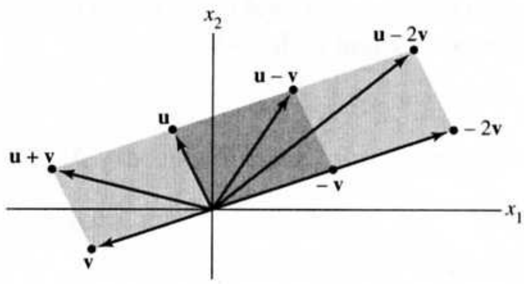
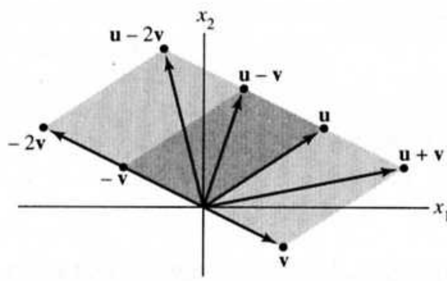
$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}+(-2)\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}(-2)(2)\\(-2)(-1)\end{bmatrix}=\begin{bmatrix}3+(-4)\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$, or, more quickly,

$$\mathbf{u}-2\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}-2\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}3-4\\2+2\end{bmatrix}=\begin{bmatrix}-1\\4\end{bmatrix}$$. The intermediate step is often not written.

3.



4.



5. $x_1\begin{bmatrix}6\\-1\\5\end{bmatrix}+x_2\begin{bmatrix}-3\\4\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$, $\begin{bmatrix}6x_1\\-x_1\\5x_1\end{bmatrix}+\begin{bmatrix}-3x_2\\4x_2\\0\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$, $\begin{bmatrix}6x_1-3x_2\\-x_1+4x_2\\5x_1\end{bmatrix}=\begin{bmatrix}1\\-7\\-5\end{bmatrix}$

$$\begin{aligned}6x_1 - 3x_2 &= 1\\ -x_1 + 4x_2 &= -7\\ 5x_1 &= -5\end{aligned}$$

Usually the intermediate steps are not displayed.

6. $x_1\begin{bmatrix}-2\\3\end{bmatrix}+x_2\begin{bmatrix}8\\5\end{bmatrix}+x_3\begin{bmatrix}1\\-6\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-2x_1\\3x_1\end{bmatrix}+\begin{bmatrix}8x_2\\5x_2\end{bmatrix}+\begin{bmatrix}x_3\\-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-2x_1+8x_2+x_3\\3x_1+5x_2-6x_3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{aligned}-2x_2 + 8x_2 + x_3 &= 0\\ 3x_1 + 5x_2 - 6x_3 &= 0\end{aligned}$$

Usually the intermediate steps are not displayed.

7. See the figure below. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

To write a vector **a** as a linear combination of **u** and **v**, imagine walking from the origin to **a** along the grid "streets" and keep track of how many "blocks" you travel in the **u**-direction and how many in the **v**-direction.

a. To reach **a** from the origin, you might travel 1 unit in the **u**-direction and –2 units in the **v**-direction (that is, 2 units in the negative **v**-direction). Hence **a** = **u** – 2**v**.

b. To reach **b** from the origin, travel 2 units in the **u**-direction and –2 units in the **v**-direction. So $\mathbf{b} = 2\mathbf{u} - 2\mathbf{v}$. Or, use the fact that **b** is 1 unit in the **u**-direction from **a**, so that

$$\mathbf{b} = \mathbf{a} + \mathbf{u} = (\mathbf{u} - 2\mathbf{v}) + \mathbf{u} = 2\mathbf{u} - 2\mathbf{v}$$

c. The vector **c** is –1.5 units from **b** in the **v**-direction, so

$$\mathbf{c} = \mathbf{b} - 1.5\mathbf{v} = (2\mathbf{u} - 2\mathbf{v}) - 1.5\mathbf{v} = 2\mathbf{u} - 3.5\mathbf{v}$$

d. The "map" suggests that you can reach **d** if you travel 3 units in the **u**-direction and –4 units in the **v**-direction. If you prefer to stay on the paths displayed on the map, you might travel from the origin to $-3\mathbf{v}$, then move 3 units in the **u**-direction, and finally move –1 unit in the **v**-direction. So

$$\mathbf{d} = -3\mathbf{v} + 3\mathbf{u} - \mathbf{v} = 3\mathbf{u} - 4\mathbf{v}$$

