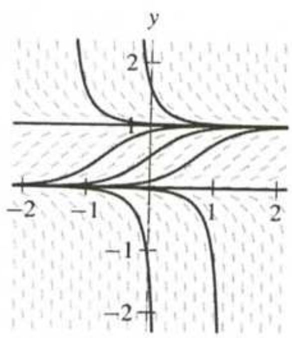
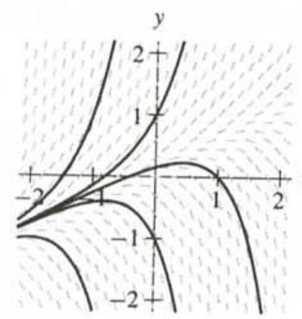
7. **(a)**



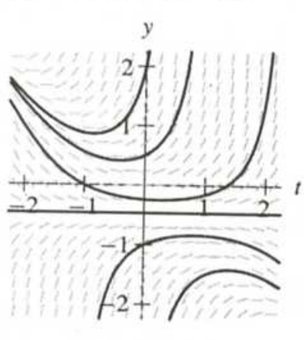
(b) The solution with $y(0) = 1/2$ approaches the equilibrium value $y = 1$ from below as t increases. It decreases toward $y = 0$ as t decreases.

8. **(a)**



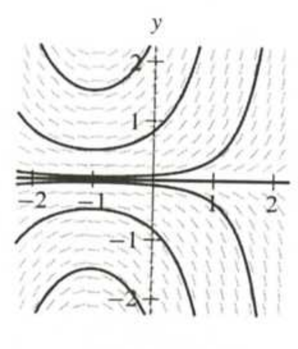
(b) The solution $y(t)$ with $y(0) = 1/2$ increases with $y(t) \to \infty$ as t increases. As t decreases, $y(t) \to -\infty$.

9. **(a)**



(b) The solution $y(t)$ with $y(0) = 1/2$ has $y(t) \to \infty$ both as t increases and as t decreases.

10. **(a)**

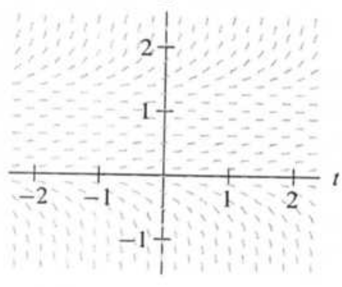


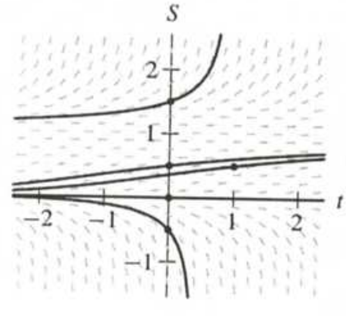
(b) The solution $y(t)$ with $y(0) = 1/2$ has $y(t) \to \infty$ both as t increases and as t decreases.

11. **(a)** On the line $y = 3$ in the ty-plane, all of the slope marks have slope -1.

(b) Because f is continuous, if y is close to 3, then $f(t, y) < 0$. So any solution close to $y = 3$ must be decreasing. Therefore, solutions $y(t)$ that satisfy $y(0) < 3$ can never be larger than 3 for $t > 0$, and consequently $y(t) < 3$ for all t.

12.





13. The slope field in t

14. Because f depends
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corresponding lines
slope field have neg

15. (a) Note that the sl
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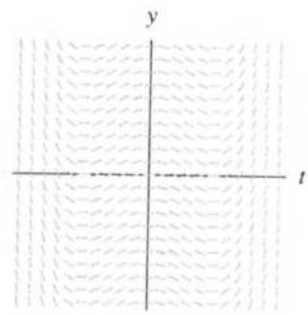
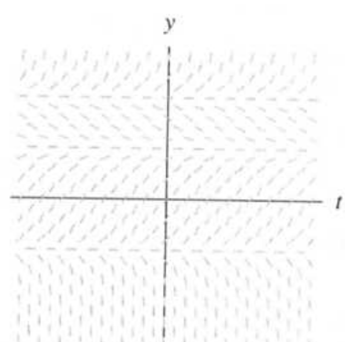
field must corre
(b) This slope field
correspond to e
solution corresp
(c) This slope field
to either equatio
correspond to eq
(d) This slope field
When $t = 0$ the

