9.4 The Cross Product

[redacted] **1.** (a) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the dot product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b} \cdot \mathbf{c}$ is a scalar, so $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b} \times \mathbf{c}$ is a vector, the cross product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is meaningful and results in another vector.

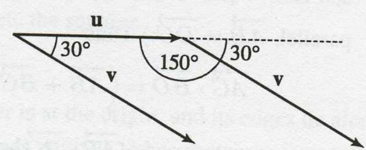
(d) $\mathbf{a} \cdot \mathbf{b}$ is a scalar, so the cross product $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ is meaningless.

(e) Since $(\mathbf{a} \cdot \mathbf{b})$ and $(\mathbf{c} \cdot \mathbf{d})$ are both scalars, the cross product $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$ is meaningless.

(f) $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are both vectors, so the dot product $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ is meaningful and is a scalar.

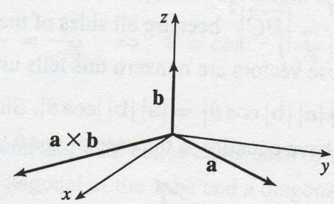
2. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = (5)(10)\sin 60^\circ = 25\sqrt{3}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.

3. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 30°, so $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 30^\circ = (6)(8)\left(\frac{1}{2}\right) = 24$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed into the page.



4. (a) $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta = 3 \cdot 2 \cdot \sin\frac{\pi}{2} = 6$

(b) $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{k}$, so it lies in the xy-plane, and its z-coordinate is 0. By the right-hand rule, its y-component is negative and its x-component is positive.



[redacted] **5.** The magnitude of the torque is
$|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.18\text{ m})(60\text{ N})\sin(70+10)^\circ = 10.8\sin 80^\circ \approx 10.6\text{ J}$.

6. $|\mathbf{r}| = \sqrt{4^2+4^2} = 4\sqrt{2}$ ft. A line drawn from the point P to the point of application of the force makes an angle of $180^\circ - (45+30)^\circ = 105^\circ$ with the force vector. Therefore, $|\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (4\sqrt{2})(36)\sin 105^\circ \approx 197$ ft-lb.

7. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}\mathbf{k} = (-1-0)\,\mathbf{i} - (1-0)\,\mathbf{j} + [2-(-3)]\,\mathbf{k} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle -1, -1, 5\rangle \cdot \langle 1, -1, 0\rangle = -1 + 1 + 0 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle -1, -1, 5\rangle \cdot \langle 3, 2, 1\rangle = -3 - 2 + 5 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$.

[redacted] **8.** $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & 2 \\ 6 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 2 \\ 6 & 3 \end{vmatrix}\mathbf{k}$$
$$= (2-6)\,\mathbf{i} - (-3-12)\,\mathbf{j} + (-9-12)\,\mathbf{k} = -4\mathbf{i} + 15\mathbf{j} - 21\mathbf{k}$$

Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = \langle -4, 15, -21\rangle \cdot \langle -3, 2, 2\rangle = 12 + 30 - 42 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$.
Since $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \langle -4, 15, -21\rangle \cdot \langle 6, 3, 1\rangle = -24 + 45 - 21 = 0$, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{b}$.

9. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}\mathbf{k}$

$= (3t^4 - 2t^4)\,\mathbf{i} - (3t^3 - t^3)\,\mathbf{j} + (2t^2 - t^2)\,\mathbf{k} = t^4\,\mathbf{i} - 2t^3\,\mathbf{j} + t^2\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle t^4, -2t^3, t^2\rangle \cdot \langle t, t^2, t^3\rangle = t^5 - 2t^5 + t^5 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle t^4, -2t^3, t^2\rangle \cdot \langle 1, 2t, 3t^2\rangle = t^4 - 4t^4 + 3t^4 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

10. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^t & e^{-t} \\ 2 & e^t & -e^{-t} \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & e^{-t} \\ 2 & -e^{-t} \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & e^t \\ 2 & e^t \end{vmatrix}\mathbf{k}$

$= (-1-1)\,\mathbf{i} - (-e^{-t} - 2e^{-t})\,\mathbf{j} + (e^t - 2e^t)\,\mathbf{k} = -2\,\mathbf{i} + 3e^{-t}\,\mathbf{j} - e^t\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (-2\,\mathbf{i} + 3e^{-t}\,\mathbf{j} - e^t\,\mathbf{k})\cdot(\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}) = -2 + 3 - 1 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (-2\,\mathbf{i} + 3e^{-t}\,\mathbf{j} - e^t\,\mathbf{k})\cdot(2\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}) = -4 + 3 + 1 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

11. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{k}$

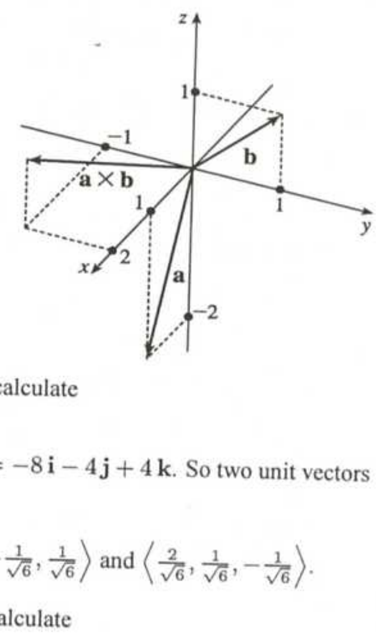
$\mathbf{a}\times\mathbf{b} = [-6 - (-8)]\,\mathbf{i} - (-9 - 4)\,\mathbf{j} + (-6 - 2)\,\mathbf{k} = 2\,\mathbf{i} + 13\,\mathbf{j} - 8\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (2\,\mathbf{i} + 13\,\mathbf{j} - 8\,\mathbf{k})\cdot(3\,\mathbf{i} + 2\,\mathbf{j} + 4\,\mathbf{k}) = 6 + 26 - 32 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (2\,\mathbf{i} + 13\,\mathbf{j} - 8\,\mathbf{k})\cdot(\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}) = 2 - 26 + 24 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

12. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= 2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$



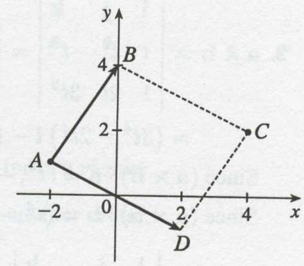
13. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\langle 1, -1, 1\rangle \times \langle 0, 4, 4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 4 & 4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}\mathbf{k} = -8\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$. So two unit vectors orthogonal to both are $\pm\dfrac{\langle -8, -4, 4\rangle}{\sqrt{64 + 16 + 16}} = \pm\dfrac{\langle -8, -4, 4\rangle}{4\sqrt{6}}$, that is, $\left\langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle$ and $\left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\rangle$.

14. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$. Thus, two unit vectors orthogonal to both are $\pm\frac{1}{\sqrt{6}}\langle 1, -1, -2\rangle$, that is, $\left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right\rangle$ and $\left\langle -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\rangle$.

15. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2, 3\rangle$ and $\overrightarrow{AD} = \langle 4, -2\rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2, 3, 0\rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 4 & -2 & 0 \end{matrix}\right\| = |(0)\mathbf{i} - (0)\mathbf{j} + (-4 - 12)\,\mathbf{k}| = |-16\,\mathbf{k}| = 16.$$

16. The parallelogram is determined by the vectors $\overrightarrow{KL} = \langle 0, 1, 3\rangle$ and $\overrightarrow{KN} = \langle 2, 5, 0\rangle$, so the area of parallelogram $KLMN$ is

$$\left|\overrightarrow{KL} \times \overrightarrow{KN}\right| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{matrix}\right\| = |(-15)\,\mathbf{i} - (-6)\mathbf{j} + (-2)\mathbf{k}| = |-15\,\mathbf{i} + 6\,\mathbf{j} - 2\,\mathbf{k}| = \sqrt{265} \approx 16.28.$$

17. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -1, 2, 0\rangle$ and $\overrightarrow{PR} = \langle -1, 0, 3\rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (2)(3) - (0)(0), (0)(-1) - (-1)(3), (-1)(0) - (2)(-1)\rangle = \langle 6, 3, 2\rangle$$

Therefore, $\langle 6, 3, 2\rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P, Q, and R.

(b) Note that the area of the triangle determined by P, Q, and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle 6, 3, 2\rangle| = \sqrt{36 + 9 + 4} = 7$, so the area of the triangle is $\frac{1}{2}(7) = \frac{7}{2}$.

18. (a) $\overrightarrow{PQ} = \langle 1, 1, 3\rangle$ and $\overrightarrow{PR} = \langle 3, 2, 5\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)\,(5) - (3)(2), (3)(3) - (1)(5), (1)\,(2) - (1)(3)\rangle = \langle -1, 4, -1\rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle -1, 4, -1\rangle| = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}$, so the area of triangle PQR is $\frac{1}{2} \cdot 3\sqrt{2} = \frac{3}{2}\sqrt{2}$.