Another solution is

$$\mathbf{d} = \mathbf{b} - 2\mathbf{v} + \mathbf{u} = (2\mathbf{u} - 2\mathbf{v}) - 2\mathbf{v} + \mathbf{u} = 3\mathbf{u} - 4\mathbf{v}$$

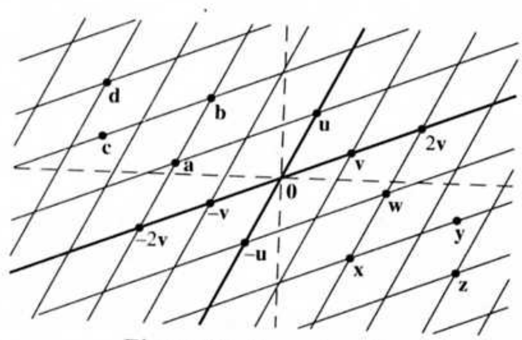


Figure for Exercises 7 and 8

8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, $\mathbf{w} = (-1)\mathbf{u} + 2\mathbf{v}$, or $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So $\mathbf{z} = 4\mathbf{v} - 3\mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$. If you prefer to stay on the paths displayed on the "map," you might travel from the origin to $-2\mathbf{u}$, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$$

9. $$\begin{array}{rcrcrcl} & & x_2 & + & 5x_3 & = & 0 \\ 4x_1 & + & 6x_2 & - & x_3 & = & 0 \\ -x_1 & + & 3x_2 & - & 8x_3 & = & 0 \end{array}, \qquad \begin{bmatrix} x_2 + 5x_3 \\ 4x_1 + 6x_2 - x_3 \\ -x_1 + 3x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 6x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} 5x_3 \\ -x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Usually, the intermediate calculations are not displayed.

Note: The *Study Guide* says, "Check with your instructor whether you need to "show work" on a problem such as Exercise 9."

10. $\begin{array}{rcrcrcr} 4x_1 & + & x_2 & + & 3x_3 & = & 9 \\ x_1 & - & 7x_2 & - & 2x_3 & = & 2 \\ 8x_1 & + & 6x_2 & - & 5x_3 & = & 15 \end{array}$, $\begin{bmatrix} 4x_1 + x_2 + 3x_3 \\ x_1 - 7x_2 - 2x_3 \\ 8x_1 + 6x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

$\begin{bmatrix} 4x_1 \\ x_1 \\ 8x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -7x_2 \\ 6x_2 \end{bmatrix} + \begin{bmatrix} 3x_3 \\ -2x_3 \\ -5x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$, $x_1\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$

Usually, the intermediate calculations are not displayed.

11. The question

Is **b** a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ have a solution?

The equation

$$\underset{\mathbf{a}_1}{\underset{\uparrow}{x_1\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}} + \underset{\mathbf{a}_2}{\underset{\uparrow}{x_2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}} + \underset{\mathbf{a}_3}{\underset{\uparrow}{x_3\begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}}} = \underset{\mathbf{b}}{\underset{\uparrow}{\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}}} \qquad (*)$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 5 & 2 \\ 0 & \textcircled{1} & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system corresponding to M *has* a solution, so the vector equation (*) has a solution, and therefore **b** *is* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

12. The equation

$$\underset{\mathbf{a}_1}{\underset{\uparrow}{x_1\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}}} + \underset{\mathbf{a}_2}{\underset{\uparrow}{x_2\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}}} + \underset{\mathbf{a}_3}{\underset{\uparrow}{x_3\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}}} = \underset{\mathbf{b}}{\underset{\uparrow}{\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}}} \qquad (*)$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 2 & -5 \\ 0 & \textcircled{5} & 4 & 1 \\ 0 & 0 & 0 & \textcircled{2} \end{bmatrix}$$

The linear system corresponding to M has *no* solution, so the vector equation (*) has no solution, ar therefore **b** is *not* a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

13. Denote the columns of A by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. To determine if **b** is a linear combination of these columns, us boxed fact on page 34. Row reduced the augmented matrix until you reach echelon form:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -4 & 2 & 3 \\ 0 & \textcircled{3} & 5 & -7 \\ 0 & 0 & 0 & \textcircled{3} \end{bmatrix}$$

The system for this augmented matrix is inconsistent, so **b** is *not* a linear combination of the columns of A.