16. (a) Because the slop
entire slope field

13. The slope field in the ty-plane is constant along vertical lines.



14. Because f depends only on y (the equation is autonomous), the slope field is constant along horizontal lines in the ty-plane. The roots of f correspond to equilibrium solutions. If $f(y) > 0$, the corresponding lines in the slope field have positive slope. If $f(y) < 0$, the corresponding lines in the slope field have negative slope.



15. **(a)** Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation must depend only on t. The two such choices are equations (i) and (iv). Because the slope is negative for $t > 1$ and positive for $t < 1$, this slope field must correspond to equation (iv).

(b) This slope field has an equilibrium solution corresponding to the line $y = 1$, so it must be correspond to either equation (ii), (v), or (viii). Both (ii) and (viii) have another equilibrium solution corresponding to $y = -1$, so this slope field must correspond to equation (v).

(c) This slope field has equilibrium solutions corresponding to $y = \pm 1$. Hence it corresponds to either equation (ii) or (viii). Since dy/dt is negative along $y = 0$, this slope field must correspond to equation (viii).

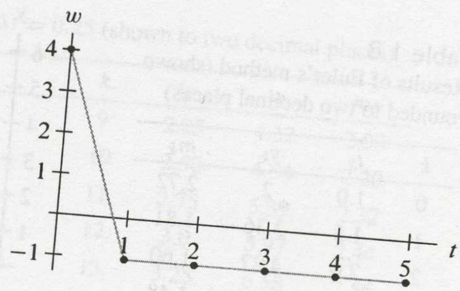
(d) This slope field depends both on y and on t, so it can only correspond to equation (iii) or (vi). When $t = 0$ the slopes are positive, so the slope field must correspond to equation (iii).

16. **(a)** Because the slope field is constant on vertical lines, the given information is enough to draw the entire slope field.

5.

Table 1.5
Results of Euler's method

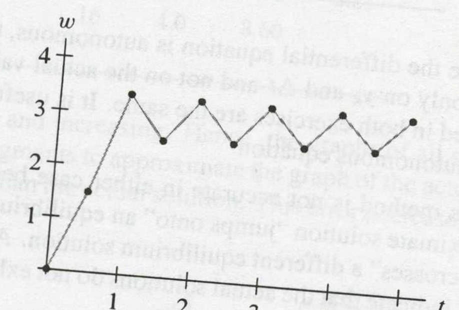
k	t_k	w_k	m_k
0	0	4	−5
1	1	−1	0
2	2	−1	0
3	3	−1	0
4	4	−1	0
5	5	−1	



6.

Table 1.6
Results of Euler's method (shown rounded to two decimal places)

k	t_k	w_k	m_k
0	0	0	3
1	0.5	1.5	3.75
2	1.0	3.38	−1.64
3	1.5	2.55	1.58
4	2.0	3.35	−1.50
5	2.5	2.59	1.46
6	3.0	3.32	−1.40
7	3.5	2.62	1.36
8	4.0	3.31	−1.31
9	4.5	2.65	1.28
10	5.0	3.29	



7.