19. Using the notation of (1), $\mathbf{r} = \langle 0, 0.3, 0\rangle$ and $\mathbf{F}$ has direction $\langle 0, 3, -4\rangle$. The angle θ between them can be determined by $\cos\theta = \dfrac{\langle 0, 0.3, 0\rangle \cdot \langle 0, 3, -4\rangle}{|\langle 0, 0.3, 0\rangle|\,|\langle 0, 3, -4\rangle|} \;\Rightarrow\; \cos\theta = \dfrac{0.9}{(0.3)(5)} \;\Rightarrow\; \cos\theta = 0.6 \;\Rightarrow\; \theta \approx 53.1^\circ$.
Then $|\tau| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta \;\Rightarrow\; 100 = 0.3\,|\mathbf{F}|\sin 53.1^\circ \;\Rightarrow\; |\mathbf{F}| \approx 417\text{ N}$.

20. Since $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, $0\le\theta\le\pi$, $|\mathbf{u}\times\mathbf{v}|$ achieves its maximum value for $\sin\theta = 1 \;\Rightarrow\; \theta = \frac{\pi}{2}$, in which case $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| = 15$. The minimum value is zero, which occurs when $\sin\theta = 0 \;\Rightarrow\; \theta = 0$ or π, so when $\mathbf{u}$, $\mathbf{v}$ are parallel. Thus, when $\mathbf{u}$ points in the same direction as $\mathbf{v}$, so $\mathbf{u} = 3\,\mathbf{j}$, $|\mathbf{u}\times\mathbf{v}| = 0$. As $\mathbf{u}$ rotates counterclockwise, $\mathbf{u}\times\mathbf{v}$ is directed in the negative z-direction (by the right-hand rule) and the length increases until $\theta = \frac{\pi}{2}$, in which case $\mathbf{u} = -3\,\mathbf{i}$ and $|\mathbf{u}\times\mathbf{v}| = 15$. As $\mathbf{u}$ rotates to the negative y-axis, $\mathbf{u}\times\mathbf{v}$ remains pointed in the negative z-direction and the length of $\mathbf{u}\times\mathbf{v}$ decreases to 0, after which the direction of $\mathbf{u}\times\mathbf{v}$ reverses to point in the positive z-direction and $|\mathbf{u}\times\mathbf{v}|$ increases. When $\mathbf{u} = 3\,\mathbf{i}$ (so $\theta = \frac{\pi}{2}$), $|\mathbf{u}\times\mathbf{v}|$ again reaches its maximum of 15, after which $|\mathbf{u}\times\mathbf{v}|$ decreases to 0 as $\mathbf{u}$ rotates to the positive y-axis.

21. We know that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product, which is

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix}$$
$$= 6(5+4) - 3(0-8) - (0-4) = 82$$

Thus the volume of the parallelepiped is 82 cubic units.

22. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -6 - 9 - 4 = -19$. So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is $|-19| = 19$ cubic units.

23. $\mathbf{a} = \overrightarrow{PQ} = \langle 1,-1,2\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 3,0,6\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 2,-2,-3\rangle$.
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 6 \\ 2 & -2 & -3 \end{vmatrix} = 1\begin{vmatrix} 0 & 6 \\ -2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 6 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 2 & -2 \end{vmatrix} = 12 - 21 - 12 = -21$, so the volume of the parallelepiped is 21 cubic units.

24. $\mathbf{a} = \overrightarrow{PQ} = \langle 2,3,3\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle -1,-1,-1\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 6,-2,2\rangle$.
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 2 & 3 & 3 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} = -8 - 12 + 24 = 4$, so the volume of the parallelepiped is 4 cubic units.

25. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 7 & 3 \end{vmatrix} = -4 - 6 + 10 = 0$, which says that the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 0, and thus these three vectors are coplanar.

26. $\mathbf{a} = \overrightarrow{PQ} = \langle 1,4,5\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2,-1,1\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 5,2,7\rangle$.
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & 2 & 7 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = -9 - 36 + 45 = 0$, so the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 0, which says these vectors lie in the same plane. Therefore, their initial and terminal points P, Q, R and S also lie in the same plane.

30. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, so $\mathbf{b} \times \mathbf{c} = \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle$ and

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), \\
&\qquad a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \rangle \\
&= \langle a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3, a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, \\
&\qquad a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \rangle \\
&= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1, (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2, \\
&\qquad (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 \rangle \\
(\star) \quad &= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 + a_1b_1c_1 - a_1b_1c_1, \\
&\qquad (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2 + a_2b_2c_2 - a_2b_2c_2, \\
&\qquad (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 + a_3b_3c_3 - a_3b_3c_3 \rangle \\
&= \langle (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1, \\
&\qquad (a_1c_1 + a_2c_2 + a_3c_3)b_2 - (a_1b_1 + a_2b_2 + a_3b_3)c_2, \\
&\qquad (a_1c_1 + a_2c_2 + a_3c_3)b_3 - (a_1b_1 + a_2b_2 + a_3b_3)c_3 \rangle \\
&= (a_1c_1 + a_2c_2 + a_3c_3)\langle b_1, b_2, b_3 \rangle - (a_1b_1 + a_2b_2 + a_3b_3)\langle c_1, c_2, c_3 \rangle \\
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}
\end{aligned}$$

$(\star)$ Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don't change the value of the component.

31. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$

$$\begin{aligned}
&= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}] + [(\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] + [(\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}] && \text{by Exercise 30} \\
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} + (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = \mathbf{0}
\end{aligned}$$

32. Let $\mathbf{c} \times \mathbf{d} = \mathbf{v}$. Then

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{v} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{v}) && \text{by (6)} \\
&= \mathbf{a} \cdot [\mathbf{b} \times (\mathbf{c} \times \mathbf{d})] \\
&= \mathbf{a} \cdot [(\mathbf{b} \cdot \mathbf{d})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}] && \text{by Exercise 30} \\
&= (\mathbf{b} \cdot \mathbf{d})(\mathbf{a} \cdot \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) && \text{by Properties 3 and 4 of the dot product} \\
&= \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}
\end{aligned}$$

33. (a) No. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, which can happen if $\mathbf{b} \neq \mathbf{c}$. For example, let $\mathbf{a} = \langle 1, 1, 1 \rangle$, $\mathbf{b} = \langle 1, 0, 0 \rangle$ and $\mathbf{c} = \langle 0, 1, 0 \rangle$.

(b) No. If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ then $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$, which of course can happen if $\mathbf{b} \neq \mathbf{c}$.

(c) Yes. Since $\mathbf{a} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b} - \mathbf{c}$. Thus since $\mathbf{a} \neq \mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b} - \mathbf{c}$, we have $\mathbf{b} - \mathbf{c} = \mathbf{0}$, so $\mathbf{b} = \mathbf{c}$.

9.5 Equations of Lines and Planes

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

(b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

(d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

(e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

(g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

(i) True; see Figure 9 and the accompanying discussion.

(j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0^\circ \le \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = \mathbf{i} - 3\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (\mathbf{i} - 3\,\mathbf{k}) + t\,(2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}) = (1 + 2t)\,\mathbf{i} - 4t\,\mathbf{j} + (-3 + 5t)\,\mathbf{k}$ and parametric equations are $x = 1 + 2t$, $y = -4t$, $z = -3 + 5t$.

3. For this line, we have $\mathbf{r}_0 = -2\,\mathbf{i} + 4\,\mathbf{j} + 10\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + \mathbf{j} - 8\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (-2\,\mathbf{i} + 4\,\mathbf{j} + 10\,\mathbf{k}) + t(3\,\mathbf{i} + \mathbf{j} - 8\,\mathbf{k}) = (-2 + 3t)\,\mathbf{i} + (4 + t)\,\mathbf{j} + (10 - 8t)\,\mathbf{k}$ and parametric equations are $x = -2 + 3t$, $y = 4 + t$, $z = 10 - 8t$.

4. This line has the same direction as the given line, $\mathbf{v} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Here $\mathbf{r}_0 = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$, so a vector equation is $\mathbf{r} = (0\,\mathbf{i} + 0\mathbf{j} + 0\,\mathbf{k}) + t\,(2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}) = 2t\,\mathbf{i} - t\,\mathbf{j} + 3t\,\mathbf{k}$ and parametric equations are $x = 2t, y = -t, z = 3t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1 \rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\,\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\,\mathbf{k}) + t(\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) = (1 + t)\,\mathbf{i} + 3t\,\mathbf{j} + (6 + t)\,\mathbf{k}$, and parametric equations are $x = 1 + t, y = 3t, z = 6 + t$.

6. The vector $\mathbf{v} = \langle 1 - 0, 2 - 0, 3 - 0 \rangle = \langle 1, 2, 3 \rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 0 + 1 \cdot t = t$, $y = 0 + 2 \cdot t = 2t$, $z = 0 + 3 \cdot t = 3t$, while symmetric equations are $x = \dfrac{y}{2} = \dfrac{z}{3}$.

7. $\mathbf{v} = \langle 3 - 3, 2 - 1, -6 - (-1) \rangle = \langle 0, 1, -5 \rangle$, and letting $P_0 = (3, 1, -1)$, parametric equations are $x = 3$, $y = 1 + t$, $z = -1 - 5t$, while symmetric equations are $x = 3$, $y - 1 = (z + 1)/(-5)$. Notice here that the direction number $a = 0$, so rather than writing $(x - 3)/0$ in the symmetric equation we must write the equation $x = 3$ separately.

