8. The standard row operations are:

$$\begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 9 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The solution set contains one solution: (0, 0, 0).

9. The system has already been reduced to triangular form. Begin by scaling the fourth row by 1/2 and then replacing R3 by R3 + (3)R4:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & 7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Next, replace R2 by R2 + (3)R3. Finally, replace R1 by R1 + R2:

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The solution set contains one solution: (4, 8, 5, 2).

10. The system has already been reduced to triangular form. Use the 1 in the fourth row to change the –4 and 3 above it to zeros. That is, replace R2 by R2 + (4)R4 and replace R1 by R1 + (–3)R4. For the final step, replace R1 by R1 + (2)R2.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & -2 \\ 0 & 1 & 0 & -4 & 7 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$$

The solution set contains one solution: (–3, –5, 6, –3).

11. First, swap R1 and R2. Then replace R3 by R3 + (–3)R1. Finally, replace R3 by R3 + (2)R2.

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 2 if there were a solution. The solution set is empty.

12. Replace R2 by R2 + (–3)R1 and replace R3 by R3 + (4)R1. Finally, replace R3 by R3 + (3)R2.

$$\begin{bmatrix} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The system is inconsistent, because the last row would require that 0 = 3 if there were a solution. The solution set is empty.

13. $\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is (5, 3, –1).

14. $\begin{bmatrix} 1 & -3 & 0 & 5 \\ -1 & 1 & 5 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & -2 & 5 & 7 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 7 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. The solution is (2, –1, 1).

15. First, replace R4 by R4 + (–3)R1, then replace R3 by R3 + (2)R2, and finally replace R4 by R4 + (3)R3.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

The resulting triangular system indicates that a solution exists. In fact, using the argument from Example 2, one can see that the solution is unique.

16. First replace R4 by R4 + (2)R1 and replace R4 by R4 + (–3/2)R2. (One could also scale R2 before adding to R4, but the arithmetic is rather easy keeping R2 unchanged.) Finally, replace R4 by R4 + R3.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is now in triangular form and has a solution. The next section discusses how to continue with this type of system.

25. $\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$

Let b denote the number $k + 2g + h$. Then the third equation represented by the augmented matrix above is $0 = b$. This equation is possible if and only if b is zero. So the original system has a solution if and only if $k + 2g + h = 0$.

26. A basic principle of this section is that row operations do not affect the solution set of a linear system. Begin with a simple augmented matrix for which the solution is obviously $(-2, 1, 0)$, and then perform any elementary row operations to produce other augmented matrices. Here are three examples. The fact that they are all row equivalent proves that they all have the solution set $(-2, 1, 0)$.

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 2 & 1 & 0 & -3 \\ 2 & 0 & 1 & -4 \end{bmatrix}$$

27. Study the augmented matrix for the given system, replacing R2 by R2 + $(-c)$R1:

$$\begin{bmatrix} 1 & 3 & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & f \\ 0 & d-3c & g-cf \end{bmatrix}$$

This shows that shows $d - 3c$ must be nonzero, since f and g are arbitrary. Otherwise, for some choices of f and g the second row would correspond to an equation of the form $0 = b$, where b is nonzero. Thus $d \neq 3c$.

28. Row reduce the augmented matrix for the given system. Scale the first row by $1/a$, which is possible since a is nonzero. Then replace R2 by R2 + $(-c)$R1.

$$\begin{bmatrix} a & b & f \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ c & d & g \end{bmatrix} \sim \begin{bmatrix} 1 & b/a & f/a \\ 0 & d-c(b/a) & g-c(f/a) \end{bmatrix}$$

The quantity $d - c(b/a)$ must be nonzero, in order for the system to be consistent when the quantity $g - c(f/a)$ is nonzero (which can certainly happen). The condition that $d - c(b/a) \neq 0$ can also be written as $ad - bc \neq 0$, or $ad \neq bc$.

