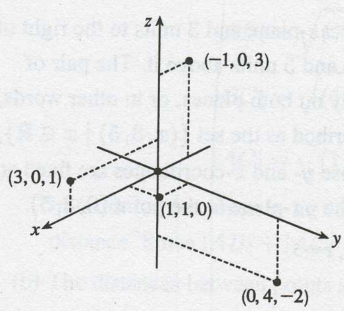
9 Vectors and the Geometry of Space

9.1 Three-Dimensional Coordinate Systems

1. We start at the origin, which has coordinates $(0, 0, 0)$. First we move 4 units along the positive x-axis, affecting only the x-coordinate, bringing us to the point $(4, 0, 0)$. We then move 3 units straight downward, in the negative z-direction. Thus only the z-coordinate is affected, and we arrive at $(4, 0, -3)$.

2.

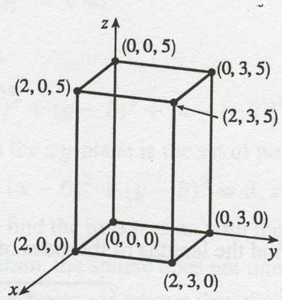


3. The distance from a point to the xz-plane is the absolute value of the y-coordinate of the point. $Q(-5, -1, 4)$ has the y-coordinate with the smallest absolute value, so Q is the point closest to the xz-plane. $R(0, 3, 8)$ must lie in the yz-plane since the distance from R to the yz-plane, given by the x-coordinate of R, is 0.

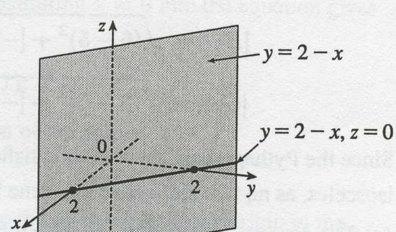
4. The projection of $(2, 3, 5)$ on the xy-plane is $(2, 3, 0)$; on the yz-plane, $(0, 3, 5)$; on the xz-plane, $(2, 0, 5)$.

The length of the diagonal of the box is the distance between the origin and $(2, 3, 5)$, given by

$$\sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38} \approx 6.16.$$



5. The equation $x + y = 2$ represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates have a sum of 2, or equivalently where $y = 2 - x$. This is the set $\{(x, 2 - x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$.



(c) The distance is the absolute value of the y-coordinate of the point: $|7| = 7$.

(d) The point on the x-axis closest to $(3, 7, -5)$ is the point $(3, 0, 0)$. (Approach the x-axis perpendicularly.) The distance from $(3, 7, -5)$ to the x-axis is the distance between these two points: $\sqrt{(3-3)^2 + (7-0)^2 + (-5-0)^2} = \sqrt{74} \approx 8.60$.

(e) The point on the y-axis closest to $(3, 7, -5)$ is $(0, 7, 0)$. The distance between these points is $\sqrt{(3-0)^2 + (7-7)^2 + (-5-0)^2} = \sqrt{34} \approx 5.83$.

(f) The point on the z-axis closest to $(3, 7, -5)$ is $(0, 0, -5)$. The distance between these points is $\sqrt{(3-0)^2 + (7-0)^2 + [-5-(-5)]^2} = \sqrt{58} \approx 7.62$.

9. (a) First we find the distances between points:

$$|AB| = \sqrt{(7-5)^2 + (9-1)^2 + (-1-3)^2} = \sqrt{84} = 2\sqrt{21}$$

$$|BC| = \sqrt{(1-7)^2 + (-15-9)^2 + [11-(-1)]^2} = \sqrt{756} = 6\sqrt{21}$$

$$|AC| = \sqrt{(1-5)^2 + (-15-1)^2 + (11-3)^2} = \sqrt{336} = 4\sqrt{21}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $|AB| + |AC| = |BC|$, the three points lie on a straight line.

(b) The distances between points are

$$|KL| = \sqrt{(1-0)^2 + (2-3)^2 + [-2-(-4)]^2} = \sqrt{6}$$

$$|LM| = \sqrt{(3-1)^2 + (0-2)^2 + [1-(-2)]^2} = \sqrt{17}$$

$$|KM| = \sqrt{(3-0)^2 + (0-3)^2 + [1-(-4)]^2} = \sqrt{43}$$

Since $\sqrt{6} + \sqrt{17} \neq \sqrt{43}$, the three points do not lie on a straight line.

10. An equation of the sphere with center $(6, 5, -2)$ and radius $\sqrt{7}$ is $(x-6)^2 + (y-5)^2 + [z-(-2)]^2 = \left(\sqrt{7}\right)^2$ or $(x-6)^2 + (y-5) + (z+2)^2 = 7$. The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z = 0$ into the equation, we have $(x-6)^2 + (y-5)^2 = 3$, $z = 0$ which represents a circle in the xy-plane with center $(6, 5, 0)$ and radius $\sqrt{3}$. To find the intersection with the xz-plane, we set $y = 0$: $(x-6)^2 + (z+2)^2 = -18$. Since no points satisfy this equation, the sphere does not intersect the xz-plane. (Also note that the distance from the center of the sphere to the xz-plane is greater than the radius of the sphere.) Similarly, the sphere does not intersect the yz-plane since substituting $x = 0$ into the equation gives $(y-5)^2 + (z+2)^2 = -29$.

