Non-Linear Dynamics Homework Solutions

Week 6

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Please email me at smachr09@evergreen.edu with any questions or concerns reguarding these solutions.

6.8.3 Locate annd find the index of the system given by

$$\begin{aligned} \dot{x} &= y - x \\ \dot{y} &= x^2 \end{aligned}$$

First we find the x-nullcline to be $y = x$ and the y-nullcline to be $x = 0$. Next plot the nullclines and figure out what direction the field flows along each nullcline. Then we fill in the vector field by remembering that the field inbetween any two nullclines points in a direction which is inbetween the direction of the field at those nullclines. Now we use the hint and draw a closed curve around the fixed point and arrive at Figure 1.

Now as we start at zero radians on the circle, and move counterclockwise around it, we see that the arrows never make ar full revolution, implying that the index must be zero.

6.8.6 A closed orbit in the phase plane encircles S saddles, N nodes, F spirals and C centers, all of the ususal type. Show that $N + F + C = 1 + S$.

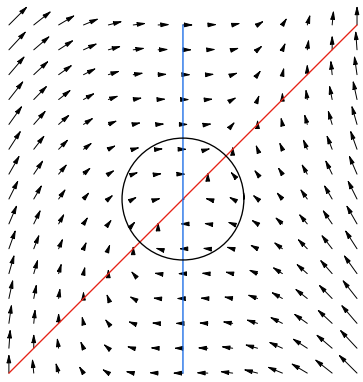

Figure 1: Plot of Vector Field With Closed Curve

Let I_S be the index of a spiral, I_N the index of a node, and so on. Closed orbits have an index of 1, so by Theorem 6.8.1, from Strogatz,

$$\begin{aligned} 1 &= SI_S + NI_N + FI_F + CI_C \\ &= -S + N + C + F. \end{aligned}$$

Rearanging this equation we get the desired result.

6.8.9 Our claim is that the statement is false. We show this by providing the following counter-example

$$\begin{aligned} \dot{r} &= r(r - R_o)(r - R_i) \\ \dot{\theta} &= R - (R_o - R_i)/2 \end{aligned}$$

This system has the desired properties. It is smooth, has an outer stable orbit of R_o and an inner one at R_i, and halfway between the sign of $\dot{\theta}$ switches, so that one of the stable orbits flows one way and the other one flows the other. However, the only fixed point in this system is at the origin.

6.8.12 *Matter and Anitmatter Problem* We explore the analogy between particle anti-particle collisions and bifurcations of fixed points by studying a two dimensional version of the saddle node bifurcation given by the system

$$\begin{aligned} \dot{x} &= a + x^2 \\ \dot{y} &= -y \end{aligned}$$

a) Find and classify all fixed points of the system.
First we find x-nullclines at $x = \pm\sqrt{-a}$ and y-nullclines at $y = 0$, Thus for $a = 0$ we have precisely one fixed point and for $a < 0$ we have precisely two, indicating a saddle-node (blue-sky) bifurcation. When we linearize (first finding the Jacobian and then evaluating at the fixed points) we get the linearization matrices

$$A = \begin{pmatrix} \pm 2\sqrt{-a} & 0 \\ 0 & -1 \end{pmatrix}$$

with the plus sign for the positive square root and the minus for the negative. These have $D = \mp 2\sqrt{-a}$ and $T = \pm 2\sqrt{-a} - 1$ with the plus and minus signs assigned as before. For $(+\sqrt{-a}, 0)$, we therefore always have a negative determinant, implying that this fixed point is always a saddle. For $(-\sqrt{-a}, 0)$, the determinant is always positive, implying that the point is always either a spiral or a node. Furthermore, the trace is always negative, so whatever it is, it's always stable. To figure when it is what, we compute

$$T^2 - 4D = -4a + 4\sqrt{-a} + 1$$

and note that since our fixed point only exists for negative values of a, this is always positive, implying that $D < T^2/4$ and that consequently the point is always an unstable node.

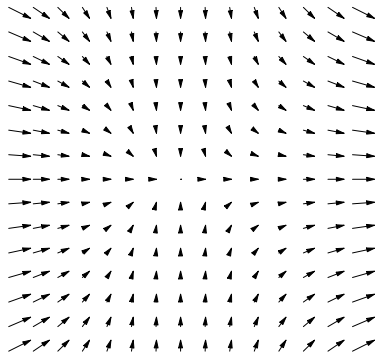

Figure 2: From problem 6.8.12

When $a = 0$, linear analysis fails, since it predicts a whole line of nonisolated fixed points, which we have already shown does not exist. For this case, we look at Figure 2, which shows the vector field for $a = 0$.

