie along $x^* = 0$ and $x^* = (1/r)-1$ (when $r \neq 0$). When we plot these two solution sets in Figure 7, we can see that there apears to be a transcritical bifurcation.

To check that this is the case, note that if $\dot{x} = f(x)$, then $f'(x) = r - (1+x)^{-2}$. When $x^* = 0$, this simplifies to $f'(x) = r - 1$. This implies that this constant fixed point goes from being unstable for $r < 1$ to stable for $r > 1$ (this follows from the signs of the derivatives there). This implies that at $r_c = 1$ there is a transcritical bifurcation. Figure 8 shows a complete bifurcation diagram.

3.4.7

$$\dot{x} = 5 - re^{-x^2}$$

To begin, we find that all fixed points must satisfy $x^* = \pm\sqrt{-\ln 5/r}$ by solving for x in the equality $\dot{x} = 0$. Note that this expression is only defined when $r > 5$ since otherwise we might not be able to divide by r and certainly wouldn't be able to take the natural log of 5 divided by r. Also note that under these conditions $-\ln 5/r > 0$ making the square root well defined. For $r \leq 5$, $\dot{x} \neq 0$ since the maximum value that the term e^{-x^2} can take on is 1.

By graphing this solution curve, and considering our discussion of the vlues of r corresponding to fixed points, we see that the bifurcation occuring at the critical value of $r_c = 5$ is of the blue-sky variety (see Figure 9 for bifurcation diagram and Figure 10 for a 3D plot of $\dot{x}$ vs. r and x, which justifies the stabilities as given).

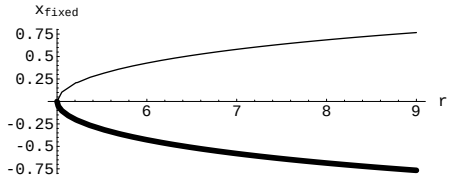

Figure 9: Bifurcation Diagram

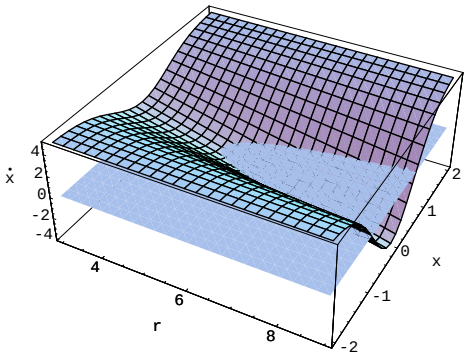

Figure 10: 3D Plot of $\dot{x}$ vs. r and x

3.4.10

$$\dot{x} = x(r + \frac{x^2}{1+x^2})$$

By factoring our expression for $\dot{x}$, we see that fixed points occurr iff $x = 0$ or $r + (x^2)/(1 + x^2) = 0$. If the latter holds then $x = \pm\sqrt{-r/(1+r)}$. This implies that r must be such that $-1 < r \leq 0$, since otherwise $-r/(1+r) < 0$, in which case the square root will not produce real outputs.

As $r \to 0$ from the left the solutions $x = \pm\sqrt{-r/(1+r)} \to 0$. This indicates that the number of solutions goes from 3 to 1 as r is varied continuously from just above zero to just below it. We conclude that a subcritical pitchfork bifurcation occurs at the the critial value $r_c = 0$. We also get some kind of bizzare bifurcation at $r = 1$, which might not have a name yet. I'm gonna call it a ω-♦ *ski-slope bifurcation*, since two infitely high and steep "ski slopes" come out of nowhere as r becomes greater than -1. See Figures 11 and 12 for bifurcation diagram and 3D plot of $\dot{x}$ vs r and x.

3.5.7 We nondimensionalize the system given by the logistic equation

$$\dot{N} = rN(1 - N/K), \qquad N(0) = N_0.$$

a) Find the dimension of the parameters r, k and N_0.
The parameter K must be in terms of population elements (such as people or tigers or whatever population is in question) so that the N/K term is unitless (this is necessary for $1 - N/K$ to make any sense. The parameter r must make the dimensions of $\dot{N}$ be in terms of population elements per unit time, so r must have units of per unit time since $N(1 - N/K)$ has units of populaton elements.