Table 1.7
Results of Euler's method (shown rounded to two decimal places)

k	t_k	y_k	m_k
0	0	2	2.72
1	0.5	3.36	1.81
2	1.0	4.27	1.60
3	1.5	5.06	1.48
4	2.0	5.81	

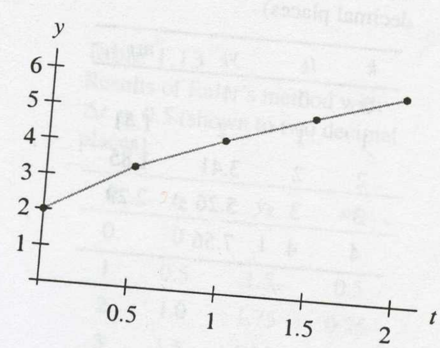
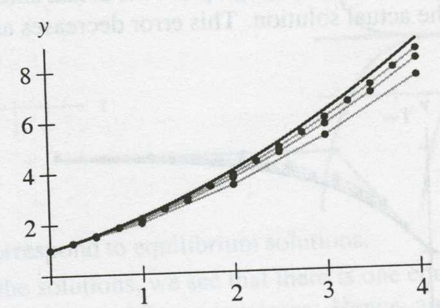


Table 1.11
Results of Euler's method with $\Delta t = 0.25$ (shown to two decimal places)

k	t_k	y_k	m_k	k	t_k	y_k	m_k
0	0	1	1	9	2.25	4.32	2.08
1	0.25	1.25	1.12	10	2.50	4.84	2.20
2	0.50	1.53	1.24	11	2.75	5.39	2.32
3	0.75	1.84	1.36	12	3.0	5.97	2.44
4	1.0	2.18	1.48	13	3.25	6.58	2.56
5	1.25	2.55	1.60	14	3.50	7.23	2.69
6	1.50	2.94	1.72	15	3.75	7.90	2.81
7	1.75	3.37	1.84	16	4.0	8.60	
8	2.0	3.83	1.96				

The slopes in the slope field are positive and increasing. Hence, the graphs of all solutions are concave up. Since Euler's method uses line segments to approximate the graph of the actual solution, the approximate solutions will always be less than the actual solution. This error decreases as the step size decreases.



13.

Table 1.12
Results of Euler's method with $\Delta t = 1.0$ (shown to two decimal places)

k	t_k	y_k	m_k
0	0	1	1
1	1	2	0
2	2	2	0
3	3	2	0
4	4	2	

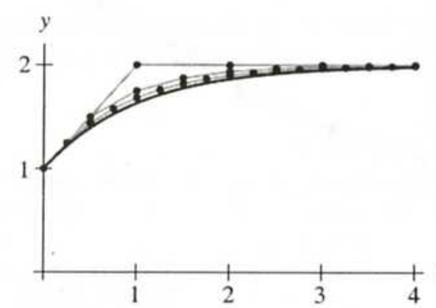
Table 1.13
Results of Euler's method with $\Delta t = 0.5$ (shown to two decimal places)

k	t_k	y_k	m_k
0	0	1	1
1	0.5	1.5	0.5
2	1.0	1.75	0.26
3	1.5	1.88	0.12
4	2.0	1.94	0.06
5	2.5	1.97	0.02
6	3.0	1.98	0.02
7	3.5	1.99	0.02
8	4.0	2.0	

Table 1.14
Results of Euler's method with $\Delta t = 0.25$ (shown to two decimal places)

k	t_k	y_k	m_k	k	t_k	y_k	m_k
0	0	1	1	9	2.25	1.92	0.08
1	0.25	1.25	0.76	10	2.50	1.94	0.06
2	0.50	1.44	0.56	11	2.75	1.96	0.04
3	0.75	1.58	0.40	12	3.0	1.97	0.03
4	1.0	1.68	0.32	13	3.25	1.98	0.02
5	1.25	1.76	0.24	14	3.50	1.98	0.02
6	1.50	1.82	0.18	15	3.75	1.99	0.01
7	1.75	1.87	0.13	16	4.0	1.99	
8	2.0	1.90	0.10				

From the differential equation, we see that dy/dt is positive and decreasing as long as $y(0) = 1$ and $y(t) < 2$ for $t > 0$. Therefore, $y(t)$ is increasing, and its graph is concave down. Since Euler's method uses line segments to approximate the graph of the actual solution, the approximate solutions will always be greater than the actual solution. This error decreases as the step size decreases.