11. The radius of the sphere is the distance between $(4, 3, -1)$ and $(3, 8, 1)$: $r = \sqrt{(3-4)^2 + (8-3)^2 + [1-(-1)]^2} = \sqrt{30}$. Thus, an equation of the sphere is $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

12. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1, 2, 3)$: $r = \sqrt{(1-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{14}$. Then an equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$.

13. Completing squares in the equation gives $\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) + \left(z^2 - z + \frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \quad \Rightarrow$ $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{3}{4}$ which we recognize as an equation of a sphere with center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and radius $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

14. Completing squares in the equation gives $4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 1 + 4 + 16 \quad \Rightarrow$ $4(x-1)^2 + 4(y+2)^2 + 4z^2 = 21 \quad \Rightarrow \quad (x-1)^2 + (y+2)^2 + z^2 = \frac{21}{4}$, which we recognize as an equation of a sphere with center $(1, -2, 0)$ and radius $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

15. (a) If the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is $Q = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\begin{aligned}
|P_1Q| &= \sqrt{\left[\tfrac{1}{2}(x_1 + x_2) - x_1\right]^2 + \left[\tfrac{1}{2}(y_1 + y_2) - y_1\right]^2 + \left[\tfrac{1}{2}(z_1 + z_2) - z_1\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \tfrac{1}{2}\,|P_1P_2|
\end{aligned}$$

$$\begin{aligned}
|QP_2| &= \sqrt{\left[x_2 - \tfrac{1}{2}(x_1 + x_2)\right]^2 + \left[y_2 - \tfrac{1}{2}(y_1 + y_2)\right]^2 + \left[z_2 - \tfrac{1}{2}(z_1 + z_2)\right]^2} \\
&= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\
&= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} \\
&= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\
&= \tfrac{1}{2}\,|P_1P_2|
\end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1\left(-\frac{1}{2}, 1, 4\right)$, $P_2\left(1, \frac{1}{2}, 5\right)$ and $P_3\left(\frac{5}{2}, \frac{3}{2}, 4\right)$. (Recall that a median of a triangle is a line segment from a vertex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$|AP_2| = \sqrt{0^2 + \left(\tfrac{1}{2} - 2\right)^2 + (5 - 3)^2} = \sqrt{\tfrac{9}{4} + 4} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2}$$

$$|BP_3| = \sqrt{\left(\tfrac{5}{2} + 2\right)^2 + \left(\tfrac{3}{2}\right)^2 + (4 - 5)^2} = \sqrt{\tfrac{81}{4} + \tfrac{9}{4} + 1} = \sqrt{\tfrac{94}{4}} = \tfrac{1}{2}\sqrt{94}$$

$$|CP_1| = \sqrt{\left(-\tfrac{1}{2} - 4\right)^2 + (1 - 1)^2 + (4 - 5)^2} = \sqrt{\tfrac{81}{4} + 1} = \tfrac{1}{2}\sqrt{85}$$

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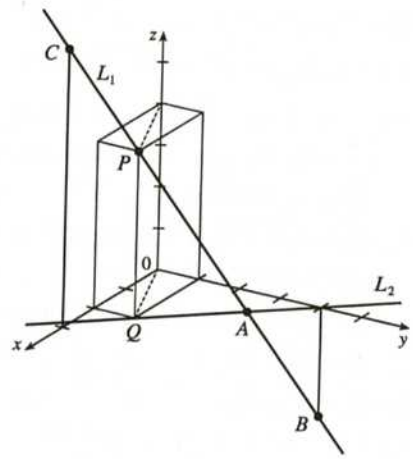
33. (a) To find the x- and y-coordinates of the point P, we project it onto L_2 and project the resulting point Q onto the x- and y-axes. To find the z-coordinate, we project P onto either the xz-plane or the yz-plane (using our knowledge of its x- or y-coordinate) and then project the resulting point onto the z-axis. (Or, we could draw a line parallel to QO from P to the z-axis.) The coordinates of P are $(2, 1, 4)$.

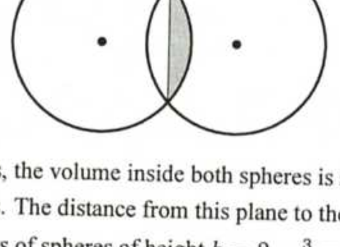
(b) A is the intersection of L_1 and L_2, B is directly below the y-intercept of L_2, and C is directly above the x-intercept of L_2.



34. Let $P = (x, y, z)$. Then $2\,|PB| = |PA| \;\Leftrightarrow\; 4\,|PB|^2 = |PA|^2 \;\Leftrightarrow$
$4\left((x-6)^2 + (y-2)^2 + (z+2)^2\right) = (x+1)^2 + (y-5)^2 + (z-3)^2 \;\Leftrightarrow$
$4(x^2 - 12x + 36) - x^2 - 2x + 4(y^2 - 4y + 4) - y^2 + 10y + 4(z^2 + 4z + 4) - z^2 + 6z = 35 \;\Leftrightarrow$
$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \;\Leftrightarrow\; x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$. By completing the square three times we get $\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

35. We need to find a set of points $\{P(x, y, z) \mid |AP| = |BP|\}$.
$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \;\Rightarrow$
$(x+1)^2 + (y-5) + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2 \;\Rightarrow$
$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 \;\Rightarrow$
$14x - 6y - 10z = 9$. Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

36. Completing the square three times in the first equation gives $(x+2)^2 + (y-1)^2 + (z+2)^2 = 2^2$, a sphere with center $(-2, 1, 2)$ and radius 2. The second equation is that of a sphere with center $(0, 0, 0)$ and radius 2. The distance between the centers of the spheres is $\sqrt{(-2-0)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{4+1+4} = 3$.



Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h = 2 - \frac{3}{2} = \frac{1}{2}$. From Exercise 6.2.19, the volume of a cap of a sphere is $V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(3 \cdot 2 - \frac{1}{2}\right) = \frac{11\pi}{24}$. So the total volume is $2 \cdot \frac{11\pi}{24} = \frac{11\pi}{12}$.

9.2 Vec

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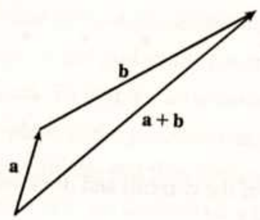
Section 1.2

6. (a)



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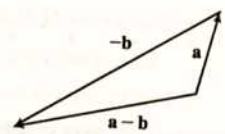

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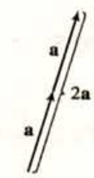

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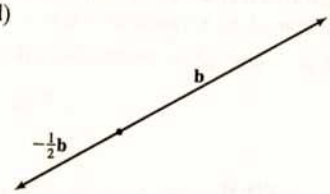

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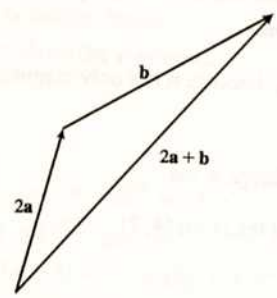

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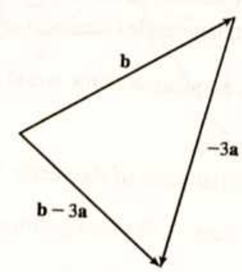
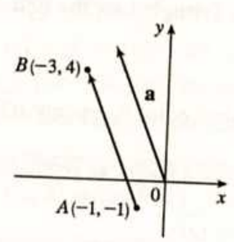
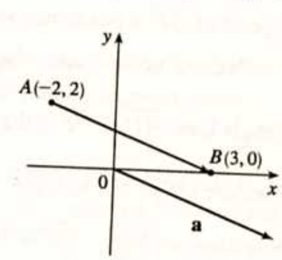


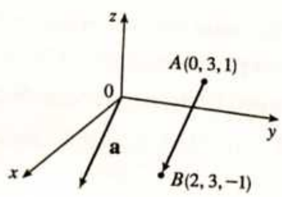
7. $\mathbf{a} = \langle -3-(-1), 4-(-1)\rangle = \langle -2, 5\rangle$



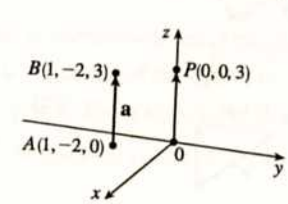
8. $\mathbf{a} = \langle 3-(-2), 0-2\rangle = \langle 5, -2\rangle$



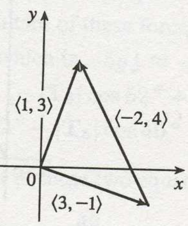
9. $\mathbf{a} = \langle 2-0, 3-3, -1-1\rangle = \langle 2, 0, -2\rangle$



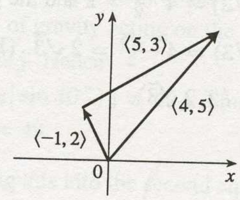
10. $\mathbf{a} = \langle 1-1, -2+2, 3-0\rangle = \langle 0, 0, 3\rangle$



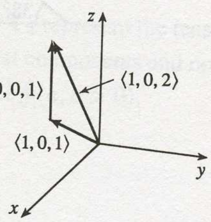
11. $\langle 3,-1\rangle + \langle -2,4\rangle = \langle 3+(-2),-1+4\rangle = \langle 1,3\rangle$



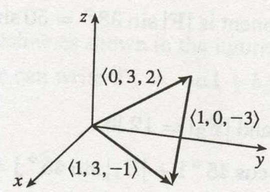
12. $\langle -1,2\rangle + \langle 5,3\rangle = \langle -1+5,2+3\rangle = \langle 4,5\rangle$



13. $\langle 1,0,1\rangle + \langle 0,0,1\rangle = \langle 1+0,0+0,1+1\rangle$
$= \langle 1,0,2\rangle$



14. $\langle 0,3,2\rangle + \langle 1,0,-3\rangle = \langle 1,3,-1\rangle$



15. $|\mathbf{a}| = \sqrt{(-4)^2+3^2} = \sqrt{25} = 5$

$\mathbf{a}+\mathbf{b} = \langle -4+6, 3+2\rangle = \langle 2,5\rangle$

$\mathbf{a}-\mathbf{b} = \langle -4-6, 3-2\rangle = \langle -10,1\rangle$

$2\mathbf{a} = \langle 2(-4), 2(3)\rangle = \langle -8,6\rangle$

$3\mathbf{a}+4\mathbf{b} = \langle -12,9\rangle + \langle 24,8\rangle = \langle 12,17\rangle$

16. $|\mathbf{a}| = \sqrt{2^2+(-3)^2} = \sqrt{13}$

$\mathbf{a}+\mathbf{b} = (2\,\mathbf{i}-3\,\mathbf{j}) + (\mathbf{i}+5\,\mathbf{j}) = 3\,\mathbf{i}+2\,\mathbf{j}$