29. Swap R1 and R2; swap R1 and R2.

30. Multiply R2 by $-1/2$; multiply R2 by -2.

31. Replace R3 by R3 + (-4)R1; replace R3 by R3 + (4)R1.

32. Replace R3 by R3 + (3)R2; replace R3 by R3 + (-3)R2.

33. The first equation was given. The others are:

$$T_2 = (T_1 + 20 + 40 + T_3)/4, \quad \text{or} \quad 4T_2 - T_1 - T_3 = 60$$
$$T_3 = (T_4 + T_2 + 40 + 30)/4, \quad \text{or} \quad 4T_3 - T_4 - T_2 = 70$$
$$T_4 = (10 + T_1 + T_3 + 30)/4, \quad \text{or} \quad 4T_4 - T_1 - T_3 = 40$$

Rearranging,

$$\begin{array}{rcrcrcrcl} 4T_1 & - & T_2 & & & - & T_4 & = & 30 \\ -T_1 & + & 4T_2 & - & T_3 & & & = & 60 \\ & & -T_2 & + & 4T_3 & - & T_4 & = & 70 \\ -T_1 & & & - & T_3 & + & 4T_4 & = & 40 \end{array}$$

34. Begin by interchanging R1 and R4, then create zeros in the first column:

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 4 & 40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 4 & 40 \\ 0 & 4 & 0 & -4 & 20 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix}$$

Scale R1 by –1 and R2 by 1/4, create zeros in the second column, and replace R4 by R4 + R3:

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & -4 & 14 & 195 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 12 & 270 \end{bmatrix}$$

Scale R4 by 1/12, use R4 to create zeros in column 4, and then scale R3 by 1/4:

$$\sim \begin{bmatrix} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 4 & 0 & 120 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix}$$

The last step is to replace R1 by R1 + (–1)R3:

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 20.0 \\ 0 & 1 & 0 & 0 & 27.5 \\ 0 & 0 & 1 & 0 & 30.0 \\ 0 & 0 & 0 & 1 & 22.5 \end{bmatrix}.$$ The solution is (20, 27.5, 30, 22.5).

Notes: The *Study Guide* includes a "Mathematical Note" about statements, "If … , then … ."

This early in the course, students typically use single row operations to reduce a matrix. As a result, even the small grid for Exercise 34 leads to about 25 multiplications or additions (not counting operations with zero). This exercise should give students an appreciation for matrix programs such as MATLAB. Exercise 14 in Section 1.10 returns to this problem and states the solution in case students have not already solved the system of equations. Exercise 31 in Section 2.5 uses this same type of problem in connection with an LU factorization.

For instructors who wish to use technology in the course, the *Study Guide* provides boxed MATLAB notes at the ends of many sections. Parallel notes for Maple, Mathematica, and the TI-83+/86/89 and HP-48G calculators appear in separate appendices at the end of the *Study Guide*. The MATLAB box for Section 1.1 describes how to access the data that is available for all numerical exercises in the text. This feature has the ability to save students time if they regularly have their matrix program at hand when studying linear algebra. The MATLAB box also explains the basic commands **`replace`**, **`swap`**, and **`scale`**. These commands are included in the text data sets, available from the text web site, www.laylinalgebra.com.

1.2 SOLUTIONS

Notes: The key exercises are 1–20 and 23–28. (Students should work at least four or five from Exercises 7–14, in preparation for Section 1.5.)