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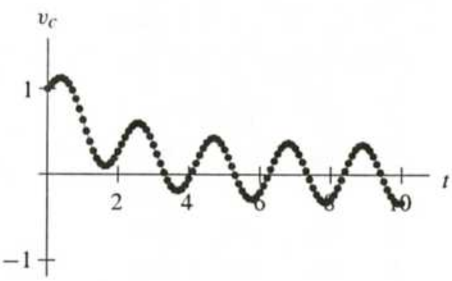
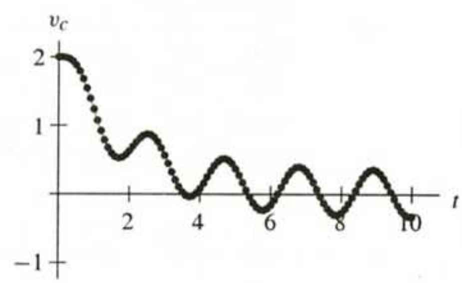


Graph of approximate solution obtained using Euler's method with $\Delta t = 0.1$.

15.



Graph of approximate solution obtained using Euler's method with $\Delta t = 0.1$.

other root can be found using Euler's method with a negative Δt. Since solutions tend toward the root, we can use a fairly large step size to obtain the required accuracy, but since this is only an approximate solution, it is best to double check the location of the root by substituting the value into $p(y)$. The approximate roots are $y \approx 2.115$, $y \approx -1.861$ and $y \approx -0.254$.

EXERCISES FOR SECTION 1.5

1. Since the constant function $y_1(t) = 3$ for all t is a solution, then the graph of any other solution $y(t)$ with $y(0) < 3$ cannot cross the line $y = 3$ by the Uniqueness Theorem. So $y(t) < 3$ for all t in the domain of $y(t)$.

2. Since $y(0) = 1$ is between the equilibrium solutions $y_2(t) = 0$ and $y_3(t) = 2$, we must have $0 < y(t) < 2$ for all t because the Uniqueness Theorem implies that graphs of solutions cannot cross (or even touch in this case).

3. Because $y_2(0) < y(0) < y_1(0)$, we know that

$$-t^2 = y_2(t) < y(t) < y_1(t) = t + 2$$

for all t. This restricts how large positive or negative $y(t)$ can be for a given value of t (that is, between $-t^2$ and $t + 2$). As $t \to -\infty$, $y(t) \to -\infty$ between $-t^2$ and $t + 2$ ($y(t) \to -\infty$ as $t \to -\infty$ at least linearly, but no faster than quadratically).

4. Because $y_1(0) < y(0) < y_2(0)$, the solution $y(t)$ must satisfy $y_1(t) < y(t) < y_2(t)$ for all t by the Uniqueness Theorem. Hence $-1 < y(t) < 1 + t^2$ for all t.

5. The Existence Theorem implies that a solution with this initial condition exists, at least for a small t-interval about $t = 0$. This differential equation has equilibrium solutions $y_1(t) = 0$, $y_2(t) = 2$, and $y_3(t) = 3$. Since $y(0) = 4$, the Uniqueness Theorem implies that $y(t) > 3$ for all t in the domain of $y(t)$. Also, $dy/dt > 0$ for all $y > 3$, so the solution $y(t)$ is increasing for all t in its domain.

6. Note that $dy/dt = 0$ if $y = 3$. Hence, $y_1(t) = 3$ for all t is an equilibrium solution. By the Uniqueness Theorem, this is the only solution that is 3 at $t = 0$. Therefore, $y(t) = 3$ for all t.

7. Because $0 < y(0) < 2$ and $y_1(t) = 0$ and $y_2(t) = 2$ are equilibrium solutions of the differential equation, we know that $0 < y(t) < 2$ for all t by the Uniqueness Theorem. Also, $dy/dt > 0$ for $0 < y < 2$, so dy/dt is always positive for this solution. Hence, $y(t) \to 2$ as $t \to \infty$, and $y(t) \to 0$ as $t \to -\infty$.