This is not any of the kinds of fixed points that we have talked about, so if I may, I would like to take the liberty to call it a Wal-Mart fixed point, since there are infinitely many ways in but only one way out.

b) Show that the sum of indices is conserved as a is varied.

One of the two distinct fixed points is a saddle, and consequently always has an index of -1, whereas the other is always a node and so always has an index of 1. When $a = 0$, the two fixed points have collided, giving us the vector field of Figure 2, which we can readily verify has an index of zero, by keeping track of the direction that vectors point in as we move around a circle centered at the origin; doing this, the vectors never make a full revolution, so the index must be zero. For $a > 0$, there are no fixed points, and consequently there is a net index of zero. Thus we see that the sum of all indices is conserved, and always equal to zero.

c) State and prove a generalization of the result for systems of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a)$, where $\mathbf{x} \in \mathbf{R}^2$ and $a \in \mathbf{R}$.

Claim: The sum of all indices involved in any bifurcation for a system of the form above, if it is a smooth vector field, is conserved through the bifurcation.

Proof. Consider a closed curve which is made to continuously deform as the parameter a is varied in such a manner that the curve encloses only the fixed points involved in the bifurcation. Then if one varies a continuously, the index will vary continuously, and since it takes on only integer values, it must stay constant as a function of a. Thus, we have completed the proof. □

7.1.2 Sketch the phase portrait (in polar coordinates) of the following system

$$\begin{aligned} \dot{r} &= r(1 - r^2)(9 - r^2) \\ \dot{\theta} &= 1 \end{aligned}$$

See Figure 3 for the phase portrait.

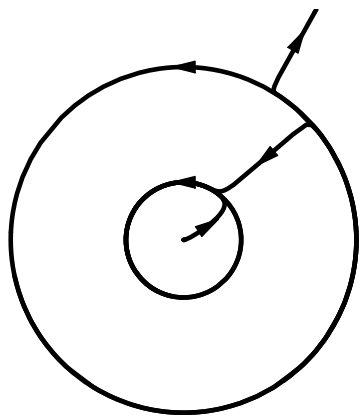

Figure 3: Phase Portrait for Problem 7.1.2

7.2.7 Consider the system

$$\begin{aligned} \dot{x} &= y + 2xy \\ \dot{y} &= x + x^2 - y^2 \end{aligned}$$

a) Show that $\partial f/\partial y = \partial g/\partial x$.
We assume that f and g are defined as in 7.2.5. Given this, we find that

$$\partial f/\partial y = 1 + 2x = 1 + 2x = \partial g/\partial x.$$

b) Find V.
Recall that V is the function which the system is the gradient of. Thus

$$\begin{aligned} \frac{\partial V}{\partial x} &= y + 2xy \\ \frac{\partial V}{\partial y} &= x + x^2 - y^2. \end{aligned}$$

From these equations we obtain two integral expressions for V by seperation of variables. This leads to the equalities

$$\begin{aligned} V &= yx + x^2y + c(y) \\ V &= xy + x^2y - y^3/3 + d(x) \end{aligned}$$

We can see that the two equations agree iff we set $c(y) = -y^3/3$ and $d(x) = 0$. Then

$$V = xy + x^2y - y^3/3.$$

c) Sketch the phase portrait.

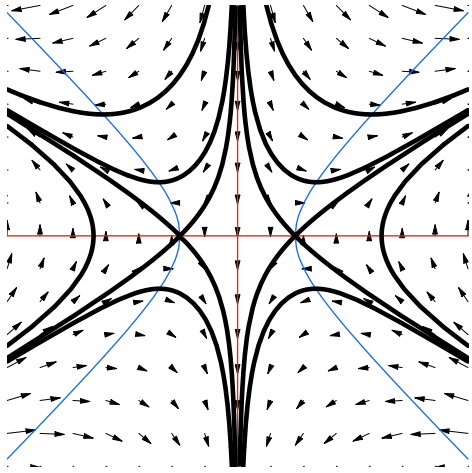

Figure 4: Nullclines, Vector Field and Phase Portrait

7.2.10 By constructing a Liapunov function of the form $V = ax^2 + by^2$, for suitable a and b, show that there exist no closed orbits in the following system

$$\begin{aligned} \dot{x} &= y - x^3 \\ \dot{y} &= -x - y^3 \end{aligned}$$

First notice that there exists only one fixed point. This follows from the fact that our nullclines are $y = x^3$ and $x = -y^3$. We could resort to a graphical argument to see that the only intersection is at the origin, or we could plugg the first equation into the latter to get the equality $x = -x^9$. So $x = 0$ is clearly a solution, but if $x \neq 0$ then we can divide through so that $x^8 = -1$, which has no real solutions since no real number raised to the power of an even number is negative. Thus, as claimed, the origin is the only fixed point.

