For solving homogeneous systems, the text recommends working with the augmented matrix, although no calculations take place in the augmented column. See the *Study Guide* comments on Exercise 7 that illustrate two common student errors.

All students need the practice of Exercises 1–14. (Assign all odd, all even, or a mixture. If you do not assign Exercise 7, be sure to assign both 8 and 10.) Otherwise, a few students may be unable later to find a basis for a null space or an eigenspace. Exercises 29–34 are important. Exercises 33 and 34 help students later understand how solutions of $A\mathbf{x} = \mathbf{0}$ encode linear dependence relations among the columns of A. Exercises 35–38 are more challenging. Exercise 37 will help students avoid the standard mistake of forgetting that Theorem 6 applies only to a *consistent* equation $A\mathbf{x} = \mathbf{b}$.

1. Reduce the augmented matrix to echelon form and circle the pivot positions. If a column of the *coefficient* matrix is not a pivot column, the corresponding variable is free and the system of equations has a nontrivial solution. Otherwise, the system has *only* the trivial solution.

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -5 & 8 & 0 \\ 0 & \textcircled{-12} & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The variable x_3 is free, so the system has a nontrivial solution.

2. $$\begin{bmatrix} 1 & -3 & 7 & 0 \\ -2 & 1 & -4 & 0 \\ 1 & 2 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 7 & 0 \\ 0 & -5 & 10 & 0 \\ 0 & 5 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 7 & 0 \\ 0 & \textcircled{-5} & 10 & 0 \\ 0 & 0 & \textcircled{12} & 0 \end{bmatrix}$$

There is no free variable; the system has only the trivial solution.

3. $$\begin{bmatrix} -3 & 5 & -7 & 0 \\ -6 & 7 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & 5 & -7 & 0 \\ 0 & \textcircled{-3} & 15 & 0 \end{bmatrix}$$ The variable x_3 is free; the system has nontrivial solutions.

An alert student will realize that row operations are unnecessary. With only two equations, there can be at most two basic variables. One variable *must* be free. Refer to Exercise 31 in Section 1.2.

4. $$\begin{bmatrix} -5 & 7 & 9 & 0 \\ 1 & -2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 6 & 0 \\ -5 & 7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 6 & 0 \\ 0 & \textcircled{-3} & 39 & 0 \end{bmatrix}$$ x_3 is a free variable; the system has nontrivial solutions. As in Exercise 3, row operations are unnecessary.

5. $$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & 0 \\ 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} \quad - 5x_3 &= 0 \\ \textcircled{x_2} + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

The variable x_3 is free, $x_1 = 5x_3$, and $x_2 = -2x_3$.

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 4 & 0 \\ 0 & \textcircled{1} & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\textcircled{x_1} \quad + 4x_3 = 0$

$\textcircled{x_2} - 3x_3 = 0$. The variable x_3 is free, $x_1 = -4x_3$, and $x_2 = 3x_3$.

$0 = 0$

In parametric vector form, the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 9 & -8 & 0 \\ 0 & \textcircled{1} & -4 & 5 & 0 \end{bmatrix}$, $\begin{aligned} \textcircled{x_1} \quad + 9x_3 - 8x_4 &= 0 \\ \textcircled{x_2} - 4x_3 + 5x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

8. $\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -5 & -7 & 0 \\ 0 & \textcircled{1} & 2 & -6 & 0 \end{bmatrix}$, $\begin{aligned} \textcircled{x_1} \quad - 5x_3 - 7x_4 &= 0 \\ \textcircled{x_2} + 2x_3 - 6x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = 5x_3 + 7x_4$ and $x_2 = -2x_3 + 6x_4$. The general solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

9. $\begin{bmatrix} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 0 \\ 3 & -9 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} \textcircled{x_1} - 3x_2 + 2x_3 &= 0 \\ 0 &= 0 \end{aligned}$

The solution is $x_1 = 3x_2 - 2x_3$, with x_2 and x_3 free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

10. $\begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 2 & 6 & 0 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{aligned} \textcircled{x_1} - 3x_2 \quad - 4x_4 &= 0 \\ 0 &= 0 \end{aligned}$

The only basic variable is x_1, so x_2, x_3, and x_4 are free. (Note that x_3 is not zero.) Also, $x_1 = 3x_2 + 4x_4$. The general solution is

The solution set is the line through $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 5.