8. $\mathbf{v} = \langle 4-(-1), -3-0, 3-5\rangle = \langle 5, -3, -2\rangle$, and letting $P_0 = (-1, 0, 5)$, parametric equations are $x = -1 + 5t$, $y = -3t$, $z = 5 - 2t$, while symmetric equations are $\dfrac{x+1}{5} = \dfrac{y}{-3} = \dfrac{z-5}{-2}$.

9. $\mathbf{v} = \left\langle 2-0, 1-\frac{1}{2}, -3-1\right\rangle = \left\langle 2, \frac{1}{2}, -4\right\rangle$, and letting $P_0 = (2, 1, -3)$, parametric equations are $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$, $z = -3 - 4t$, while symmetric equations are $\dfrac{x-2}{2} = \dfrac{y-1}{1/2} = \dfrac{z+3}{-4}$ or $\dfrac{x-2}{2} = 2y - 2 = \dfrac{z+3}{-4}$.

10. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1\rangle \times \langle 1, 0, 1\rangle = \langle 1, 0, -1\rangle$ is the direction of this line. Taking the point $(0, 1, 0)$ as P_0, parametric equations are $x = t$, $y = 1$, $z = -t$, and symmetric equations are $x = -z$, $y = 1$.

11. Direction vectors of the lines are $\mathbf{v}_1 = \langle 6, 9, 12\rangle$ and $\mathbf{v}_2 = \langle 4, 6, 8\rangle$, and since $\mathbf{v}_1 = \frac{3}{2}\mathbf{v}_2$, the direction vectors and thus the lines are parallel.

12. Direction vectors of the lines are $\mathbf{v}_1 = \langle 1, -2, 5\rangle$ and $\mathbf{v}_2 = \langle 3, 4, 1\rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 - 8 + 5 = 0$, the direction vectors and thus the lines are perpendicular.

13. (a) A direction vector of the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$ is $\mathbf{v} = \langle 2, 3, -7\rangle$ and the desired parallel line must also have $\mathbf{v}$ as a direction vector. Here $P_0 = (0, 2, -1)$, so symmetric equations for the line are $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{z+1}{-7}$.

(b) The line intersects the xy-plane when $z = 0$, so we need $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{1}{-7}$ or $x = -\frac{2}{7}$, $y = \frac{11}{7}$. Thus the point of intersection with the xy-plane is $\left(-\frac{2}{7}, \frac{11}{7}, 0\right)$. Similarly for the yz-plane, we need $x = 0 \Leftrightarrow 0 = \dfrac{y-2}{3} = \dfrac{z+1}{-7} \Leftrightarrow y = 2$, $z = -1$. Thus the line intersects the yz-plane at $(0, 2, -1)$. For the xz-plane, we need $y = 0 \Leftrightarrow \dfrac{x}{2} = -\dfrac{2}{3} = \dfrac{z+1}{-7} \Leftrightarrow x = -\frac{4}{3}$, $z = \frac{11}{3}$. So the line intersects the xz-plane at $\left(-\frac{4}{3}, 0, \frac{11}{3}\right)$.

14. (a) A vector normal to the plane $2x - y + z = 1$ is $\mathbf{n} = \langle 2, -1, 1\rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 5 + 2t$, $y = 1 - t$, $z = t$.

(b) On the xy-plane, $z = 0$. So $z = t = 0$ in the parametric equations of the line, and therefore $x = 5$ and $y = 1$, giving the point of intersection $(5, 1, 0)$. For the yz-plane, $x = 0$ which implies $t = -\frac{5}{2}$, so $y = \frac{7}{2}$ and $z = -\frac{5}{2}$ and the point is $\left(0, \frac{7}{2}, -\frac{5}{2}\right)$. For the xz-plane, $y = 0$ which implies $t = 1$, so $x = 7$ and $z = 1$ and the point of intersection is $(7, 0, 1)$.

15. The lines aren't parallel since the direction vectors $\langle 2, 4, -3\rangle$ and $\langle 1, 3, 2\rangle$ aren't parallel, so we check to see if the lines intersect. The parametric equations of the lines are L_1: $x = 4 + 2t$, $y = -5 + 4t$, $z = 1 - 3t$ and L_2: $x = 2 + s$, $y = -1 + 3s$, $z = 2s$. For the lines to intersect we must be able to find one value of t and one value of s satisfying the following three equations: $4 + 2t = 2 + s$, $-5 + 4t = -1 + 3s$, $1 - 3t = 2s$. Solving the first two equations we get $t = -5$, $s = -8$ and checking, we see that these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

16. Since the direction vectors $\langle 2, 1, 4\rangle$ and $\langle 1, 2, 3\rangle$ aren't parallel, the lines aren't parallel. Here the parametric equations are L_1: $x = 1 + 2t$, $y = t$, $z = 1 + 4t$; L_2: $x = s$, $y = -2 + 2s$, $z = -2 + 3s$. Thus, for the lines to intersect, the three equations $1 + 2t = s$, $t = -2 + 2s$ and $1 + 4t = -2 + 3s$ must be satisfied simultaneously. Solving the first two equations gives $t = 0$, $s = 1$ and, checking, we see these values do satisfy the third equation, so the lines intersect when $t = 0$ and $s = 1$, that is, at the point $(1, 0, 1)$.