$\mathbf{a}-\mathbf{b} = (2\,\mathbf{i}-3\,\mathbf{j}) - (\mathbf{i}+5\,\mathbf{j}) = \mathbf{i}-8\,\mathbf{j}$

$2\mathbf{a} = 2(2\,\mathbf{i}-3\,\mathbf{j}) = 4\,\mathbf{i}-6\,\mathbf{j}$

$3\mathbf{a}+4\mathbf{b} = 3(2\,\mathbf{i}-3\,\mathbf{j}) + 4(\mathbf{i}+5\,\mathbf{j})$
$= 6\,\mathbf{i}-9\,\mathbf{j}+4\,\mathbf{i}+20\,\mathbf{j} = 10\,\mathbf{i}+11\,\mathbf{j}$

17. $|\mathbf{a}| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$

$\mathbf{a}+\mathbf{b} = (\mathbf{i}-2\,\mathbf{j}+\mathbf{k}) + (\mathbf{j}+2\,\mathbf{k}) = \mathbf{i}-\mathbf{j}+3\,\mathbf{k}$

$\mathbf{a}-\mathbf{b} = (\mathbf{i}-2\,\mathbf{j}+\mathbf{k}) - (\mathbf{j}+2\,\mathbf{k}) = \mathbf{i}-3\,\mathbf{j}-\mathbf{k}$

$2\mathbf{a} = 2(\mathbf{i}-2\,\mathbf{j}+\mathbf{k}) = 2\,\mathbf{i}-4\,\mathbf{j}+2\,\mathbf{k}$

$3\mathbf{a}+4\mathbf{b} = 3(\mathbf{i}-2\,\mathbf{j}+\mathbf{k}) + 4(\mathbf{j}+2\,\mathbf{k})$
$= 3\,\mathbf{i}-6\,\mathbf{j}+3\,\mathbf{k}+4\,\mathbf{j}+8\,\mathbf{k}$
$= 3\,\mathbf{i}-2\,\mathbf{j}+11\,\mathbf{k}$

18. $|\mathbf{a}| = \sqrt{3^2+0^2+(-2)^2} = \sqrt{13}$

$\mathbf{a}+\mathbf{b} = (3\,\mathbf{i}-2\,\mathbf{k}) + (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 4\,\mathbf{i}-\mathbf{j}-\mathbf{k}$

$\mathbf{a}-\mathbf{b} = (3\,\mathbf{i}-2\,\mathbf{k}) - (\mathbf{i}-\mathbf{j}+\mathbf{k}) = 2\,\mathbf{i}+\mathbf{j}-3\,\mathbf{k}$

$2\mathbf{a} = 2(3\,\mathbf{i}-2\,\mathbf{k}) = 6\,\mathbf{i}-4\,\mathbf{k}$

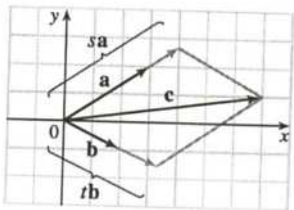
$3\mathbf{a}+4\mathbf{b} = 3(3\,\mathbf{i}-2\,\mathbf{k}) + 4(\mathbf{i}-\mathbf{j}+\mathbf{k})$
$= 9\,\mathbf{i}-6\,\mathbf{k}+4\,\mathbf{i}-4\,\mathbf{j}+4\,\mathbf{k}$
$= 13\,\mathbf{i}-4\,\mathbf{j}-2\,\mathbf{k}$

19. The vector $8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}$ has length $|8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}| = \sqrt{8^2+(-1)^2+4^2} = \sqrt{81} = 9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}) = \frac{8}{9}\,\mathbf{i}-\frac{1}{9}\,\mathbf{j}+\frac{4}{9}\,\mathbf{k}$.

20. $|\langle -2,4,2\rangle| = \sqrt{(-2)^2+4^2+2^2} = \sqrt{24} = 2\sqrt{6}$, so a unit vector in the direction of $\langle -2,4,2\rangle$ is $\mathbf{u} = \dfrac{1}{2\sqrt{6}}\langle -2,4,2\rangle$. A vector in the same direction but with length 6 is

$$6\mathbf{u} = 6\cdot\frac{1}{2\sqrt{6}}\langle -2,4,2\rangle = \left\langle -\frac{6}{\sqrt{6}}, \frac{12}{\sqrt{6}}, \frac{6}{\sqrt{6}}\right\rangle \text{ or } \langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6}\rangle.$$

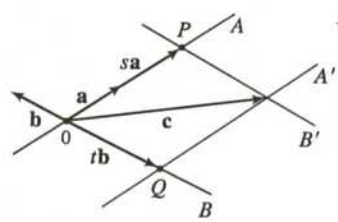
29. (a), (b)



(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \quad \Leftrightarrow \quad 7 = 3s + 2t$ and $1 = 2s - t$.
Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

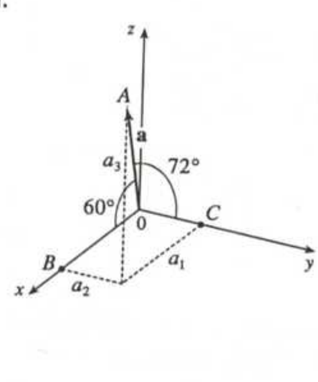
30. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin. Extend $\mathbf{a}$ and $\mathbf{b}$ to form lines A and B, and draw lines A' and B' parallel to these two lines through the terminal point of $\mathbf{c}$.



Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}$, so if $s = \dfrac{\left|\overrightarrow{OP}\right|}{|\mathbf{a}|}$ (or its negative, if $\mathbf{a}$ points in the direction opposite $\overrightarrow{OP}$) and $t = \dfrac{\left|\overrightarrow{OQ}\right|}{|\mathbf{b}|}$ (or its negative, as in the diagram), then $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, and $\mathbf{c} = \langle c_1, c_2\rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \dfrac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}$. Similarly $s = \dfrac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}$. Since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, the denominator is not zero.

31.



Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, as shown in the figure. Since $|\mathbf{a}| = 1$ and triangle ABO is a right triangle, we have $\cos 60^\circ = \dfrac{a_1}{1} \Rightarrow a_1 = \cos 60^\circ$. Similarly, triangle ACO is a right triangle, so $a_2 = \cos 72^\circ$. Finally, since $|\mathbf{a}| = 1$ we have

$\sqrt{(\cos 60^\circ)^2 + (\cos 72^\circ)^2 + a_3^2} = 1 \Rightarrow$

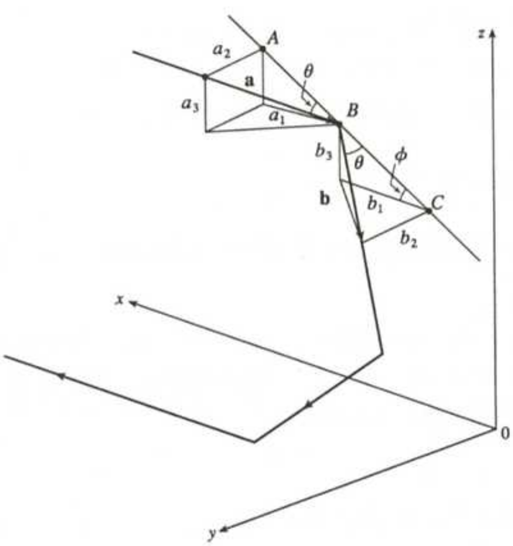
$a_3^2 = 1 - (\cos 60^\circ)^2 - (\cos 72^\circ)^2 \Rightarrow$

$a_3 = \sqrt{1 - (\cos 60^\circ)^2 - (\cos 72^\circ)^2}$. Thus

$$\mathbf{a} = \left\langle \cos 60^\circ, \cos 72^\circ, \sqrt{1 - (\cos 60^\circ)^2 - (\cos 72^\circ)^2} \right\rangle$$
$$\approx \langle 0.50, 0.31, 0.81\rangle$$

37. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (1) and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ (2). However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

38.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\dfrac{|b_2|}{|\mathbf{b}|} = \sin\theta = \dfrac{|a_2|}{|\mathbf{a}|} \Rightarrow |b_2| = |a_2|$. From the diagram $b_2\,\mathbf{j}$ and $a_2\,\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|AB| = |BC|$, so $|b_3| = \sin\phi\,|BC| = \sin\phi\,|AB| = |a_3|$, and $|b_1| = \cos\phi\,|BC| = \cos\phi\,|AB| = |a_1|$. $b_3\,\mathbf{k}$ and $a_3\,\mathbf{k}$ have the same direction, as do $b_1\,\mathbf{i}$ and $a_1\,\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3 \rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3 \rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

9.3 The Dot Product

1. (a) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ has no meaning.

(b) $(\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b} \cdot \mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b} \cdot \mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$ are vectors, so the dot product $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ has meaning.

(e) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}| \cdot (\mathbf{b} + \mathbf{c})$ has no meaning.

2. Let the vectors be $\mathbf{a}$ and $\mathbf{b}$. Then by definition of the dot product,
$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (6)\left(\frac{1}{3}\right)\cos\frac{\pi}{4} = \frac{6}{3\sqrt{2}} = \sqrt{2}$.

3. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta = (12)(15)\cos\frac{\pi}{6} = 180 \cdot \frac{\sqrt{3}}{2} = 90\sqrt{3} \approx 155.9$

4. $\mathbf{a} \cdot \mathbf{b} = \left\langle \frac{1}{2}, 4 \right\rangle \cdot \langle -8, -3 \rangle = \left(\frac{1}{2}\right)(-8) + (4)(-3) = -16$

7. $\mathbf{a}\cdot\mathbf{b} = (\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k})\cdot(5\,\mathbf{i} + 9\,\mathbf{k}) = (1)(5) + (-2)(0) + (3)(9) = 32$

8. $\mathbf{a}\cdot\mathbf{b} = (4\,\mathbf{j} - 3\,\mathbf{k})\cdot(2\,\mathbf{i} + 4\,\mathbf{j} + 6\,\mathbf{k}) = (0)(2) + (4)(4) + (-3)(6) = -2$

9. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u}\cdot\mathbf{w} = |\mathbf{u}|\,|\mathbf{w}|\cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

10. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45°, we have $|\mathbf{v}| = |\mathbf{u}|\cos 45^\circ = \frac{\sqrt{2}}{2}$. Then $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 45^\circ = (1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{1}{2}$. Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u}\cdot\mathbf{w} = 0$.

11. (a) $\mathbf{i}\cdot\mathbf{j} = \langle 1,0,0\rangle\cdot\langle 0,1,0\rangle = (1)(0) + (0)(1) + (0)(0) = 0$. Similarly $\mathbf{j}\cdot\mathbf{k} = (0)(0) + (1)(0) + (0)(1) = 0$ and $\mathbf{k}\cdot\mathbf{i} = (0)(1) + (0)(0) + (1)(0) = 0$.