8. Note that $y(0) < 0$. Since $y_1(t) = 0$ is an equilibrium solution, the Uniqueness Theorem implies that $y(t) < 0$ for all t. Also, $dy/dt < 0$ if $y < 0$, so $y(t)$ is decreasing for all t, and $y(t) \to -\infty$ as t increases. As $t \to -\infty$, $y(t) \to 0$.

9. **(a)** To check that $y_1(t) = t^2$ is a solution, we compute

$$\frac{dy_1}{dt} = 2t$$

where c is a constant. Substituting the initial condition $y(0) = 0$ and solving for c, we have

$$0 = -1 \pm \sqrt{1 + \ln(4c)},$$

and thus $c = 1/4$. The desired solution is therefore

$$y(t) = -1 + \sqrt{1 + \ln((1 - t/2)^2)}$$

(b) The solution is defined only when $1 + \ln((1 - t/2)^2) \geq 0$, that is, when $|t - 2| \geq 2/\sqrt{e}$. Therefore, the domain of the solution is

$$t \leq 2(1 - 1/\sqrt{e}).$$

(c) As $t \to 2(1 - 1/\sqrt{e})$, then $1 + \ln((1 - t/2)^2) \to 0$. Thus

$$\lim_{t \to 2(1-1/\sqrt{e})} y(t) = -1.$$

Note that the differential equation is not defined at $y = -1$. Also, note that

$$\lim_{t \to -\infty} y(t) = \infty.$$

16. **(a)** The equation is separable. We separate, integrate

$$\int (y + 2)^2 \, dy = \int dt,$$

and solve for y to obtain the general solution

$$y(t) = (3t + c)^{1/3} - 2,$$

where c is any constant. To obtain the desired solution, we use the initial condition $y(0) = 1$ and solve

$$1 = (3 \cdot 0 + c)^{1/3} - 2$$

for c to obtain $c = 27$. So the solution to the given initial-value problem is

$$y(t) = (3t + 27)^{1/3} - 2.$$

(b) This function is defined for all t. However, $y(-9) = -2$, and the differential equation is not defined at $y = -2$. Strictly speaking, the solution exists only for $t > -9$.

(c) As $t \to \infty$, $y(t) \to \infty$. As $t \to -9^+$, $y(t) \to -2$.

17. **(a)** The equation is separable. Separating variables we obtain

$$\int (y - 2) \, dy = \int t \, dt.$$

Solving for y with help from the quadratic formula yields the general solution

$$y(t) = 2 \pm \sqrt{t^2 + c}.$$

To find c, we let $t = -1$ and $y = 0$, and we obtain $c = 3$. The desired solution is therefore $y(t) = 2 - \sqrt{t^2 + 3}$

(b) Since $t^2 + 2$ is always positive and $y(t) < 2$ for all t, the solution $y(t)$ is defined for all real numbers.

(c) As $t \to \pm\infty$, $t^2 + 3 \to \infty$. Therefore,

$$\lim_{t \to \pm\infty} y(t) = -\infty.$$

18. **(a)** The partial derivative with respect to v of dv/dt does not exist at $v = 0$. Hence the Uniqueness Theorem tells us nothing about the uniqueness of solutions that involve $v = 0$. In fact, if we use the techniques described in the section related to the uniqueness of solutions for $dy/dt = 3y^{2/3}$, we can find infinitely many solutions with this initial condition.

(b) Since it does not make sense to talk about rain drops with negative volume, we always have $v \geq 0$. Once $v > 0$, the evolution of the drop is completely determined by the differential equation.

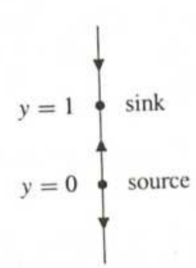
What is the physical significance of a drop with $v = 0$? It is tempting to interpret the fact that solutions can have $v = 0$ for an arbitrary amount of time before beginning to grow as a statement that the rain drops can spontaneously begin to grow at any time. Since the model gives no information about when a solution with $v = 0$ starts to grow, it is not very useful for the understanding the initial formation of rain drops. The safest assertion is to say is the model breaks down if $v = 0$.