Let $V = ax^2 + by^2$. Now so long as $a, b > 0$, V is *positive definite* (see Strogatz, pg. 201). To see what further restrictions must be placed on V so that $\dot{V} < 0$ for all $\vec{x} \neq \vec{x}^*$ (the other condition on pg. 201 that must be satisfied), we find that

$$\begin{aligned} \dot{V} &= 2ax\dot{x} + 2by\dot{y} \\ &= 2axy - 2bxy - 2ax^4 - 2by^4. \end{aligned}$$

So if we let $a = b$ we get $V = -2a(x^4 + y^4)$ which for any positive a is always negative, so long as we are not evaluating at the origin. Thus, $V = x^2 + y^2$ is a Liapunov function for this system. Consequently, there exist no closed orbits in this system.

7.3.1 Consider the system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + 5y^2) \\ \dot{y} &= x + y - y(x^2 + y^2). \end{aligned}$$

a) Classify the fixed point at the origin.

We first compute the Linearization matrix using the Jacobian

$$J|_O = \begin{pmatrix} 1-3x^2-5y^2 & -1-10yx \\ 1-2yx & 1-x^2-3y^2 \end{pmatrix}_O = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

which has $T = 2$ and $D = 2$, implying that the origin is an unstable spiral since $T^2 - 4D = -4 < 0$.

b) Rewrite the system in polar coordinates using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

From the first equation we get

$$\begin{aligned} r\dot{r} &= x^2 - xy - x^2(x^2+5y^2) + yx + y^2 - y^2(x^2+y^2) \\ &= x^2 + y^2 - x^4 - y^4 - 6x^2y^2 \\ &= r^2 - ((x^2+y^2)^2 - 2x^2y^2) - 6x^2y^2 \\ &= r^2 - r^4 - 4r^4\cos^2\theta\sin^2\theta \end{aligned}$$

Note that in this simplification we have used the facts that $x^2 + y^2 = r^2$, $x = r\cos\theta$ and $y = r\sin\theta$. Dividing this final equation through by r^2we find that

$$\dot{r} = r - r^3(1 + 4\cos^2\theta\sin^2\theta).$$

We also find in a similar manner that

$$\begin{aligned} r^2\dot{\theta} &= (x^2 + yx - yx(x^2+y^2)) - (yx - y^2 - yx(x^2+5y^2)) \\ &= r^2 + 4xy^3 \\ \dot{\theta} &= 1 + 4r^2\cos\theta\sin^3\theta \end{aligned}$$

c) Determine the circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially *outward* component on it.

This task is equivalent to finding the maximum radius for which $\dot{r} > 0$ for all θ. If we look at our equation for $\dot{r}$, we can see that the bigger the periodic term $(1+4\cos^2\theta\sin^2\theta)$ is, the harder it will be to make $\dot{r} > 0$, so we want to find the maximum of the term inside the parenthesis, since we'll our r_1 to make $\dot{r} > 0$ for that maximum, but will also be guarenteed that $\dot{r} > 0$ for all other θ values, since they would only make the periodic term less, and hence, make $\dot{r}$ even larger. To find this maximal value, we could resort to analytic techniques, ie. find the derivitive of the periodic term with respect to θ, set that equal to zero and then figure out which of the resulting critical points give us maxima and which give us minima. There may even be an easier analytic method involving geometric of trigonometric aproaches, but a graphic argument is really easy and quite convincing. Taking a look at Figure 4, we can se that the maximum value of this term is 2. We can therefore set that term equal to 2 in our equation and see what values of r makes $\dot{r}$ positive. This brings us to solving the inequality