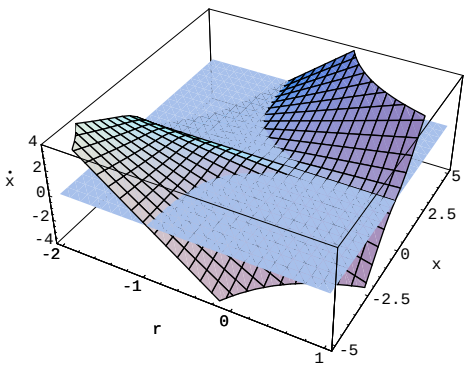

Figure 11: Bifurcation Diagram

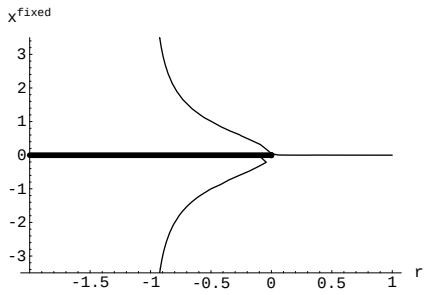

Figure 12: 3D Plot of $\dot{x}$ vs. r and x

b) Rewrite the system in the form

$$\frac{dx}{d\tau} = x(1-x), \qquad\qquad x(0) = x_0.$$

To do this we are going to define $x = N/K$, so that the term $(1 - N/K) = (1 - x)$, which is part of what we need. Next, we want to come up with a definition for τ that gets rid of the need for r in our expression. Note that whatever choice we make, by the chain rule we must have that

$$\frac{dx}{d\tau} = \frac{dx}{dN} \cdot \frac{dN}{dt} \cdot \frac{dt}{d\tau},$$

which evaluates to $dx/d\tau = rx(1-x) \cdot dt/d\tau$. We want to have $dt/d\tau \cdot r = 1$. Setting $\tau = rt$ we get precisely that. Since at $\tau = 0$ we have $t = 0$ (we are assuming here that $r > 0$), $x_0 = x|_{\tau=0} = N/K|_{\tau=0} = N/K|_{\tau=0} = N_0/K$.

c) Find a different nondimensionalization in terms of u and τ such that $u_0 = 1$ always.

To do this we assume that $N_0 \leq 0$ so that we can set $u = N/N_0$ and $\tau = rt$. Using the chain rule as in part b) we find that $du/d\tau = u(1 - N_0/(Ku))$. Note that this is indeed nondimensionalized, since N_0/K is unitless (since both K and N_0 have the same units). Next we find that $u_0 = u|_{\tau=0} = N/N_0|_{\tau=0} = N/N_0|_{\tau=0} = N_0/N_0 = 1$, as it should.

d) What are the advantages of one nondimensionalization over the other?

The 1st nondimensionalization is easier in that $x^* = 1$ and $x^* = 0$ are the only fixed points and don't vary given specific parameters. The 2nd nondimensionalization is easier in that the initial condition is always 1, which is only a possible translation if $N_0 \leq 0$.

3.6.3 Consider the system given by

$$\dot{x} = rx + ax^2 - x^3, \qquad\qquad a, r \in \mathbf{R}.$$

a) Sketch all qualitatively different bifurcation diagrams that can be obtained by varying a.

Since $\dot{x} = x(r - ax - x^2)$, we get that fixed points occur at $x^* = 0$ and $x^* = \frac{a}{2} \pm \frac{\sqrt{a^2+4r}}{2}$. Thus when $r = 0$, There are two fixed points, one at $x^* = 0$ and the other at $x^* = a$. Also, the vertex of the parabolic shape formed by the parametric plot of $x^* = \frac{a}{2} \pm \frac{\sqrt{a^2+4r}}{2}$ occurs when the term after the $\pm$ sign is zero, which occurs when $r = \frac{-a^2}{4}$. This curve in (r, a) corresponds to blue sky bifurcations, except when $r = a = 0$ which corresponds to a supercritical pitchfork bifurcation. All other places in the phase space where $r = 0$ are transcritical bifurcations. Finally we see that the qualitively different bifurcation diagrams of x^* vs r can be classified according to whether $a < 0$, $a = 0$ or $a > 0$. Figure 13 shows three dimensional plots of $\dot{x}$ vs r and x for a values in each of these regions, and Figure 14 shows the corresponding bifurcation diagrams in each of those regions.

b) In the (r, a) plane plot the regions which correspond to qualitatively different classes of vector fields. Identify and classify these regions.

We've already done the grunt work, all we need to do is make the graph. See Figure 15.