16. Row reduce the augmented matrix for the system:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 4 & -5 \\ 0 & \textcircled{1} & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \textcircled{x_1} \quad + 4x_3 &= -5 \\ \textcircled{x_2} - 3x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -5 - 4x_3$, $x_2 = 3 + 3x_3$, and x_3 is free. In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5-4x_3 \\ 3+3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

The solution set is the line through $\begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$, parallel to the line that is the solution set of the homogeneous system in Exercise 6.

17. Solve $x_1 + 9x_2 - 4x_3 = -2$ for the basic variable: $x_1 = -2 - 9x_2 + 4x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -9x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 + 9x_2 - 4x_3 = 0$ is $x_1 = -9x_2 + 4x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2+4x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by $\mathbf{u}$ and $\mathbf{v}$. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$.

18. Solve $x_1 - 3x_2 + 5x_3 = 4$ for the basic variable: $x_1 = 4 + 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form, the solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4+3x_2-5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

The solution of $x_1 - 3x_2 + 5x_3 = 0$ is $x_1 = 3x_2 - 5x_3$, with x_2 and x_3 free. In vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5x_3 \\ 0 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_3\mathbf{v}$$

The solution set of the homogeneous equation is the plane through the origin in $\mathbf{R}^3$ spanned by **u** and **v**. The solution set of the nonhomogeneous equation is parallel to this plane and passes through the point $\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

19. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\,\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \end{bmatrix}, \text{ or } \begin{cases} x_1 = -2 - 5t \\ x_2 = 3t \end{cases}$$

20. The line through **a** parallel to **b** can be written as $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, where t represents a parameter:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + t\begin{bmatrix} -7 \\ 8 \end{bmatrix}, \text{ or } \begin{cases} x_1 = 3 - 7t \\ x_2 = -4 + 8t \end{cases}$$

21. The line through **p** and **q** is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, form

$\mathbf{q} - \mathbf{p} = \begin{bmatrix} -3-2 \\ 1-(-5) \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$, and write the line as $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + t\begin{bmatrix} -5 \\ 6 \end{bmatrix}$.

22. The line through **p** and **q** is parallel to $\mathbf{q} - \mathbf{p}$. So, given $\mathbf{p} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$, form

$\mathbf{q} - \mathbf{p} = \begin{bmatrix} 0-(-6) \\ -4-3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \end{bmatrix}$, and write the line as $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \begin{bmatrix} -6 \\ 3 \end{bmatrix} + t\begin{bmatrix} 6 \\ -7 \end{bmatrix}$

Note: Exercises 21 and 22 prepare for Exercise 27 in Section 1.8.

23. a. True. See the first paragraph of the subsection titled *Homogeneous Linear Systems*.

b. False. The equation $A\mathbf{x} = \mathbf{0}$ gives an *implicit* description of its solution set. See the subsection entitled *Parametric Vector Form*.

c. False. The equation $A\mathbf{x} = \mathbf{0}$ *always* has the trivial solution. The box before Example 1 uses the word *nontrivial* instead of *trivial*.

d. False. The line goes through **p** parallel to **v**. See the paragraph that precedes Fig. 5.

e. False. The solution set could be *empty*! The statement (from Theorem 6) is true only when there exists a vector **p** such that $A\mathbf{p} = \mathbf{b}$.

24. a. False. A nontrivial solution of $A\mathbf{x} = \mathbf{0}$ is any nonzero **x** that satisfies the equation. See the sentence before Example 2.

b. True. See Example 2 and the paragraph following it.

33. Look at $x_1\begin{bmatrix}-2\\7\\-3\end{bmatrix}+x_2\begin{bmatrix}-6\\21\\-9\end{bmatrix}$ and notice that the second column is 3 times the first. So suitable values for x_1 and x_2 would be 3 and –1 respectively. (Another pair would be 6 and –2, etc.) Thus $\mathbf{x}=\begin{bmatrix}3\\-1\end{bmatrix}$ satisfies $A\mathbf{x}=\mathbf{0}$.

34. Inspect how the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are related. The second column is –3/2 times the first. Put another way, $3\mathbf{a}_1+2\mathbf{a}_2=\mathbf{0}$. Thus $\begin{bmatrix}3\\2\end{bmatrix}$ satisfies $A\mathbf{x}=\mathbf{0}$.

Note: Exercises 33 and 34 set the stage for the concept of linear dependence.

35. Look for $A=[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1+1\mathbf{a}_2+1\cdot\mathbf{a}_3=\mathbf{0}$. That is, construct A so that each row sum (the sum of the entries in a row) is zero.

