EXERCISES FOR SECTION 1.2

1. **(a)** Let's check Bob's solution first. Since $dy/dt = 1$ and

$$\frac{y(t)+1}{t+1} = \frac{t+1}{t+1} = 1,$$

Bob's answer is correct.

Now let's check Glen's solution. Since $dy/dt = 2$ and

$$\frac{y(t)+1}{t+1} = \frac{2t+2}{t+1} = 2,$$

Glen's solution is also correct.

Finally let's check Paul's solution. We have $dy/dt = 2t$ on one hand and

$$\frac{y(t)+1}{t+1} = \frac{t^2-1}{t+1} = t-1$$

on the other. Therefore, Paul is wrong.

(b) At first glance, they should have seen the equilibrium solution $y(t) = -1$ for all t because $dy/dt = 0$ for any constant function and $y = -1$ implies that

$$\frac{y+1}{t+1} = 0$$

independent of t.

Strictly speaking the differential equation is not defined for $t = -1$, and hence the solutions are not defined for $t = -1$.

2. We note that $dy/dt = 2e^{2t}$ for $y(t) = e^{2t}$. If $y(t) = e^{2t}$ is a solution to the differential equation, then we must have

$$\begin{aligned} 2e^{2t} &= 2y(t) - t + g(y(t)) \\ &= 2e^{2t} - t + g(e^{2t}). \end{aligned}$$

Hence, we need

$$g(e^{2t}) = t.$$

This equation is satisfied if we let $g(y) = (\ln y)/2$. In other words, $y(t) = e^{2t}$ is a solution of the differential equation

$$\frac{dy}{dt} = 2y - t + \frac{\ln y}{2}.$$

3. In order to find one such $f(t, y)$, we compute the derivative of $y(t)$. We obtain

$$\frac{dy}{dt} = \frac{de^{t^3}}{dt} = 3t^2 e^{t^3}.$$

Now we replace e^{t^3} in the last expression by y and get the differential equation

$$\frac{dy}{dt} = 3t^2 y.$$

4. For example, if you start with $y(t) = t^3$, then $dy/dt = 3t^2$. We can write $dy/dt = 3t^2$ as $3t^3/t$ and replace t^3 with y. Hence, the differential equation

$$\frac{dy}{dt} = \frac{3y}{t}$$

has $y(t) = t^3$ as a solution.

Note that, if you already know the solution $y(t)$, it is easier to derive an equation of the form $dy/dt = f(t, y)$ than it is to figure out what $y(t)$ might satisfy a given equation.

5. The constant function $y(t) = 0$ is an equilibrium solution.

For $y \neq 0$ we separate the variables and integrate

$$\int \frac{dy}{y} = \int t\,dt$$

$$\ln|y| = \frac{t^2}{2} + c$$

$$|y| = c_1 e^{t^2/2}$$

where $c_1 = e^c$ is an arbitrary positive constant.

If $y > 0$, then $|y| = y$ and we can just drop the absolute value signs in this calculation. If $y < 0$, then $|y| = -y$, so $-y = c_1 e^{t^2/2}$. Hence, $y = -c_1 e^{t^2/2}$. Therefore,

$$y = k e^{t^2/2}$$

where $k = \pm c_1$. Moreover, if $k = 0$, we get the equilibrium solution. Thus, $y = k e^{t^2/2}$ yields all solutions to the differential equation if we let k be any real number. (Strickly speaking we need a theorem from Section 1.5 to justify the assertion that this formula provides all solutions.)

6. Separating variables and integrating, we obtain

$$\int \frac{1}{y}\,dy = \int t^4\,dt$$

$$\ln|y| = \frac{t^5}{5} + c$$

$$|y| = c_1 e^{t^5/5},$$

where $c_1 = e^c$. As in Exercise 5, we can eliminate the absolute values by replacing the positive constant c_1 with $k = \pm c_1$. Hence, the general solution is

$$y(t) = k e^{t^5/5},$$

where k is any real number. Note that $k = 0$ gives the equilibrium solution.