Another Method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product is $\cos\frac{\pi}{2} = 0$.

(b) By Property 1 of the dot product, $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j}\cdot\mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k}\cdot\mathbf{k} = |\mathbf{k}|^2 = 1$.

12. The dot product $\mathbf{A}\cdot\mathbf{P}$ is

$$\begin{aligned}\langle a,b,c\rangle\cdot\langle 2, 1.5, 1\rangle &= a(2) + b(1.5) + c(1)\\ &= (\text{number of hamburgers sold})(\text{price per hamburger})\\ &\quad + (\text{number of hot dogs sold})(\text{price per hot dog})\\ &\quad + (\text{number of soft drinks sold})(\text{price per soft drink})\end{aligned}$$

so it is equal to the vendor's total revenue for that day.

13. $|\mathbf{a}| = \sqrt{3^2 + 4^2} = 5$, $|\mathbf{b}| = \sqrt{5^2 + 12^2} = 13$, and $\mathbf{a}\cdot\mathbf{b} = (3)(5) + (4)(12) = 63$. From the definition of the dot product, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{63}{5\cdot 13} = \dfrac{63}{65}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{63}{65}\right) \approx 14^\circ$.

14. $|\mathbf{a}| = \sqrt{6^2 + (-3)^2 + 2^2} = 7$, $|\mathbf{b}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, and $\mathbf{a}\cdot\mathbf{b} = (6)(2) + (-3)(1) + (2)(-2) = 5$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{7\cdot 3} = \dfrac{5}{21}$ and $\theta = \cos^{-1}\left(\frac{5}{21}\right) \approx 76^\circ$.

15. $|\mathbf{a}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$, $|\mathbf{b}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$, and $\mathbf{a}\cdot\mathbf{b} = (0)(1) + (1)(2) + (1)(-3) = -1$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{2}\cdot\sqrt{14}} = \dfrac{-1}{2\sqrt{7}}$ and $\theta = \cos^{-1}\left(-\dfrac{1}{2\sqrt{7}}\right) \approx 101^\circ$.

16. Let p, q and r be the angles at vertices P, Q and R. Then p is the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, q is the angle between vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and r is the angle between vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$. Thus

$$\cos p = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|} = \frac{\langle 2,2,-9\rangle\cdot\langle 5,5,-4\rangle}{\sqrt{89}\,\sqrt{66}} = \frac{56}{\sqrt{5874}},\ \text{so } p = \cos^{-1}\frac{56}{\sqrt{5874}} \approx 43^\circ;$$

$\cos q = \dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|} = \dfrac{\langle -2,-2,9\rangle \cdot \langle 3,3,5\rangle}{\sqrt{89}\,\sqrt{43}} = \dfrac{33}{\sqrt{3827}}$, so $q = \cos^{-1}\dfrac{33}{\sqrt{3827}} \approx 58^\circ$; and

$r \approx 180^\circ - (43^\circ + 58^\circ) = 79^\circ$.

Alternate Solution: Apply the Law of Cosines three times as follows:

$\cos p = \dfrac{\left|\overrightarrow{QR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2}{2\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|}$, $\cos q = \dfrac{\left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PQ}\right|\left|\overrightarrow{QR}\right|}$, $\cos r = \dfrac{\left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PR}\right|\left|\overrightarrow{QR}\right|}$.

17. (a) $\mathbf{a} \cdot \mathbf{b} = (-5)(6) + (3)(-8) + (7)(2) = -40 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(b) $\mathbf{a} \cdot \mathbf{b} = (4)(-3) + (6)(2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(c) $\mathbf{a} \cdot \mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(d) Because $\mathbf{a} = -\frac{2}{3}\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

18. $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ are orthogonal when $\langle -6, b, 2\rangle \cdot \langle b, b^2, b\rangle = 0 \Leftrightarrow (-6)(b) + (b)(b^2) + (2)(b) = 0$ $\Leftrightarrow b^3 - 4b = 0 \Leftrightarrow b(b+2)(b-2) = 0 \Leftrightarrow b = 0$ or $b = \pm 2$.

19. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ be a vector orthogonal to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$. Then $\mathbf{a} \cdot (\mathbf{i} + \mathbf{j}) = 0 \Leftrightarrow a_1 + a_2 = 0$ and $\mathbf{a} \cdot (\mathbf{i} + \mathbf{k}) = 0 \Leftrightarrow a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ are two such unit vectors.

20. According to the definition of the dot product, we need $\langle 1,2,1\rangle \cdot \langle 1,0,c\rangle = |\langle 1,2,1\rangle|\,|\langle 1,0,c\rangle| \cos 60^\circ \Leftrightarrow 1 + c = \sqrt{6}\,\sqrt{1+c^2} \cdot \frac{1}{2} \Leftrightarrow 2(1+c) = \sqrt{6}\,\sqrt{1+c^2}$. Squaring both sides gives $6(1+c^2) = 4(1 + 2c + c^2)$. Thus $6 + 6c^2 = 4 + 8c + 4c^2$ or $2c^2 - 8c + 2 = 0$ and $c = \dfrac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. Each of these values for c can be checked to show it gives a solution.

21. $|\mathbf{a}| = \sqrt{4+9} = \sqrt{13}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{2 \cdot 4 + 3 \cdot 1}{\sqrt{13}} = \dfrac{11}{\sqrt{13}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{11}{\sqrt{13}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{11}{\sqrt{13}} \cdot \frac{1}{\sqrt{13}}\langle 2,3\rangle = \frac{11}{13}\langle 2,3\rangle = \left\langle \frac{22}{13}, \frac{33}{13}\right\rangle$.