36. Look for $A=[\mathbf{a}_1\ \ \mathbf{a}_2\ \ \mathbf{a}_3]$ such that $1\cdot\mathbf{a}_1-2\cdot\mathbf{a}_2+1\cdot\mathbf{a}_3=\mathbf{0}$. That is, construct A so that the sum of the first and third columns is twice the second column.

37. Since the solution set of $A\mathbf{x}=\mathbf{0}$ contains the point (4,1), the vector $\mathbf{x}=(4,1)$ satisfies $A\mathbf{x}=\mathbf{0}$. Write this equation as a vector equation, using $\mathbf{a}_1$ and $\mathbf{a}_2$ for the columns of A:

$$4\cdot\mathbf{a}_1+1\cdot\mathbf{a}_2=\mathbf{0}$$

Then $\mathbf{a}_2=-4\mathbf{a}_1$. So choose any nonzero vector for the first column of A and multiply that column by – 4 to get the second column of A. For example, set $A=\begin{bmatrix}1&-4\\1&-4\end{bmatrix}$.

Finally, the only way the solution set of $A\mathbf{x}=\mathbf{b}$ could *not* be parallel to the line through (1,4) and the origin is for the solution set of $A\mathbf{x}=\mathbf{b}$ to be *empty*. This does not contradict Theorem 6, because that theorem applies only to the case when the equation $A\mathbf{x}=\mathbf{b}$ has a nonempty solution set. For $\mathbf{b}$, take any vector that is *not* a multiple of the columns of A.

Note: In the *Study Guide*, a "Checkpoint" for Section 1.5 will help students with Exercise 37.

38. No. If $A\mathbf{x}=\mathbf{y}$ has no solution, then A cannot have a pivot in each row. Since A is 3×3, it has at most two pivot positions. So the equation $A\mathbf{x}=\mathbf{z}$ for any $\mathbf{z}$ has at most two basic variables and at least one free variable. Thus, the solution set for $A\mathbf{x}=\mathbf{z}$ is either empty or has infinitely many elements.

39. If $\mathbf{u}$ satisfies $A\mathbf{x}=\mathbf{0}$, then $A\mathbf{u}=\mathbf{0}$. For any scalar c, Theorem 5(b) in Section 1.4 shows that $A(c\mathbf{u})=cA\mathbf{u}=c\cdot\mathbf{0}=\mathbf{0}$.

40. Suppose $A\mathbf{u}=\mathbf{0}$ and $A\mathbf{v}=\mathbf{0}$. Then, since $A(\mathbf{u}+\mathbf{v})=A\mathbf{u}+A\mathbf{v}$ by Theorem 5(a) in Section 1.4,

$$A(\mathbf{u}+\mathbf{v})=A\mathbf{u}+A\mathbf{v}=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Now, let c and d be scalars. Using both parts of Theorem 5,

$$A(c\mathbf{u}+d\mathbf{v})=A(c\mathbf{u})+A(d\mathbf{v})=cA\mathbf{u}+dA\mathbf{v}=c\mathbf{0}+d\mathbf{0}=\mathbf{0}.$$

Note: The MATLAB box in the *Study Guide* introduces the `zeros` command, in order to augment a matrix with a column of zeros.

$$\sim\cdots\sim\begin{bmatrix}1&0&0&0&0&-1&100\\0&1&0&0&0&-1&0\\0&0&1&0&0&-1&50\\0&0&0&1&0&-1&-70\\0&0&0&0&1&-1&80\\0&0&0&0&0&0&0\end{bmatrix}. \text{ The general solution is } \begin{cases}x_1=100+x_6\\x_2=x_6\\x_3=50+x_6\\x_4=-70+x_6\\x_5=80+x_6\\x_6 \text{ is free}\end{cases}$$

Since x_4 cannot be negative, the minimum value of x_6 is 70.

Note: The MATLAB box in the *Study Guide* discusses rational calculations, needed for balancing the chemical equations in Exercises 9 and 10. As usual, the appendices cover this material for Maple, Mathematica, and the TI and HP graphic calculators.

1.7 SOLUTIONS

Note: Key exercises are 9–20 and 23–30. Exercise 30 states a result that could be a theorem in the text. There is a danger, however, that students will memorize the result without understanding the proof, and then later mix up the words row and column. Exercises 37 and 38 anticipate the discussion in Section 1.9 of one-to-one transformations. Exercise 44 is fairly difficult for my students.