7. We separate variables and integrate to obtain

$$\int \frac{dy}{2y+1} = \int dt.$$

We get

$$\frac{1}{2}\ln|2y+1| = t + c$$

$$|2y+1| = c_1 e^{2t},$$

where $c_1 = e^{2c}$. As in Exercise 5, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k_1. Hence, we have

$$2y + 1 = k_1 e^{2t}$$

$$y = \frac{1}{2}\left(k_1 e^{2t} - 1\right),$$

and letting $k = k_1/2$, $y(t) = ke^{2t} - 1/2$. Note that, for $k = 0$, we get the equilibrium solution.

8. Separating variables and integrating, we obtain

$$\int \frac{1}{2-y}\,dy = \int dt$$

$$-\ln|2-y| = t + c$$

$$\ln|2-y| = -t + c_1,$$

where we have replaced $-c$ with c_1. Then

$$|2-y| = k_1 e^{-t},$$

where $k_1 = e_1^c$. We can drop the absolute value signs if we replace $\pm k_1$ with k_2, that is, if we allow k_2 to be either positive or negative. Then we have

$$2 - y = k_2 e^{-t}$$
$$y = 2 - k_2 e^{-t}.$$

This could also be written as $y(t) = ke^{-t} + 2$, where we replace $-k_2$ with k. Note that $k = 0$ gives the equilibrium solution.

9. We separate variables and integrate to obtain

$$\int e^y\,dy = \int dt$$

$$e^y = t + c,$$

where c is any constant. We obtain $y(t) = \ln(t + c)$.

10. Separating variables and integrating, we obtain

$$\int \frac{1}{y^2}\,dy = \int t^2\,dt$$

13. First note that the differential equation is not defined for $y = -1/2$. We separate variables and integrate to obtain

$$\int (2y+1)\,dy = \int dt$$

$$y^2 + y = t + k,$$

where k is any constant. So

$$y(t) = \frac{-1 \pm \sqrt{4t+4k+1}}{2} = \frac{-1 \pm \sqrt{4t+c}}{2},$$

where c is any constant and the $\pm$ sign is determined by the initial condition.

We can rewrite the answer in the more simple form

$$y(t) = -\frac{1}{2} \pm \sqrt{t + c_1}$$

where $c_1 = k + 1/4$. If k can be any possible constant, then c_1 can be as well.

14. Separating variables and integrating we obtain

$$\int (1+y^2)\,dy = \int t\,dt$$

$$y + \frac{y^3}{3} = \frac{t^2}{2} + c.$$

To express y as a function of t, we must solve a cubic. The equation for the roots of a cubic can be found in old algebra books or by asking a computer algebra program. The result is

$$y(t) = \frac{-3 \cdot 2^{1/3}}{\left(81(c+t^2/2) + \sqrt{2916 + 6561(c+t^2/2)^2}\right)^{1/3}}$$

$$+ \frac{\left(81(c+t^2/2) + \sqrt{2916+6561(c+t^2/2)^2}\right)^{1/3}}{3 \cdot 2^{1/3}}$$

15. First of all, the equilibrium solutions are $y = 0$ and $y = 1$. Now suppose $y \neq 0$ and $y \neq 1$. We separate variables to obtain

$$\int \frac{1}{y(1-y)}\,dy = \int dt = t + c,$$

where c is any constant. To integrate, we use partial fractions. Write

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

We must have $A = 1$ and $-A + B = 0$. Hence, $A = B = 1$ and

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

and solving for c_1, we obtain $c_1 = -2$. So

$$y(t) = \frac{-2}{t^2 + 4t - 2}.$$

27. Separating variables and integrating, we obtain

$$\int \frac{dy}{y^2} = -\int dt$$

$$-\frac{1}{y} = -t + c.$$

So we get

$$y = \frac{1}{t - c}.$$

Now we need to find the constant c so that $y(0) = 1/2$. To do this we solve

$$\frac{1}{2} = \frac{1}{0 - c}$$

and get $c = -2$. The solution of the initial-value problem is

$$y(t) = \frac{1}{t + 2}.$$

28. Separating variables and integrating, we obtain

$$\int \frac{1}{x}\, dx = \int -t\, dt$$

$$\ln|x| = -\frac{t^2}{2} + c$$

$$|x| = ke^{-t^2/2},$$

where $k = e^c$. Allowing k to be positive or negative, we can remove the absolute values signs, so the general solution is