17. Since the direction vectors are $\mathbf{v}_1 = \langle -6, 9, -3 \rangle$ and $\mathbf{v}_2 = \langle 2, -3, 1 \rangle$, we have $\mathbf{v}_1 = -3\mathbf{v}_2$ so the lines are parallel.

18. Since the direction vectors are $\langle 1, -1, 3 \rangle$ and $\langle -1, 2, 1 \rangle$, the lines aren't parallel. For the lines to intersect, the three equations $1 + t = 2 - s$, $2 - t = 1 + 2s$, $3t = 4 + s$ must be satisfied simultaneously. Solving the first two equations gives $t = 1$, $s = 0$ and, checking, we see these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

19. Since the plane is perpendicular to the vector $\langle -2, 1, 5 \rangle$, we can take $\langle -2, 1, 5 \rangle$ as a normal vector to the plane. $(6, 3, 2)$ is a point on the plane, so setting $a = -2$, $b = 1$, $c = 5$ and $x_0 = 6$, $y_0 = 3$, $z_0 = 2$ in Equation 6 gives $-2(x - 6) + 1(y - 3) + 5(z - 2) = 0$ or $-2x + y + 5z = 1$ to be an equation of the plane.

20. $\mathbf{j} + 2\,\mathbf{k} = \langle 0, 1, 2 \rangle$ is a normal vector to the plane and $(4, 0, -3)$ is a point on the plane, so setting $a = 0$, $b = 1$, $c = 2$, $x_0 = 4$, $y_0 = 0$, $z_0 = -3$ in Equation 6 gives $0(x - 4) + 1(y - 0) + 2\,[z - (-3)] = 0$ or $y + 2z = -6$ to be an equation of the plane.

21. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 2, -1, 3 \rangle$, and an equation of the plane is $2(x - 0) - 1(y - 0) + 3(z - 0) = 0$ or $2x - y + 3z = 0$.

22. First, a normal vector for the plane $2x + 4y + 8z = 17$ is $\mathbf{n} = \langle 2, 4, 8 \rangle$. A direction vector for the line is $\mathbf{v} = \langle 2, 1, -1 \rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$ we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know the point $(3, 0, 8)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 2, 4, 8 \rangle$, so an equation of the plane is $2(x - 3) + 4(y - 0) + 8(z - 8) = 0$ or $x + 2y + 4z = 35$.

23. Here the vectors $\mathbf{a} = \langle 1 - 0, 0 - 1, 1 - 1 \rangle = \langle 1, -1, 0 \rangle$ and $\mathbf{b} = \langle 1 - 0, 1 - 1, 0 - 1 \rangle = \langle 1, 0, -1 \rangle$ lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 - 0, 0 + 1, 0 + 1 \rangle = \langle 1, 1, 1 \rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 2$.

24. Here the vectors $\mathbf{a} = \langle 2, -4, 6 \rangle$ and $\mathbf{b} = \langle 5, 1, 3 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -12 - 6, 30 - 6, 2 + 20 \rangle = \langle -18, 24, 22 \rangle$ is a normal vector to the plane and an equation of the plane is $-18(x - 0) + 24(y - 0) + 22(z - 0) = 0$ or $-18x + 24y + 22z = 0$.

25. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4 \rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6 - 4, 0 - 3, -2 - 7 \rangle = \langle 2, -3, -9 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -45 + 12, 8 - 18, 6 - 10 \rangle = \langle -33, -10, -4 \rangle$. Thus, an equation of the plane is $-33(x - 6) - 10(y - 0) - 4\,[z - (-2)] = 0$ or $33x + 10y + 4z = 190$.

26. Since the line $x = 2y = 3z$, or $x = \dfrac{y}{1/2} = \dfrac{z}{1/3}$, lies in the plane, its direction vector $\mathbf{a} = \left\langle 1, \frac{1}{2}, \frac{1}{3} \right\rangle$ is parallel to the plane. The point $(0, 0, 0)$ is on the line (put $t = 0$), and we can verify that the given point $(1, -1, 1)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 1, -1, 1 \rangle$, is therefore parallel to the plane, but not parallel to $\langle 1, 2, 3 \rangle$. Then $\mathbf{a} \times \mathbf{b} = \left\langle \frac{1}{2} + \frac{1}{3}, \frac{1}{3} - 1, -1 - \frac{1}{2} \right\rangle = \left\langle \frac{5}{6}, -\frac{2}{3}, -\frac{3}{2} \right\rangle$ is a normal vector to the plane, and an equation of the plane is $\frac{5}{6}(x - 0) - \frac{2}{3}(y - 0) - \frac{3}{2}(z - 0) = 0$ or $5x - 4y - 9z = 0$.