EXERCISES FOR SECTION 1.6

1. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For

$$f(y) = 3y(1 - y),$$

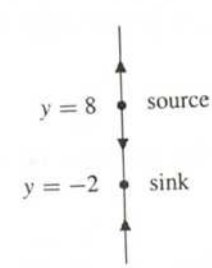
the equilibrium points are $y = 0$ and $y = 1$. Since $f(y)$ is negative for $y < 0$, positive for $0 < y < 1$, and negative for $y > 1$, the equilibrium point $y = 0$ is a source and the equilibrium point $y = 1$ is a sink.



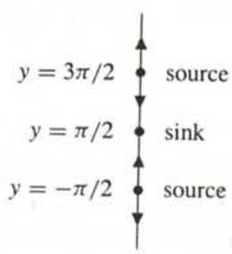
2. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For

$$f(y) = y^2 - 6y - 16 = (y + 2)(y - 8),$$

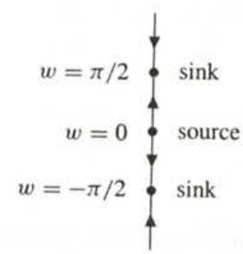
the equilibrium points are $y = -2$ and $y = 8$. Since $f(y)$ is positive for $y < -2$, negative for $-2 < y < 8$, and positive for $y > 8$, the equilibrium point $y = -2$ is a sink and the equilibrium point $y = 8$ is a source.



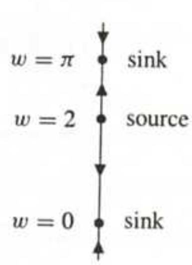
3. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = \cos y$, the equilibrium points are $y = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \dots$. Since $\cos y > 0$ for $-\pi/2 < y < \pi/2$ and $\cos y < 0$ for $\pi/2 < y < 3\pi/2$, we see that the equilibrium point at $y = \pi/2$ is a sink. Since the sign of $\cos y$ alternates between positive and negative in a period fashion, we see that the equilibrium points at $y = \pi/2 + 2n\pi$ are sinks and the equilibrium points at $y = 3\pi/2 + 2n\pi$ are sources.



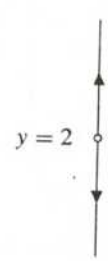
4. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = w \cos w$, the equilibrium points are $w = 0$ and $w = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \dots$. The sign of $w \cos w$ alternates positive and negative at successive zeros. It is negative for $-\pi/2 < w < 0$ and positive for $0 < w < \pi/2$. Therefore, $w = 0$ is a source, and the equilibrium points alternate back and forth between sources and sinks.



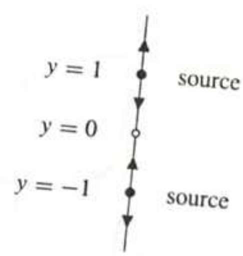
5. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = (w-2)\sin w$, the equilibrium points are $w = 2$ and $w = n\pi$, where $n = 0, \pm 1, \pm 2, \dots$. The sign of $(w-2)\sin w$ alternates between positive and negative at successive zeros. It is positive for $-\pi < w < 0$ and negative for $0 < w < 2$. Therefore, $w = 0$ is a sink, and the equilibrium points alternate back and forth between sources and sinks.



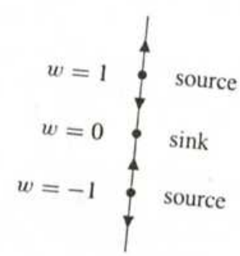
6. This equation has no equilibrium points, but the equation is not defined at $y = 2$. For $y > 2$, $dy/dt > 0$, so solutions increase. If $y < 2$, $dy/dt < 0$, so solutions decrease. The solutions approach the point $y = 2$ as time decreases and actually arrive there in finite time.



11. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = y \ln |y|$, there are equilibrium points at $y = \pm 1$. In addition, although the function $f(y)$ is technically undefined at $y = 0$, the limit of $f(y)$ as $y \to 0$ is 0. Thus we can treat $y = 0$ as another equilibrium point. Since $f(y) < 0$ for $y < -1$ and $0 < y < 1$, and $f(y) > 0$ for $y > 1$ and $-1 < y < 0$, $y = -1$ is a source, $y = 0$ is a sink, and $y = 1$ is a source.



12. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = (w^2 - 1) \arctan w$, there are equilibrium points at $w = \pm 1$ and $w = 0$. Since $f(w) > 0$ for $w > 1$ and $-1 < w < 0$, and $f(w) < 0$ for $w < -1$ and $0 < w < 1$, the equilibrium points at $w = \pm 1$ are sources, and the equilibrium point at $w = 0$ is a sink.



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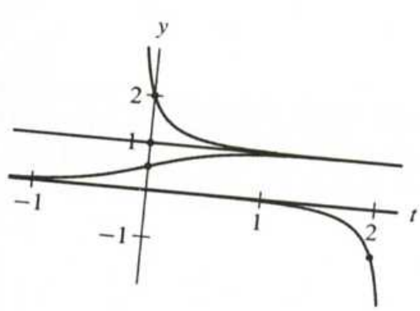

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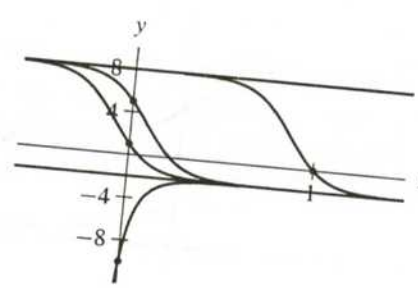

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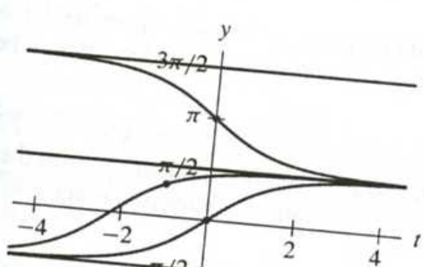
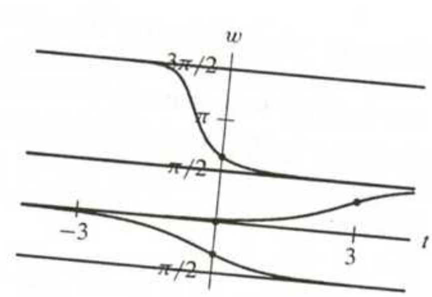


16.



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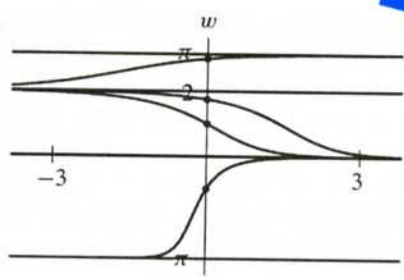

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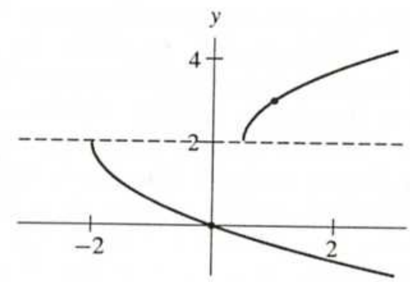
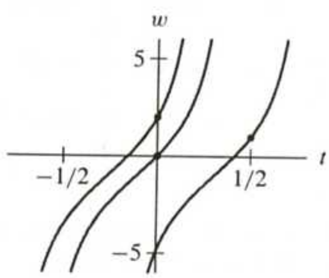
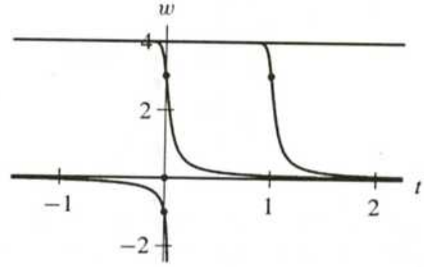


The equation is undefined at $y = 2$.