14. $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{b}] = \begin{bmatrix} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & -6 & 11 \\ 0 & \textcircled{3} & 7 & -5 \\ 0 & 0 & \textcircled{11} & -2 \end{bmatrix}$. The linear system corresponding to this matrix *has* a solution, so **b** is a linear combination of the columns of A.

15. Noninteger weights are acceptable, of course, but some simple choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 12 \\ -2 \\ -6 \end{bmatrix}$$

16. Some likely choices are $0\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \mathbf{0}$, and

$$1\cdot\mathbf{v}_1 + 0\cdot\mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad 0\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

$$1\cdot\mathbf{v}_1 + 1\cdot\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \quad 1\cdot\mathbf{v}_1 - 1\cdot\mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$$

17. $[\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{b}] = \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 4 \\ 0 & \textcircled{1} & -3 \\ 0 & 0 & h+17 \end{bmatrix}$. The vector **b** is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h + 17$ is zero, that is, when $h = -17$.

18. $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{y}] = \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & -3+2h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & h \\ 0 & \textcircled{1} & -5 \\ 0 & 0 & 7+2h \end{bmatrix}$. The vector **y** is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ when $7 + 2h$ is zero, that is, when $h = -7/2$.

19. By inspection, $\mathbf{v}_2 = (3/2)\mathbf{v}_1$. Any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is actually just a multiple of $\mathbf{v}_1$. For instance,

$$a\mathbf{v}_1 + b\mathbf{v}_2 = a\mathbf{v}_1 + b(3/2)\mathbf{v}_2 = (a + 3b/2)\mathbf{v}_1$$

So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is the set of points on the line through $\mathbf{v}_1$ and $\mathbf{0}$.

Note: Exercises 19 and 20 prepare the way for ideas in Sections 1.4 and 1.7.

20. Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ is a plane in $\mathbb{R}^3$ through the origin, because the neither vector in this problem is a multiple of the other. Every vector in the set has 0 as its second entry and so lies in the xz-plane in ordinary 3-space. So Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ *is* the xz-plane.

21. Let $\mathbf{y} = \begin{bmatrix} h \\ k \end{bmatrix}$. Then $[\mathbf{u} \ \ \mathbf{v} \ \ \mathbf{y}] = \begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & 2 & h \\ 0 & \textcircled{2} & k+h/2 \end{bmatrix}$. This augmented matrix corresponds to a consistent system for all h and k. So **y** is in Span$\{\mathbf{u}, \mathbf{v}\}$ for all h and k.

22. Construct any 3×4 matrix in echelon form that corresponds to an inconsistent system. Perform sufficient row operations on the matrix to eliminate all zero entries in the first three columns.

23. **a.** False. The alternative notation for a (column) vector is (–4, 3), using parentheses and commas.

b. False. Plot the points to verify this. Or, see the statement preceding Example 3. If $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ were on the line through $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and the origin, then $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ would have to be a multiple of $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$, which is not the case.

c. True. See the line displayed just before Example 4.

d. True. See the box that discusses the matrix in (5).

e. False. The statement is often true, but Span$\{\mathbf{u}, \mathbf{v}\}$ is not a plane when **v** is a multiple of **u**, or when **u** is the zero vector.

24. **a.** True. See the beginning of the subsection *Vectors in* $\mathbb{R}^n$.

b. True. Use Fig. 7 to draw the parallelogram determined by $\mathbf{u} - \mathbf{v}$ and **v**.

c. False. See the first paragraph of the subsection *Linear Combinations*.

d. True. See the statement that refers to Fig. 11.

e. True. See the paragraph following the definition of Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

25. **a.** There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\mathbf{b}$ is not one of them.

b. There are infinitely many vectors in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if $\mathbf{b}$ is in W, use the method of Exercise 13.