1. Use an augmented matrix to study the solution set of $x_1\mathbf{u}+x_2\mathbf{v}+x_3\mathbf{w}=\mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix}5&7&9&0\\0&2&4&0\\0&-6&-8&0\end{bmatrix}\sim\begin{bmatrix}\textcircled{5}&7&9&0\\0&\textcircled{2}&4&0\\0&0&\textcircled{4}&0\end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

2. Use an augmented matrix to study the solution set of $x_1\mathbf{u}+x_2\mathbf{v}+x_3\mathbf{w}=\mathbf{0}$ (*), where **u**, **v**, and **w** are the three given vectors. Since $\begin{bmatrix}0&0&-3&0\\0&5&4&0\\2&-8&1&0\end{bmatrix}\sim\begin{bmatrix}\textcircled{2}&-8&1&0\\0&\textcircled{5}&4&0\\0&0&\textcircled{-3}&0\end{bmatrix}$, there are no free variables. So the homogeneous equation (*) has only the trivial solution. The vectors are linearly independent.

3. Use the method of Example 3 (or the box following the example). By comparing entries of the vectors, one sees that the second vector is –3 times the first vector. Thus, the two vectors are linearly dependent.

4. From the first entries in the vectors, it seems that the second vector of the pair $\begin{bmatrix}-1\\4\end{bmatrix},\begin{bmatrix}-2\\-8\end{bmatrix}$ may be 2 times the first vector. But there is a sign problem with the second entries. So neither of the vectors is a multiple of the other. The vectors are linearly independent.

5. Use the method of Example 2. Row reduce the augmented matrix for $A\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix}0&-8&5&0\\3&-7&4&0\\-1&5&-4&0\\1&-3&2&0\end{bmatrix}\sim\begin{bmatrix}1&-3&2&0\\3&-7&4&0\\-1&5&-4&0\\0&-8&5&0\end{bmatrix}\sim\begin{bmatrix}1&-3&2&0\\0&2&-2&0\\0&2&-2&0\\0&-8&5&0\end{bmatrix}\sim\begin{bmatrix}1&-3&2&0\\0&2&-2&0\\0&0&0&0\\0&0&-3&0\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-3&2&0\\0&\textcircled{2}&-2&0\\0&0&\textcircled{-3}&0\\0&0&0&0\end{bmatrix}$$

10. a. The vector $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{v}_3$ has a solution. To find out, row reduce $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3]$, considered as an augmented matrix:

$$\begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 2 \\ 0 & 0 & \textcircled{1} \\ 0 & 0 & h+6 \end{bmatrix}$$

At this point, the equation $0 = 1$ shows that the original vector equation has no solution. So $\mathbf{v}_3$ is in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for *no* value of h.

b. For $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent, the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ must have only the trivial solution. Row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$

$$\begin{bmatrix} 1 & -2 & 2 & 0 \\ -5 & 10 & -9 & 0 \\ -3 & 6 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & h+6 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 2 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For every value of h, x_2 is a free variable, and so the homogeneous equation has a nontrivial solution. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set for all h.

11. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & -5 & h+4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 3 & -1 & 0 \\ 0 & \textcircled{-2} & 4 & 0 \\ 0 & 0 & h-6 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h - 6 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = 6$.

12. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$:

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -6 & 8 & 0 \\ 0 & \textcircled{-5} & h+16 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

13. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 3 & 0 \\ 0 & \textcircled{1} & h-15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a free variable and hence a nontrivial solution no matter what the value of h. So the vectors are linearly dependent for all values of h.

14. To study the linear dependence of three vectors, say $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, row reduce the augmented matrix $[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{0}]$:

$$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ -3 & 8 & h & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -7 & h+3 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -5 & 1 & 0 \\ 0 & \textcircled{2} & 2 & 0 \\ 0 & 0 & h+10 & 0 \end{bmatrix}$$

The equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a nontrivial solution if and only if $h + 10 = 0$ (which corresponds to x_3 being a free variable). Thus, the vectors are linearly dependent if and only if $h = -10$.

15. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

16. The set is linearly dependent because the second vector is 3/2 times the first vector.

17. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

18. The set is linearly dependent, by Theorem 8, because there are four vectors in the set but only two entries in each vector.

19. The set is linearly independent because neither vector is a multiple of the other vector. [Two of the entries in the first vector are – 4 times the corresponding entry in the second vector. But this multiple does not work for the third entries.]