19.



20.



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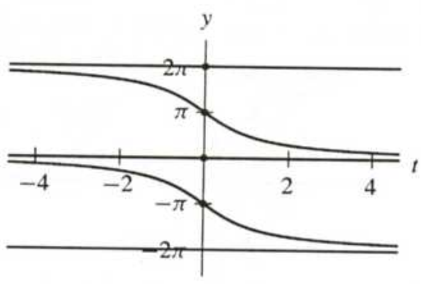

22. The initial value $y(0) = 0$ is below the two equilibrium points $y = 2 \pm \sqrt{2}$. Since $dy/dt > 0$ for $y < 2 - \sqrt{2}$, this solution is increasing, and $y(t) \to 2 - \sqrt{2}$ as $t \to \infty$. As t decreases, it becomes unbounded in the negative direction in finite time.

23. The initial value $y(0) = 1$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2 - \sqrt{2} < y < 2 + \sqrt{2}$. Hence the solution is decreasing and tends toward the smaller equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward the larger equilibrium point $y = 2 + \sqrt{2}$ as $t \to -\infty$.

24. Because $y(0) = -1 < 2 - \sqrt{2}$, this solution increases toward $2 - \sqrt{2}$ as t increases and decreases as t decreases. In fact, because $y(0) = -1 < 0$, this solution is always below the solution in Exercise 22.

25. The initial value $y(0) = -10$ is below both of the equilibrium points. Since dy/dt is positive for $y < 2 - \sqrt{2}$, the solution is increasing for all t and tends to the equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. As t decreases, it becomes unbounded in the negative direction in finite time.

26. The initial value $y(0) = 10$ is greater than the largest equilibrium point $2 + \sqrt{2}$, and $dy/dt > 0$ if $y > 2 + \sqrt{2}$. Hence, this solution increases without bound as t increases. (In fact, it blows up in finite time). As $t \to -\infty$, $y(t) \to 2 + \sqrt{2}$.

27. The initial value $y(3) = 1$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2 - \sqrt{2} < y < 2 + \sqrt{2}$. Hence the solution is decreasing and tends toward the smaller equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward the larger equilibrium point $y = 2 + \sqrt{2}$ as $t \to -\infty$.

28. **(a)** Any solution that has an initial value between the equilibrium points at $y = -1$ and $y = 2$ must remain between these values for all t, so $-1 < y(t) < 2$ for all t.

(b) The extra assumption implies that the solution is increasing for all t such that $-1 < y(t) < 2$. Again assuming that the Uniqueness Theorem applies, we conclude that $y(t) \to 2$ as $t \to \infty$ and $y(t) \to -1$ as $t \to -\infty$.

29. The function $f(y)$ has two zeros $\pm y_0$, where y_0 is some positive number. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) < 0$ if $-y_0 < y < y_0$ and $f(y) > 0$ if $y < -y_0$ or if $y > y_0$. Hence y_0 is a source and $-y_0$ is a sink.

30. The function $f(y)$ has three zeros. We denote them as y_1, y_2, and y_3, where $y_1 < 0 < y_2 < y_3$. So the differential equation $dy/dt = f(y)$ has three equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, $f(y) < 0$ if $y_1 < y < y_2$, and $f(y) > 0$ if $y_2 < y < y_3$ or if $y > y_3$. Hence y_1 is a sink, y_2 is a source, and y_3 is a node.

31. The function $f(y)$ has two zeros, one positive and one negative. We denote them as y_1 and y_2, where $y_1 < y_2$. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y_1 < y < y_2$ and $f(y) < 0$ if $y < y_1$ or if $y > y_2$. Hence y_1 is a source and y_2 is a sink.