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ 0 & 6 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -4 & 4 \\ 0 & \textcircled{3} & -2 & 1 \\ 0 & 0 & \textcircled{-1} & 2 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{matrix}$$

The system for this augmented matrix is consistent, so $\mathbf{b}$ is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

26. **a.** $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}] = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Yes, $\mathbf{b}$ is a linear combination of the columns of A, that is, $\mathbf{b}$ is in W.

b. The third column of A is in W because $\mathbf{a}_3 = 0 \cdot \mathbf{a}_1 + 0 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3$.

27. **a.** $5\mathbf{v}_1$ is the output of 5 days' operation of mine #1.

b. The total output is $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$, so x_1 and x_2 should satisfy $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 150 \\ 2825 \end{bmatrix}$.

c. **[M]** Reduce the augmented matrix $\begin{bmatrix} 20 & 30 & 150 \\ 550 & 500 & 2825 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 4.0 \end{bmatrix}$.

Operate mine #1 for 1.5 days and mine #2 for 4 days. (This is the exact solution.)

28. **a.** The amount of heat produced when the steam plant burns x_1 tons of anthracite and x_2 tons of bituminous coal is $27.6x_1 + 30.2x_2$ million Btu.

b. The total output produced by x_1 tons of anthracite and x_2 tons of bituminous coal is given by the vector $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix}$.

c. **[M]** The appropriate values for x_1 and x_2 satisfy $x_1\begin{bmatrix} 27.6 \\ 3100 \\ 250 \end{bmatrix} + x_2\begin{bmatrix} 30.2 \\ 6400 \\ 360 \end{bmatrix} = \begin{bmatrix} 162 \\ 23{,}610 \\ 1{,}623 \end{bmatrix}$.

To solve, row reduce the augmented matrix:

$$\begin{bmatrix} 27.6 & 30.2 & 162 \\ 3100 & 6400 & 23610 \\ 250 & 360 & 1623 \end{bmatrix} \sim \begin{bmatrix} 1.000 & 0 & 3.900 \\ 0 & 1.000 & 1.800 \\ 0 & 0 & 0 \end{bmatrix}$$

The steam plant burned 3.9 tons of anthracite coal and 1.8 tons of bituminous coal.

For your information: The unique solution of this equation is (5, 7, 3). Finding the solution by hand would be time-consuming.

Note: The skill of writing a vector equation as a matrix equation will be important for both theory and application throughout the text. See also Exercises 27 and 28.

8. The left side of the equation is a linear combination of four vectors. Write the matrix A whose columns are those four vectors, and create a variable vector with four entries:

$$A = \left[\begin{bmatrix} 4 \\ -2 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \end{bmatrix} \begin{bmatrix} -5 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix}, \text{ and } \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}. \text{ Then the equation } A\mathbf{z} = \mathbf{b}$$

is
$$\begin{bmatrix} 4 & -4 & -5 & 3 \\ -2 & 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}.$$

For your information: One solution is (7, 3, 3, 1). The general solution is $z_1 = 6 + .75z_3 - 1.25z_4$, $z_2 = 5 - .5z_3 - .5z_4$, with z_3 and z_4 free.

9. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

10. The system has the same solution set as the vector equation

$$x_1\begin{bmatrix} 8 \\ 5 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

and this equation has the same solution set as the matrix equation

$$\begin{bmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

11. To solve $A\mathbf{x} = \mathbf{b}$, row reduce the augmented matrix $[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3\ \ \mathbf{b}]$ for the corresponding linear system:

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & -3 \\ 0 & 0 & \textcircled{1} & 1 \end{bmatrix}$$

$$\sim\begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_3-5b_1+2(b_2+3b_1) \end{bmatrix} = \begin{bmatrix} \textcircled{1} & -3 & -4 & b_1 \\ 0 & \textcircled{-7} & -6 & b_2+3b_1 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such **b** is a plane through the origin in $\mathbf{R}^3$.