27. A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1 \rangle \times \langle 2, -1, 3 \rangle = \langle 2, -5, -3 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $x = 0$, the equations of the planes reduce to $y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $\left(0, \frac{7}{2}, \frac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle$. Then a normal vector to the plane is $\mathbf{n} = \langle 2, -5, -3 \rangle \times \left\langle -1, -\frac{3}{2}, -\frac{1}{2} \right\rangle = \langle -2, 4, -8 \rangle$ and an equation of the plane is $-2(x+1) + 4(y-2) - 8(z-1) = 0$ or $x - 2y + 4z = -1$.

28. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = \langle 3, 3, 3 \rangle$, or we can use $\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

29. Substituting the parametric equations of the line into the equation of the plane gives $2x + y - z + 5 = 2(1 + 2t) + (-1) - t + 5 = 0 \quad \Rightarrow \quad 3t + 6 = 0 \quad \Rightarrow \quad t = -2$. Therefore, the point of intersection is given by $x = 1 + 2(-2) = -3$, $y = -1$ and $z = -2$, that is, the point $(-3, -1, -2)$.

30. Substitution into the equation of the plane of the parametric expressions for x, y and z gives $z = 1 - 2x + y \quad \Rightarrow \quad (1 + t) = 1 - 2(1 - t) + t \quad \Rightarrow \quad -2 + 2t = 0 \quad \Rightarrow \quad t = 1$. Thus, $x = 1 - 1$, $y = 1$ and $z = 1 + 1$ and the point of intersection is $(0, 1, 2)$.

31. The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, 0, 1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 1 \rangle$. Thus the normal vectors (and consequently the planes) aren't parallel. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \neq 0$ so the planes aren't perpendicular. Letting θ be the angle between the two planes, we have $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{1}{\sqrt{2}\,\sqrt{2}} = \dfrac{1}{2}$ and $\theta = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$.

32. Here the normals are $\mathbf{n}_1 = \langle -8, -6, 2 \rangle$ and $\mathbf{n}_2 = \langle 4, 3, -1 \rangle$. Since $\mathbf{n}_1 = -2\mathbf{n}_2$, the normals (and thus the planes) are parallel.

33. The normals are $\mathbf{n}_1 = \langle 1, 4, -3 \rangle$ and $\mathbf{n}_2 = \langle -3, 6, 7 \rangle$, so the normals (and thus the planes) aren't parallel. But $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals (and thus the planes) are perpendicular.

34. The normals are $\mathbf{n}_1 = \langle 2, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle 6, -3, 2 \rangle$ so the planes aren't parallel. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 12 - 6 - 2 = 4 \neq 0$, so the planes aren't perpendicular. Then $\cos\theta = \dfrac{4}{\sqrt{9}\,\sqrt{49}} = \dfrac{4}{21}$ and $\theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$.

35. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will only work if the line of intersection crosses the xy-plane; otherwise, try setting x or y equal to 0.) Then the equations of the planes reduce to $x + y = 2$ and $3x - 4y = 6$. Solving these two equations gives $x = 2$, $y = 0$. So a point on the line of intersection is $(2, 0, 0)$. The direction of the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5 - 4, -3 - 5, -4 - 3 \rangle = \langle 1, -8, -7 \rangle$, and symmetric equations for the line are $x - 2 = \dfrac{y}{-8} = \dfrac{z}{-7}$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{3 - 4 - 5}{\sqrt{3}\,\sqrt{50}} = -\dfrac{\sqrt{6}}{5}$. Therefore $\theta = \cos^{-1}\left(-\frac{\sqrt{6}}{5}\right) \approx 119^\circ$ (or 61°).

36. The plane will contain all perpendicular bisectors of the line segment joining the two points. Thus, a point in the plane is $P_0 = (-1, -1, 2)$, the midpoint of the line segment joining the two given points, and a normal to the plane is $\mathbf{n} = \langle 6, -6, 2\rangle$, the vector connecting the two points. So an equation of the plane is $6(x+1) - 6(y+1) + 2(z-2) = 0$ or $3x - 3y + z = 2$.

37. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0\rangle$ and $\mathbf{b} = \langle -a, 0, c\rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab\rangle = \langle bc, ac, ab\rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. This is a good equation to remember!

38. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2)$.