1. Reduced echelon form: a and b. Echelon form: d. Not echelon: c.

2. Reduced echelon form: a. Echelon form: b and d. Not echelon: c.

3. $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & -2 \\ 0 & \textcircled{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \text{ Pivot cols 1 and 2. } \begin{bmatrix} \textcircled{1} & 2 & 3 & 4 \\ 4 & \textcircled{5} & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 0 \\ 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}. \text{ Pivot cols 1, 2, and 4 } \begin{bmatrix} \textcircled{1} & 3 & 5 & 7 \\ 3 & \textcircled{5} & 7 & 9 \\ 5 & 7 & 9 & \textcircled{1} \end{bmatrix}$$

5. $$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}$$

6. $$\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \blacksquare & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

7. $$\begin{bmatrix} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & -5 \\ 0 & 0 & \textcircled{1} & 3 \end{bmatrix}$$

Corresponding system of equations:
$$\begin{aligned} \textcircled{x_1} + 3x_2 \quad &= -5 \\ \textcircled{x_3} &= 3 \end{aligned}$$

The basic variables (corresponding to the pivot positions) are x_1 and x_3. The remaining variable x_2 is free. Solve for the basic variables in terms of the free variable. The general solution is

$$\begin{cases} x_1 = -5 - 3x_2 \\ x_2 \text{ is free} \\ x_3 = 3 \end{cases}$$

Note: Exercise 7 is paired with Exercise 10.

Basic variable: x_1; free variables x_2, x_3. General solution: $\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 \text{ is free} \\ x_3 \text{ is free} \end{cases}$

12. $\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \textcircled{1} & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{aligned} \textcircled{x_1} - 7x_2 \quad + 6x_4 &= 5 \\ \textcircled{x_3} - 2x_4 &= -3 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1 and x_3; free variables: x_2, x_4. General solution: $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$

13. $\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & -3 & 5 \\ 0 & \textcircled{1} & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & \textcircled{1} & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{aligned} \textcircled{x_1} \quad - 3x_5 &= 5 \\ \textcircled{x_2} \quad - 4x_5 &= 1 \\ \textcircled{x_4} + 9x_5 &= 4 \\ 0 &= 0 \end{aligned}$$

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution: $\begin{cases} x_1 = 5 + 3x_5 \\ x_2 = 1 + 4x_5 \\ x_3 \text{ is free} \\ x_4 = 4 - 9x_5 \\ x_5 \text{ is free} \end{cases}$

Note: The *Study Guide* discusses the common mistake $x_3 = 0$.

14. $\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 7 & 0 & 0 & -9 \\ 0 & \textcircled{1} & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Corresponding system:

$$\begin{array}{rrrrrcr} \textcircled{x_1} & & + 7x_3 & & & = & -9 \\ & \textcircled{x_2} & - 6x_3 & - 3x_4 & & = & 2 \\ & & & & \textcircled{x_5} & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

Basic variables: x_1, x_2, x_5; free variables: x_3, x_4. General solution: $\begin{cases} x_1 = -9 - 7x_3 \\ x_2 = 2 + 6x_3 + 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \\ x_5 = 0 \end{cases}$

15. **a**. The system is consistent, with a unique solution.

b. The system is inconsistent. (The rightmost column of the augmented matrix is a pivot column).

16. **a**. The system is consistent, with a unique solution.

b. The system is consistent. There are many solutions because x_2 is a free variable.

17. $\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & 3 & h \\ 0 & 0 & 7-2h \end{bmatrix}$ The system has a solution only if $7 - 2h = 0$, that is, if $h = 7/2$.

18. $\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix}$ If $h + 15$ is zero, that is, if $h = -15$, then the system has no solution, because 0 cannot equal 3. Otherwise, when $h \neq -15$, the system has a solution.

19. $\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$

a. When $h = 2$ and $k \neq 8$, the augmented column is a pivot column, and the system is inconsistent.

b. When $h \neq 2$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 2$ and $k = 8$, the system is consistent and has many solutions.

20. $\begin{bmatrix} 1 & 3 & 2 \\ 3 & h & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 2 \\ 0 & h-9 & k-6 \end{bmatrix}$

a. When $h = 9$ and $k \neq 6$, the system is inconsistent, because the augmented column is a pivot column.

b. When $h \neq 9$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 9$ and $k = 6$, the system is consistent and has many solutions.