22. $|\mathbf{a}| = \sqrt{9+1} = \sqrt{10}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{3 \cdot 2 - 1 \cdot 3}{\sqrt{10}} = \dfrac{3}{\sqrt{10}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{3}{\sqrt{10}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}}\langle 3,-1\rangle = \frac{3}{10}\langle 3,-1\rangle = \left\langle \frac{9}{10}, -\frac{3}{10}\right\rangle$.

23. $|\mathbf{a}| = \sqrt{16+4+0} = 2\sqrt{5}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{1}{2\sqrt{5}}(4+2+0) = \dfrac{3}{\sqrt{5}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{3}{\sqrt{5}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{3}{\sqrt{5}} \cdot \dfrac{1}{2\sqrt{5}}\langle 4,2,0\rangle = \frac{1}{5}\langle 6,3,0\rangle = \left\langle \frac{6}{5}, \frac{3}{5}, 0\right\rangle$.

24. $|\mathbf{a}| = \sqrt{4+9+1} = \sqrt{14}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \dfrac{2 - 18 - 2}{\sqrt{14}} = -\dfrac{18}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\operatorname{proj}_{\mathbf{a}}\mathbf{b} = -\dfrac{18}{\sqrt{14}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\dfrac{18}{\sqrt{14}} \cdot \dfrac{2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}}{\sqrt{14}} = -\frac{9}{7}(2\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k})$.

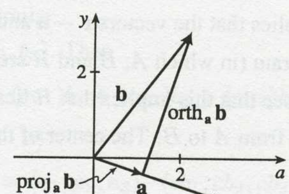
25. $(\text{orth}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a}=(\mathbf{b}-\text{proj}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}-(\text{proj}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{a}-\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}\cdot\mathbf{a}$

$$=\mathbf{b}\cdot\mathbf{a}-\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}|\mathbf{a}|^2=\mathbf{b}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}=0$$

So they are orthogonal by (2).

26. Using the result of Exercise 22, we have

$$\begin{aligned}\text{orth}_{\mathbf{a}}\,\mathbf{b}&=\mathbf{b}-\text{proj}_{\mathbf{a}}\,\mathbf{b}\\&=\langle 2,3\rangle-\left\langle \tfrac{9}{10},-\tfrac{3}{10}\right\rangle\\&=\langle 1.1,3.3\rangle\end{aligned}$$



27. $\text{comp}_{\mathbf{a}}\,\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=2\quad\Leftrightarrow\quad\mathbf{a}\cdot\mathbf{b}=2\,|\mathbf{a}|=2\sqrt{10}$. If $\mathbf{b}=\langle b_1,b_2,b_3\rangle$, then we need $3b_1+0b_2-1b_3=2\sqrt{10}$.
One possible solution is obtained by taking $b_1=0$, $b_2=0$, $b_3=-2\sqrt{10}$.
In general, $\mathbf{b}=\left\langle s,t,3s-2\sqrt{10}\right\rangle$, $s,t\in\mathbb{R}$.

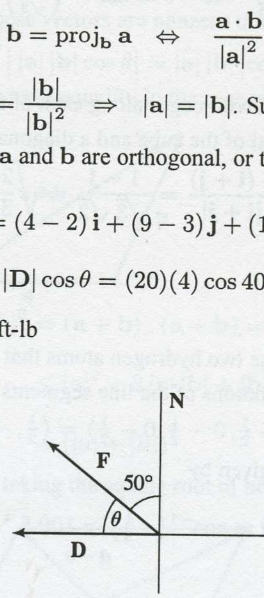
28. (a) $\text{comp}_{\mathbf{a}}\,\mathbf{b}=\text{comp}_{\mathbf{b}}\,\mathbf{a}\quad\Leftrightarrow\quad\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}=\dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|}\quad\Leftrightarrow\quad\dfrac{1}{|\mathbf{a}|}=\dfrac{1}{|\mathbf{b}|}$ or $\mathbf{a}\cdot\mathbf{b}=0\quad\Leftrightarrow\quad|\mathbf{b}|=|\mathbf{a}|$ or $\mathbf{a}\cdot\mathbf{b}=0$. That is, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or if they have the same length.

(b) $\text{proj}_{\mathbf{a}}\,\mathbf{b}=\text{proj}_{\mathbf{b}}\,\mathbf{a}\quad\Leftrightarrow\quad\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}=\dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\,\mathbf{b}\quad\Leftrightarrow\quad\mathbf{a}\cdot\mathbf{b}=0$ or $\dfrac{\mathbf{a}}{|\mathbf{a}|^2}=\dfrac{\mathbf{b}}{|\mathbf{b}|^2}$. But $\dfrac{\mathbf{a}}{|\mathbf{a}|^2}=\dfrac{\mathbf{b}}{|\mathbf{b}|^2}\quad\Rightarrow$ $\dfrac{|\mathbf{a}|}{|\mathbf{a}|^2}=\dfrac{|\mathbf{b}|}{|\mathbf{b}|^2}\quad\Rightarrow\quad|\mathbf{a}|=|\mathbf{b}|$. Substituting this into the previous equation gives $\mathbf{a}=\mathbf{b}$. So $\text{proj}_{\mathbf{a}}\,\mathbf{b}=\text{proj}_{\mathbf{b}}\,\mathbf{a}$ $\Leftrightarrow$ $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, or they are equal.