(b) The direction vectors of the lines are $\langle 1, -1, 2\rangle$ and $\langle -1, 1, 0\rangle$, so a normal vector for the plane is $\langle -1, 1, 0\rangle \times \langle 1, -1, 2\rangle = \langle 2, 2, 0\rangle$ and it contains the point $(2, 0, 2)$. Then the equation of the plane is $2(x-2) + 2(y-0) + 0(z-2) = 0 \quad\Leftrightarrow\quad x + y = 2$.

39. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1\rangle$, and a direction vector for the given line, $\langle 1, -1, 2\rangle$. So a direction vector for the required line is $\langle 1, 1, 1\rangle \times \langle 1, -1, 2\rangle = \langle 3, -1, -2\rangle$. Thus L is given by $\langle x, y, z\rangle = \langle 0, 1, 2\rangle + t\langle 3, -1, -2\rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.

40. Let L be the given line. Then $(1, 1, 0)$ is the point on L corresponding to $t = 0$. L is in the direction of $\mathbf{a} = \langle 1, -1, 2\rangle$ and $\mathbf{b} = \langle -1, 0, 2\rangle$ is the vector joining $(1, 1, 0)$ and $(0, 1, 2)$. Then

$$\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \langle -1, 0, 2\rangle - \frac{\langle 1, -1, 2\rangle \cdot \langle -1, 0, 2\rangle}{1^2 + (-1)^2 + 2^2} \langle 1, -1, 2\rangle = \langle -1, 0, 2\rangle - \tfrac{1}{2}\langle 1, -1, 2\rangle = \left\langle -\tfrac{3}{2}, \tfrac{1}{2}, 1\right\rangle$$ is a

direction vector for the required line. Thus $2\left\langle -\frac{3}{2}, \frac{1}{2}, 1\right\rangle = \langle -3, 1, 2\rangle$ is also a direction vector, and the line has parametric equations $x = -3t$, $y = 1 + t$, $z = 2 + 2t$. (Notice that this is the same line as in Exercise 39.)

41. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 4, -2, 6\rangle$, $\mathbf{n}_2 = \langle 4, -2, -2\rangle$, $\mathbf{n}_3 = \langle -6, 3, -9\rangle$, $\mathbf{n}_4 = \langle 2, -1, -1\rangle$. Now $\mathbf{n}_1 = -\frac{2}{3}\mathbf{n}_3$, so $\mathbf{n}_1$ and $\mathbf{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\mathbf{n}_2 = 2\mathbf{n}_4$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. $\left(0, 0, \frac{1}{2}\right)$ lies on P_1, but not on P_3, so they are not the same plane, but both P_2 and P_4 contain the point $(0, 0, -3)$, so these two planes are identical.

42. Let L_i have direction vector $\mathbf{v}_i$. Then $\mathbf{v}_1 = \langle 1, 1, -5\rangle$, $\mathbf{v}_2 = \langle 1, 1, -1\rangle$, $\mathbf{v}_3 = \langle 1, 1, -1\rangle$, $\mathbf{v}_4 = \langle 2, 2, -10\rangle$. $\mathbf{v}_2$ and $\mathbf{v}_3$ are equal so they're parallel. $\mathbf{v}_4 = 2\mathbf{v}_1$, so L_4 and L_1 are parallel. L_3 contains the point $(1, 4, 1)$, but this point does not lie on L_2, so they're not equal. $(2, 1, -3)$ lies on L_4, and on L_1, with $t = 1$. So L_1 and L_4 are identical.

43. Let $Q = (2, 2, 0)$ and $R = (3, -1, 5)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (1, 2, 3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1, -3, 5\rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle -1, 0, 3\rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -3, 5\rangle \times \langle -1, 0, 3\rangle|}{|\langle 1, -3, 5\rangle|} = \frac{|\langle -9, -8, -3\rangle|}{|\langle 1, -3, 5\rangle|} = \frac{\sqrt{9^2 + 8^2 + 3^2}}{\sqrt{1^2 + 3^2 + 5^2}} = \frac{\sqrt{154}}{\sqrt{35}} = \sqrt{\frac{22}{5}}.$$

44. Let $Q = (5, 0, 1)$ and $R = (4, 3, 3)$, points on the line corresponding to $t = 0$ and $t = 1$. Let $P = (1, 0, -1)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle -1, 3, 2\rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle -4, 0, -2\rangle$. The distance is

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle -1, 3, 2\rangle \times \langle -4, 0, -2\rangle|}{|\langle -1, 3, 2\rangle|} = \frac{|\langle -6, -10, 12\rangle|}{|\langle -1, 3, 2\rangle|} = \frac{2\sqrt{3^2 + 5^2 + 6^2}}{\sqrt{1^2 + 3^2 + 2^2}} = \frac{2\sqrt{70}}{\sqrt{14}} = 2\sqrt{5}.$$