21. **a**. False. See Theorem 1.

b. False. See the second paragraph of the section.

c. True. Basic variables are defined after equation (4).

d. True. This statement is at the beginning of *Parametric Descriptions of Solution Sets*.

e. False. The row shown corresponds to the equation $5x_4 = 0$, which does not by itself lead to a contradiction. So the system might be consistent or it might be inconsistent.

32. According to the numerical note in Section 1.2, when $n = 30$ the reduction to echelon form takes about $2(30)^3/3 = 18{,}000$ flops, while further reduction to reduced echelon form needs at most $(30)^2 = 900$ flops. Of the total flops, the "backward phase" is about 900/18900 = .048 or about 5%.

When $n = 300$, the estimates are $2(300)^3/3 = 18{,}000{,}000$ phase for the reduction to echelon form and $(300)^2 = 90{,}000$ flops for the backward phase. The fraction associated with the backward phase is about $(9\times10^4)/(18\times10^6) = .005$, or about .5%.

33. For a quadratic polynomial $p(t) = a_0 + a_1t + a_2t^2$ to exactly fit the data (1, 12), (2, 15), and (3, 16), the coefficients a_0, a_1, a_2 must satisfy the systems of equations given in the text. Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 12 \\ 1 & 2 & 4 & 15 \\ 1 & 3 & 9 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 2 & 8 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 12 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 13 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 7 \\ 0 & \textcircled{1} & 0 & 6 \\ 0 & 0 & \textcircled{1} & -1 \end{bmatrix}$$

The polynomial is $p(t) = 7 + 6t - t^2$.

34. **[M]** The system of equations to be solved is:

$$\begin{aligned}
a_0 + a_1\cdot 0 + a_2\cdot 0^2 + a_3\cdot 0^3 + a_4\cdot 0^4 + a_5\cdot 0^5 &= 0 \\
a_0 + a_1\cdot 2 + a_2\cdot 2^2 + a_3\cdot 2^3 + a_4\cdot 2^4 + a_5\cdot 2^5 &= 2.90 \\
a_0 + a_1\cdot 4 + a_2\cdot 4^2 + a_3\cdot 4^3 + a_4\cdot 4^4 + a_5\cdot 4^5 &= 14.8 \\
a_0 + a_1\cdot 6 + a_2\cdot 6^2 + a_3\cdot 6^3 + a_4\cdot 6^4 + a_5\cdot 6^5 &= 39.6 \\
a_0 + a_1\cdot 8 + a_2\cdot 8^2 + a_3\cdot 8^3 + a_4\cdot 8^4 + a_5\cdot 8^5 &= 74.3 \\
a_0 + a_1\cdot 10 + a_2\cdot 10^2 + a_3\cdot 10^3 + a_4\cdot 10^4 + a_5\cdot 10^5 &= 119
\end{aligned}$$

The unknowns are $a_0, a_1, \ldots, a_5$. Use technology to compute the reduced echelon of the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 1 & 4 & 16 & 64 & 256 & 1024 & 14.8 \\ 1 & 6 & 36 & 216 & 1296 & 7776 & 39.6 \\ 1 & 8 & 64 & 512 & 4096 & 32768 & 74.3 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 & 119 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 24 & 192 & 1248 & 7680 & 30.9 \\ 0 & 0 & 48 & 480 & 4032 & 32640 & 62.7 \\ 0 & 0 & 80 & 960 & 9920 & 99840 & 104.5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 192 & 2688 & 26880 & 8.7 \\ 0 & 0 & 0 & 480 & 7680 & 90240 & 14.5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 16 & 32 & 2.9 \\ 0 & 0 & 8 & 48 & 224 & 960 & 9 \\ 0 & 0 & 0 & 48 & 576 & 4800 & 3.9 \\ 0 & 0 & 0 & 0 & 384 & 7680 & -6.9 \\ 0 & 0 & 0 & 0 & 1920 & 42240 & -24.5 \end{bmatrix}$$