Differential Equations Week 1 Homework

Blanchard and Devaney Ch 1.1, Ex 1,2,8,9,13,15,19

EXERCISES FOR SECTION 1.1

[redacted] **1.** **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$ or $P = 230$.

(b) The population is increasing if $dP/dt > 0$. That is, $P(1 - P/230) > 0$. Hence, $0 < P < 230$.

(c) The population is decreasing if $dP/dt < 0$. That is, $P(1 - P/230) < 0$. Hence, $P > 230$ or $P < 0$. Since this is a population model, $P < 0$ might be considered "nonphysical."

[redacted] **2.** **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$, $P = 50$, or $P = 200$.

(b) The population is increasing if $dP/dt > 0$. That is, $P < 0$ or $50 < P < 200$. Note, $P < 0$ might be considered "nonphysical" for a population model.

(c) The population is decreasing if $dP/dt < 0$. That is, $0 < P < 50$ or $P > 200$.

They are both learning at the same rate when $t = 0$.

[redacted] **8.** **(a)** The rate of change of the amount of radioactive material is dr/dt. This rate is proportional to the amount r of material present at time t. With $-\lambda$ as the (positive) proportionality constant, we obtain the differential equation

$$\frac{dr}{dt} = -\lambda r.$$

Note that the minus sign (along with the assumption that λ is positive) means that the material decays.

(b) The only additional assumption is the initial condition $r(0) = r_0$. Consequently, the corresponding initial-value problem is

$$\frac{dr}{dt} = -\lambda r, \quad r(0) = r_0.$$

[redacted] **9.** The general solution of the differential equation $dr/dt = -\lambda r$ is $r(t) = r_0 e^{-\lambda t}$ where $r(0) = r_0$ is the initial amount.

(a) We have $r(t) = r_0 e^{-\lambda t}$ and $r(5230) = r_0/2$. Thus

$$\frac{r_0}{2} = r_0 e^{-\lambda \cdot 5230}$$

$$\frac{1}{2} = e^{-\lambda \cdot 5230}$$

$$\ln \frac{1}{2} = -\lambda \cdot 5230$$

$$-\ln 2 = -\lambda \cdot 5230$$

because $\ln 1/2 = -\ln 2$. Thus,

$$\lambda = \frac{\ln 2}{5230} \approx 0.000132533.$$

(b) We have $r(t) = r_0 e^{-\lambda t}$ and $r(8) = r_0/2$. By a computation similar to the one in part (a), we have

$$\lambda = \frac{\ln 2}{8} \approx 0.0866434.$$

(c) If $r(t)$ is the number of atoms of C-14, then the units for dr/dt is number of atoms per year. Since $dr/dt = -\lambda r$, λ is "per year." Similarly, for I-131, λ is "per day." The unit of measurement of r does not matter.

(d) We get the same answer because the original quantity, r_0, cancels from each side of the equation. We are only concerned with the proportion remaining (one-half of the original amount).

13. Let $P(t)$ be the population at time t, k be the growth-rate parameter, and N be the carrying capacity. The modified models are

(a) $dP/dt = k(1 - P/N)P - 100$

(b) $dP/dt = k(1 - P/N)P - P/3$

(c) $dP/dt = k(1 - P/N)P - a\sqrt{P}$, where a is a positive parameter.

14. **(a)** The differential equation is $dP/dt = 0.3P(1 - P/2500) - 100$. The equilibrium solutions of this equation correspond to the values of P for which $dP/dt = 0$ for all t. Using the quadratic formula, we obtain two such values, $P_1 \approx 396$ and $P_2 \approx 2104$. If $P > P_2$, $dP/dt < 0$, so $P(t)$ is decreasing. If $P_1 < P < P_2$, $dP/dt > 0$, so $P(t)$ is increasing. Hence the solution that satisfies the initial condition $P(0) = 2500$ decreases toward the equilibrium $P_2 \approx 2104$.

(b) The differential equation is $dP/dt = 0.3P(1 - P/2500) - P/3$. The equilibrium solutions of this equation are $P_1 \approx -277$ and $P_2 = 0$. If $P > 0$, $dP/dt < 0$, so $P(t)$ is decreasing. Hence, for $P(0) = 2500$, the population decreases toward $P = 0$ (extinction).

15. Several different models are possible. Let $R(t)$ denote the rhinoceros population at time t. The basic assumption is that there is a minimum threshold that the population must exceed if it is to survive. In terms of the differential equation, this assumption means that dR/dt must be negative if R is close to zero. Three models that satisfy this assumption are:

- If k is a growth-rate parameter and M is a parameter measuring when the population is "too small", then
$$\frac{dR}{dt} = kR\left(\frac{R}{M} - 1\right).$$
- If k is a growth-rate parameter and b is a parameter that determines the level the population will start to decrease ($R < b/k$), then
$$\frac{dR}{dt} = kR - b.$$
- If k is a growth-rate parameter and b is a parameter that determines the extinction threshold, then
$$\frac{dR}{dt} = aR - \frac{b}{R}.$$

In each case, if R is below a certain threshold, dR/dt is negative. Thus, the rhinos will eventually die out. The choice of which model to use depends on other assumptions. There are other equations that are also consistent with the basic assumption.

19. **(a)** We consider dx/dt in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number x of prey is equal for both systems, dx/dt is larger in system (i). Therefore, the prey in system (i) reproduce faster if there are no predators.

(b) We must see what affect the predators (represented by the y-terms) have on dx/dt in each system. Since the magnitude of the coefficient of the xy-term is larger in system (ii) than in system (i), y has a greater effect on dx/dt in system (ii). Hence the predators have a greater effect on the rate of change of the prey in system (ii).

(c) We must see what affect the prey (represented by the x-terms) have on dy/dt in each system. Since x and y are both nonnegative, it follows that

$$-2y + \tfrac{1}{2}xy < -2y + 6xy,$$

and therefore, if the number of predators is equal for both systems, dy/dt is smaller in system (i). Hence more prey are required in system (i) than in system (ii) to achieve a certain growth rate.