20. The set is linearly dependent, by Theorem 9, because the list of vectors contains a zero vector.

21. a. False. A homogeneous system *always* has the trivial solution. See the box before Example 2.
   b. False. See the warning after Theorem 7.
   c. True. See Fig. 3, after Theorem 8.
   d. True. See the remark following Example 4.

22. a. True. See Fig. 1.
   b. False. For instance, the set consisting of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is linearly dependent. See the warning after Theorem 8.
   c. True. See the remark following Example 4.
   d. False. See Example 3(a).

23. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$

24. $\begin{bmatrix} \blacksquare & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

25. $\begin{bmatrix} \blacksquare & * \\ 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \blacksquare \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

26. $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$. The columns must linearly independent, by Theorem 7, because the first column is not zero, the second column is not a multiple of the first, and the third column is not a linear combination of the preceding two columns (because $\mathbf{a}_3$ is not in Span $\{\mathbf{a}_1, \mathbf{a}_2\}$).

27. All five columns of the 7×5 matrix A must be pivot columns. Otherwise, the equation $A\mathbf{x} = \mathbf{0}$ would have a free variable, in which case the columns of A would be linearly dependent.

28. If the columns of a 5×7 matrix A span $\mathbb{R}^5$, then A has a pivot in each row, by Theorem 4. Since each pivot position is in a different column, A has five pivot columns.

29. A: any 3×2 matrix with two nonzero columns such that neither column is a multiple of the other. In this case the columns are linearly independent and so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
B: any 3×2 matrix with one column a multiple of the other.

30. a. n
 b. The columns of A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens if and only if $A\mathbf{x} = \mathbf{0}$ has no free variables, which in turn happens if and only if every variable is a basic variable, that is, if and only if every column of A is a pivot column.

31. Think of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$. The text points out that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$. Rewrite this as $\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 1, -1)$.

32. Think of $A = [\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3]$. The text points out that $\mathbf{a}_1 + 2\mathbf{a}_2 = \mathbf{a}_3$. Rewrite this as $\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$. As a matrix equation, $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = (1, 2, -1)$.

33. True, by Theorem 7. (The *Study Guide* adds another justification.)

34. True, by Theorem 9.

35. False. The vector $\mathbf{v}_1$ could be the zero vector.

36. False. Counterexample: Take $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$ all to be multiples of one vector. Take $\mathbf{v}_3$ to be *not* a multiple of that vector. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$$

37. True. A linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ may be extended to a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ by placing a zero weight on $\mathbf{v}_4$.

38. True. If the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ had a nontrivial solution (with at least one of x_1, x_2, x_3 nonzero), then so would the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0 \cdot \mathbf{v}_4 = \mathbf{0}$. But that cannot happen because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be linearly independent. This problem can also be solved using Exercise 37, if you know that the statement there is true.

3. $[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, unique solution

4. $[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 3 & -5 & -9 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 4 & -15 & -27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 6 \\ 0 & 1 & -4 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$, unique solution

5. $[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 3 \\ 0 & \textcircled{1} & 2 & 1 \end{bmatrix}$

Note that a solution is *not* $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. To avoid this common error, write the equations:

$\begin{array}{rcrcl} x_1 & & + \ 3x_3 & = & 3 \\ & x_2 & + \ 2x_3 & = & 1 \end{array}$ and solve for the basic variables: $\begin{cases} x_1 = 3 - 3x_3 \\ x_2 = 1 - 2x_3 \\ x_3 \text{ is free} \end{cases}$

General solution $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$. For a particular solution, one might choose

$x_3 = 0$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$.

6. $[A \ \ \mathbf{b}] = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & -4 & 5 & 9 \\ 0 & 1 & 1 & 3 \\ -3 & 5 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 2 & 2 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 7 \\ 0 & \textcircled{1} & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{array}{rcrcl} x_1 & & + \ 3x_3 & = & 7 \\ & x_2 & + \ x_3 & = & 3 \end{array}$. $\begin{cases} x_1 = 7 - 3x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$

General solution: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 - 3x_3 \\ 3 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$, one choice: $\begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$.

7. $a = 5$; the domain of T is $\mathbf{R}^5$, because a 6×5 matrix has 5 columns and for $A\mathbf{x}$ to be defined, $\mathbf{x}$ must be in $\mathbf{R}^5$. $b = 6$; the codomain of T is $\mathbf{R}^6$, because $A\mathbf{x}$ is a linear combination of the columns of A, and each column of A is in $\mathbf{R}^6$.