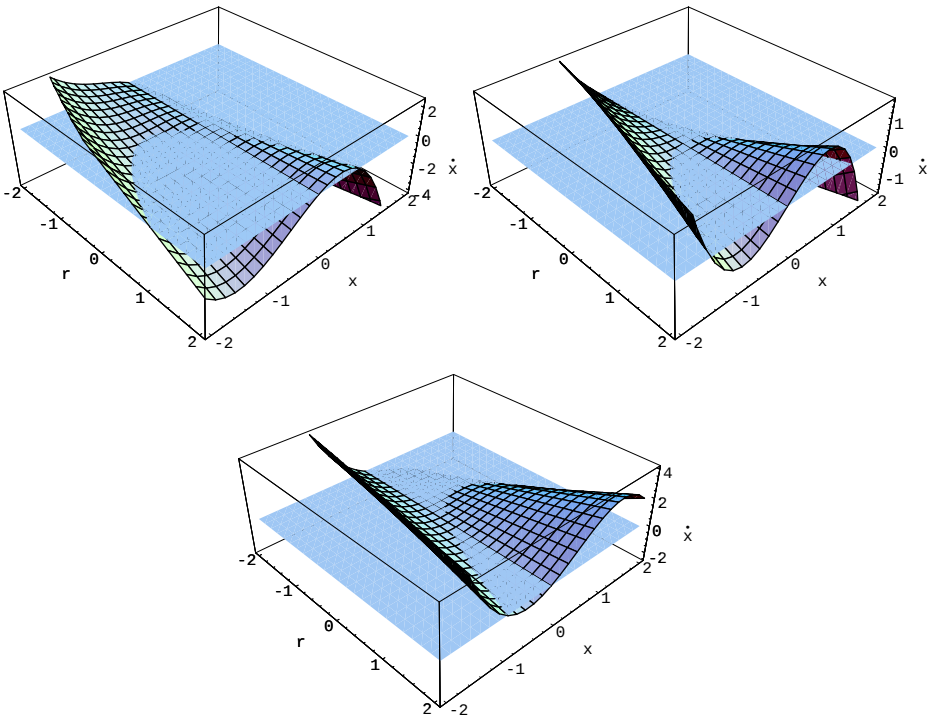

Figure 13: 3D Plot of $\dot{x}$ vs. r and x for $a < 0$, $a = 0$ and $a > 0$ (from left to right, and then down)

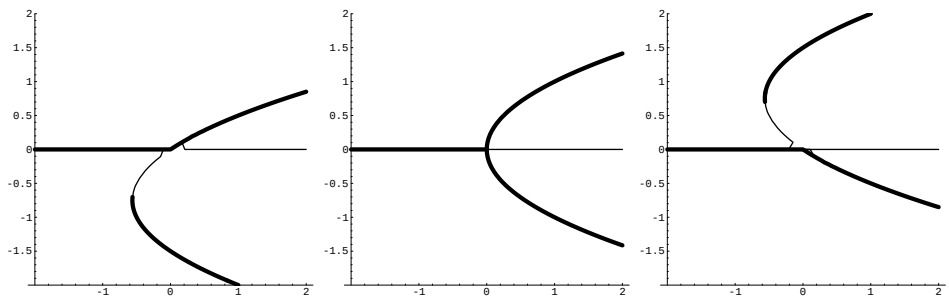

Figure 14: Bifurcation Diagrams for $a < 0$, $a = 0$ and $a > 0$ (from left to right)

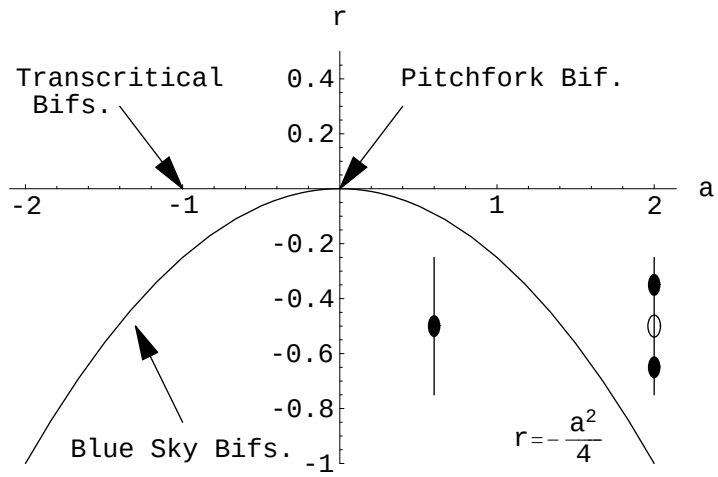

Figure 15: (r, a) Phase Plane