$$\begin{aligned} 0 < \dot{r} &= r - 2r^3 \\ &= r(1 - 2r^2) \end{aligned}$$

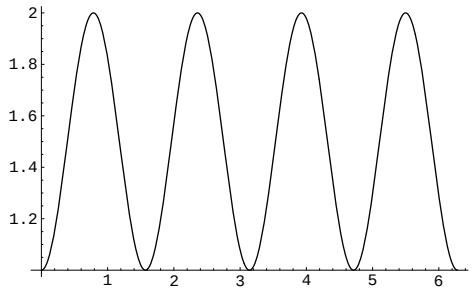

Figure 5: Plot of the periodic term from 7.3.1 vs. θ

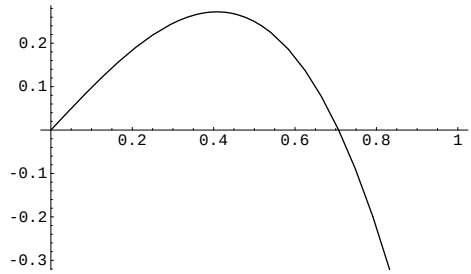

Figure 6: Plot of $\dot{r}$ versus r

Since this expression for $\dot{r}$ is a downward facing cubic with x intercepts at $r = \pm\sqrt{1/2}$ and $r = 0$ we can deduce that for all $r < \sqrt{1/2}$ but greater than zero, $\dot{r} > 0$. See Figure 5 for graphical clearification on this arguement. Thus we get the desired value by setting $r_1 = \sqrt{1/2} - \epsilon$ for any positive but small as we like ϵ.

d) Determine the circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially *inward* component on it.

We proceed here as we did in part (c), only here we want the smallest r for which $\dot{r}$ is negative for all θ. Consequently, we want to find the minimum of $(1 + 4\cos^2\theta\sin^2\theta)$ instead of the maximum. By applying analytic methods such as those described in part (c), or looking at Figure 4 again, we can see that the minimum of the term is 1. We use this in the equation for $\dot{r}$ as we did above, and find that if $\dot{r} = r(1-r^2)$ then we will have a negative value of $\dot{r}$ so long as $r > 1$. Thus we may set $r_2 = 1 + \epsilon$ for some small but positive ϵ.

e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 < r < r_2$.

Proof. By our construction we have a trapping region that satisfies the conditions of the Poincaré-Bendixson Theorem. Consequently, there exists a limit cycle in the region in question. □

7.3.7 Consider the system

$$\begin{aligned} \dot{x} &= y + ax(1 - 2b - r^2) \\ \dot{y} &= -x + ay(1 - r^2) \end{aligned}$$

where a and b are parameters ($0 < a \leq 1$, $0 \leq b < 1/2$) and $r^2 = x^2 + y^2$.

a) Rewrite this system in polar coordinates.
We proceed as we did in problem 7.3.1.

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= (ax^2 + ay^2)(1 - r) - 2bax^2 \\ \dot{r} &= ar(1 - r^2 - 2b\cos^2\theta) \end{aligned}$$

$$\begin{aligned} r^2\dot{\theta} &= -x^2 + ayx(1 - r^2) - y^2 - ayx(1 - 2b - r^2) \\ \dot{\theta} &= -1 + 2ba\cos\theta\sin\theta \end{aligned}$$

b) Prove that there is at least one limit cycle, and that if there are several, they all have the same period $T(a, b)$.
As in problem 7.3.1, we find r_1 and r_2 with the properties we need to have a trapping region. We want to maximaize the periodic term $2b\cos^2\theta$, which happens when $\theta = k\pi$ for some integer k. At these points the term in question equals $2b$, so we substitute this into our equation for $\dot{r}$ and find that $\dot{r} = (1 - r^2 - 2b) = ar((1 - 2b) - r^2)$, so if we set $r_1 = \sqrt{1 - 2b} - \epsilon$, for some small but positive ϵ we get what we need.
Since the minimum of the periodic term is 0, we can use this to find a suitable r_2. We get $\dot{r} = ar(1 - r^2)$ so if we let $r_2 = 1 + \epsilon$ for some small but positive ϵ, then we have our outer trapping region boundary.
By the Poincaré-Bendixson Theorem, and the existence of a trapping region for this system, the system must have at least one limit cycle in the region inbetween r_1 and r_2.
Reguarding the period of such trajectories, note that $\dot{\theta}$ depends only on θ and not on r. Consequently, for any two initial conditions with the same initial θ value, the trajectories will make a full revolution in the same amount of time, since their angular positions are governed by the same dynamical equations.

c) Prove that for $b = 0$ there is only one limit cycle.

Proof. When $b = 0$ our inner trapping radius will be $r_1 = 1 - \epsilon$ and our outer will be $r_2 = 1 + \epsilon$. For two limit cycles to exist, there must be a radial distance d_r between them for any given value of θ, but we can make the values of ϵ as low as we like, so low that $2\epsilon < d_r$ so that there is no way both could fall inside the trapping region. □