$$x(t) = ke^{-t^2/2}.$$

Using the initial condition to solve for k, we have

$$\frac{1}{\sqrt{\pi}} = x(0) = ke^0 = k,$$

and therefore

$$x(t) = \frac{e^{-t^2/2}}{\sqrt{\pi}}.$$

36. Rewrite the equation as

$$\frac{dC}{dt} = -k_1 C + (k_1 C_0 + k_2 E),$$

separate variables, and integrate to obtain

$$\int \frac{1}{-k_1 C + (k_1 C_0 + k_2 E)}\, dC = \int dt$$

$$-\frac{1}{k_1} \ln|-k_1 C + k_1 C_0 + k_2 E| = t + c$$

$$-k_1 C + k_1 C_0 + k_2 E = c_1 e^{-k_1 t},$$

where c_1 is a constant determined by the initial condition. Hence,

$$C(t) = C_0 + \frac{k_2}{k_1} E - c_2 e^{-k_1 t},$$

where c_2 is a constant.

(a) Substituting the given values for the parameters, we obtain

$$C(t) = 600 - c_2 e^{-0.1t},$$

and the initial condition $C(0) = 150$ gives $c_2 = 450$, which implies that

$$C(t) = 600 - 450e^{-0.1t}.$$

Hence, $C(2) \approx 232$.

(b) Using part (a), $C(5) \approx 328$.

(c) When t is very large, $e^{-0.1t}$ is very close to zero, so $C(t) \approx 600$. (We could also obtain this conclusion by doing a qualitative analysis of the solutions.)

(d) Using the new parameter values and $C(0) = 600$ yields

$$C(t) = 300 + 300e^{-0.1t},$$

so $C(1) \approx 571$, $C(5) \approx 482$, and as $t \to \infty$, $C \to 300$.

(e) Again changing the parameter values and using $C(0) = 600$, we have

$$C(t) = 500 + 100e^{-0.1t},$$

so $C(1) \approx 590$, $C(5) \approx 560$, and as $t \to \infty$, $C \to 500$.

37. **(a)** If we let k denote the proportionality constant in Newton's law of cooling, the differential equation satisfied by the temperature T of the chocolate is

$$\frac{dT}{dt} = k(T - 70).$$

We also know that $T(0) = 170$ and that $dT/dt = -20$ at $t = 0$. Therefore, we obtain k by evaluating the differential equation at $t = 0$. We have

$$-20 = k(170 - 70),$$

so $k = -0.2$. The initial-value problem is

$$\frac{dT}{dt} = -0.2(T - 70), \quad T(0) = 170.$$

(b) We can solve the initial-value problem in part (a) by separating variables. We have

$$\int \frac{dT}{T - 70} = \int -0.2\, dt$$

$$\ln|T - 70| = -0.2t + k$$

$$|T - 70| = ce^{-0.2t}.$$

Since the temperature of the chocolate cannot become lower than the temperature of the room, we can ignore the absolute value and conclude

$$T(t) = 70 + ce^{-0.2t}.$$

Now we use the initial condition $T(0) = 170$ to find the constant c because

$$170 = T(0) = 70 + ce^{-0.2(0)},$$

which implies that $c = 100$. The solution is

$$T = 70 + 100e^{-0.2t}.$$

In order to find t so that the temperature is 110° F, we solve

$$110 = 70 + 100e^{-0.2t}$$

for t obtaining

$$\tfrac{2}{5} = e^{-0.2t}$$

$$\ln \tfrac{2}{5} = -0.2t$$

so that

$$t = \frac{\ln(2/5)}{-0.2} \approx 4.6.$$

38. The easiest way to do this problem is to use the fact that $C = S/100$, so

$$\frac{dC}{dt} = \frac{1}{100}\frac{dS}{dt}.$$

Hence,

$$\frac{dC}{dt} = \frac{1}{100}\frac{dS}{dt} = \frac{1}{100}\left(20 - \frac{3S}{100}\right) = \frac{1}{100}(20 - 3C)$$

We obtain

$$\frac{dC}{dt} = \frac{20 - 3C}{100} = \frac{1}{5} - \frac{3C}{100}.$$