29. Here $\mathbf{D}=(4-2)\,\mathbf{i}+(9-3)\,\mathbf{j}+(15-0)\,\mathbf{k}=2\,\mathbf{i}+6\,\mathbf{j}+15\,\mathbf{k}$ so $W=\mathbf{F}\cdot\mathbf{D}=20+108-90=38$ joules.

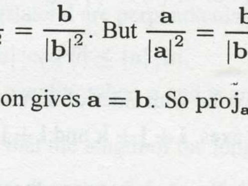
30. $W=|\mathbf{F}|\,|\mathbf{D}|\cos\theta=(20)(4)\cos 40^\circ$

≈ 61 ft-lb



31. $W=|\mathbf{F}|\,|\mathbf{D}|\cos\theta=(25)(10)\cos 20^\circ$

≈ 235 ft-lb



32. Here $|\mathbf{D}|=100$ m, $|\mathbf{F}|=50$ N, and $\theta=30^\circ$. Thus $W=|\mathbf{F}|\,|\mathbf{D}|\cos\theta=(50)(100)\left(\frac{\sqrt{3}}{2}\right)=2500\sqrt{3}$ joules.

33. First note that $\mathbf{n}=\langle a,b\rangle$ is perpendicular to the line, because if $Q_1=(a_1,b_1)$ and $Q_2=(a_2,b_2)$ lie on the line, then $\mathbf{n}\cdot\overrightarrow{Q_1Q_2}=aa_2-aa_1+bb_2-bb_1=0$, since $aa_2+bb_2=-c=aa_1+bb_1$ from the equation of the line. Let $P_2=(x_2,y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. $\text{comp}_\mathbf{n}\left(\overrightarrow{P_1P_2}\right) = \dfrac{|\mathbf{n} \cdot \langle x_2 - x_1, y_2 - y_1\rangle|}{|\mathbf{n}|} = \dfrac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2+b^2}} = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

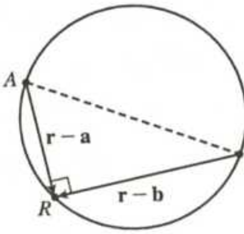
since $ax_2 + by_2 = -c$. The required distance is $\dfrac{|3 \cdot -2 + -4 \cdot 3 + 5|}{\sqrt{3^2+4^2}} = \dfrac{13}{5}$.

34. $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = 0$ implies that the vectors $\mathbf{r} - \mathbf{a}$ and $\mathbf{r} - \mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B. The center of this circle is the midpoint of AB, that is,

$\frac{1}{2}(\mathbf{a} + \mathbf{b}) = \left\langle \frac{1}{2}(a_1 + b_1), \frac{1}{2}(a_2 + b_2), \frac{1}{2}(a_3 + b_3)\right\rangle$, and its radius is

$\frac{1}{2}|\mathbf{a} - \mathbf{b}| = \frac{1}{2}\sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$.

Or: Expand the given equation, substitute $\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.



35. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1, 1, 1)$ has vector representation $\langle 1, 1, 1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x-axis [that is, $\langle 1, 0, 0\rangle$] is given by $\cos\theta = \dfrac{\langle 1,1,1\rangle \cdot \langle 1,0,0\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,0,0\rangle|} = \dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

36. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos\theta = \dfrac{(\mathbf{i}+\mathbf{j}+\mathbf{k}) \cdot (\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|\,|\mathbf{i}+\mathbf{j}|} = \dfrac{1+1}{\sqrt{3}\sqrt{2}} = \sqrt{\dfrac{2}{3}} \quad\Rightarrow$

$\theta = \cos^{-1}\sqrt{\frac{2}{3}} \approx 35^\circ$.

37. Consider the H-C-H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1, 0, 0)$ and $(0, 1, 0)$ (or any H-C-H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1 - \frac{1}{2}, 0 - \frac{1}{2}, 0 - \frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left\langle 0 - \frac{1}{2}, 1 - \frac{1}{2}, 0 - \frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$. The bond angle, θ, is therefore given by

$$\cos\theta = \frac{\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle \cdot \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\right|\left|\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle\right|} = \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(-\tfrac{1}{3}\right) \approx 109.5^\circ.$$

38. Let α be the angle between $\mathbf{a}$ and $\mathbf{c}$ and β be the angle between $\mathbf{c}$ and $\mathbf{b}$. We need to show that $\alpha = \beta$. Now

$\cos\alpha = \dfrac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a} \cdot |\mathbf{a}|\,\mathbf{b} + \mathbf{a} \cdot |\mathbf{b}|\,\mathbf{a}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|^2\,|\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a} \cdot \mathbf{b} + |\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}$. Similarly,

$\cos\beta = \dfrac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,|\mathbf{b}| + \mathbf{b} \cdot \mathbf{a}}{|\mathbf{c}|}$. Thus $\cos\alpha = \cos\beta$. However $0^\circ \le \alpha \le 180^\circ$ and $0^\circ \le \beta \le 180^\circ$, so $\alpha = \beta$ and $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

39. If $c = 0$
$c(\mathbf{a} \cdot \mathbf{b})$
$\mathbf{a}$ and $\mathbf{b}$
product
$\mathbf{a} \cdot (c\mathbf{b}$
$(c\mathbf{a}) \cdot$
Using

40. Let the
and $\overline{B}$
parall

But
both

41. $|\mathbf{a} \cdot$
Note

42. (a)

(b)

43. (a)

(b)