17. Row reduction shows that only three rows of A contain a pivot position:

$$A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & -4 & 2 & -8 \\ 0 & -6 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & 3 \\ 0 & \textcircled{2} & -1 & 4 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of A contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $A\mathbf{x} = \mathbf{b}$ does *not* have a solution for each **b** in $\mathbf{R}^4$.

18. Row reduction shows that only three rows of B contain a pivot position:

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -2 & 2 \\ 0 & \textcircled{1} & 1 & -5 \\ 0 & 0 & 0 & \textcircled{-7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because not every row of B contains a pivot position, Theorem 4 in Section 1.4 shows that the equation $B\mathbf{x} = \mathbf{y}$ does *not* have a solution for each **y** in $\mathbf{R}^4$.

19. The work in Exercise 17 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of A. Also, the columns of A do *not* span $\mathbf{R}^4$.

20. The work in Exercise 18 shows that statement (d) in Theorem 4 is false. So all four statements in Theorem 4 are false. Thus, not all vectors in $\mathbf{R}^4$ can be written as a linear combination of the columns of B. The columns of B certainly do *not* span $\mathbf{R}^3$, because each column of B is in $\mathbf{R}^4$, not $\mathbf{R}^3$. (This question was asked to alert students to a fairly common misconception among students who are just learning about spanning.)

21. Row reduce the matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 1 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ does not have a pivot in each row, so the columns of the matrix do not span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbf{R}^4$.

Note: Some students may realize that row operations are not needed, and thereby discover the principle covered in Exercises 31 and 32.

22. Row reduce the matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ to determine whether it has a pivot in each row.

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -1 \\ -2 & 8 & -5 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-2} & 8 & -5 \\ 0 & \textcircled{-3} & -1 \\ 0 & 0 & \textcircled{4} \end{bmatrix}$$

The matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ has a pivot in each row, so the columns of the matrix span $\mathbf{R}^4$, by Theorem 4. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbf{R}^4$.

23. **a.** False. See the paragraph following equation (3). The text calls $A\mathbf{x} = \mathbf{b}$ a *matrix equation.*
   b. True. See the box before Example 3.
   c. False. See the warning following Theorem 4.
   d. True. See Example 4.
   e. True. See parts (c) and (a) in Theorem 4.
   f. True. In Theorem 4, statement (a) is false if and only if statement (d) is also false.

24. **a.** True. This statement is in Theorem 3. However, the statement is true without any "proof" because, by definition, $A\mathbf{x}$ is simply a notation for $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A.
   b. True. See Example 2.
   c. True, by Theorem 3.
   d. True. See the box before Example 2. Saying that $\mathbf{b}$ is not in the set spanned by the columns of A is the same a saying that $\mathbf{b}$ is not a linear combination of the columns of A.
   e. False. See the warning that follows Theorem 4.
   f. True. In Theorem 4, statement (c) is false if and only if statement (a) is also false.

25. By definition, the matrix-vector product on the left is a linear combination of the columns of the matrix, in this case using weights –3, –1, and 2. So $c_1 = -3$, $c_2 = -1$, and $c_3 = 2$.

26. The equation in x_1 and x_2 involves the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, and it may be viewed as

$[\mathbf{u}\ \ \mathbf{v}]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}$. By definition of a matrix-vector product, $x_1\mathbf{u} + x_2\mathbf{v} = \mathbf{w}$. The stated fact that

$3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$ can be rewritten as $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$. So, a solution is $x_1 = 3$, $x_2 = -5$.

27. Place the vectors $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ into the columns of a matrix, say, Q and place the weights x_1, x_2, and x_3 into a vector, say, $\mathbf{x}$. Then the vector equation becomes

$$Q\mathbf{x} = \mathbf{v}, \text{ where } Q = [\mathbf{q}_1\ \ \mathbf{q}_2\ \ \mathbf{q}_3] \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: If your answer is the equation $A\mathbf{x} = \mathbf{b}$, you need to specify what A and $\mathbf{b}$ are.

28. The matrix equation can be written as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{v}_6$, where $c_1 = -3$, $c_2 = 2$, $c_3 = 4$, $c_4 = -1$, $c_5 = 2$, and

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 9 \\ -2 \end{bmatrix}, \mathbf{v}_5 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}, \mathbf{v}_6 = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$