8. A must have 5 rows and 4 columns. For the domain of T to be $\mathbf{R}^4$, A must have four columns so that $A\mathbf{x}$ is defined for $\mathbf{x}$ in $\mathbf{R}^4$. For the codomain of T to be $\mathbf{R}^5$, the columns of A must have five entries (in which case A must have five rows), because $A\mathbf{x}$ is a linear combination of the columns of A.

9. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -9 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} x_1 \quad - 9x_3 + 7x_4 &= 0 \\ x_2 - 4x_3 + 3x_4 &= 0, \\ 0 &= 0 \end{aligned} \quad \begin{cases} x_1 = 9x_3 - 7x_4 \\ x_2 = 4x_3 - 3x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

10. Solve $A\mathbf{x} = \mathbf{0}$. $\begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ -2 & 3 & 0 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & -6 & 0 \\ 0 & 9 & 18 & 9 & 0 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & 9 & 2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 9 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

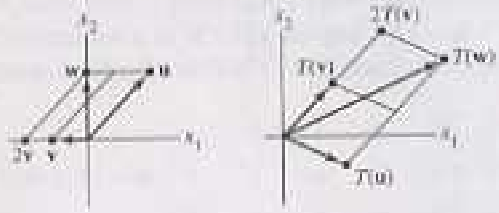
$$\begin{aligned} x_1 \quad + 3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_4 &= 0 \end{aligned} \quad \begin{cases} x_1 = -3x_3 \\ x_2 = -2x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \end{cases} \quad \mathbf{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

11. Is the system represented by $[A \;\; \mathbf{b}]$ consistent? Yes, as the following calculation shows.

$$\begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 2 & -8 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & -5 & -1 \\ 0 & 1 & -4 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, so $\mathbf{b}$ is in the range of the transformation $\mathbf{x} \mapsto A\mathbf{x}$.

18. Draw a line through **w** parallel to **v**, and draw a line through **w** parallel to **u**. See the left part of the figure below. From this, estimate that $\mathbf{w} = \mathbf{u} + 2\mathbf{v}$. Since T is linear, $T(\mathbf{w}) = T(\mathbf{u}) + 2T(\mathbf{v})$. Locate $T(\mathbf{u})$ and $2T(\mathbf{v})$ as in the right part of the figure and form the associated parallelogram to locate $T(\mathbf{w})$.



19. All we know are the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the fact that T is linear. The key idea is to write $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5\mathbf{e}_1 - 3\mathbf{e}_2$. Then, from the linearity of T, write

$$T(\mathbf{x}) = T(5\mathbf{e}_1 - 3\mathbf{e}_2) = 5T(\mathbf{e}_1) - 3T(\mathbf{e}_2) = 5\mathbf{y}_1 - 3\mathbf{y}_2 = 5\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}.$$

To find the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, observe that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) = x_1\begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

20. Use the basic definition of $A\mathbf{x}$ to construct A. Write

$$T(\mathbf{x}) = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = [\mathbf{v}_1 \ \ \mathbf{v}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}\mathbf{x}, \quad A = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}$$

21. a. True. Functions from $\mathbf{R}^n$ to $\mathbf{R}^m$ are defined before Fig. 2. A linear transformation is a function with certain properties.

b. False. The domain is $\mathbf{R}^5$. See the paragraph before Example 1.

c. False. The range is the set of all linear combinations of the columns of A. See the paragraph before Example 1.

d. False. See the paragraph after the definition of a linear transformation.

e. True. See the paragraph following the box that contains equation (4).

22. a. True. See the paragraph following the definition of a linear transformation.

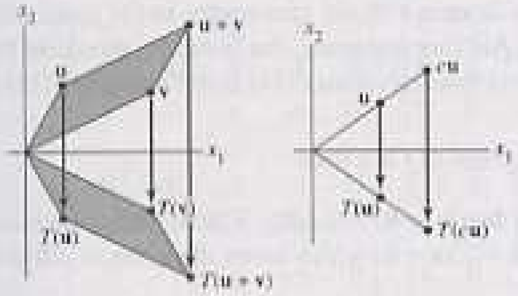
b. False. If A is an $m \times n$ matrix, the codomain is $\mathbf{R}^m$. See the paragraph before Example 1.

c. False. The question is an existence question. See the remark about Example 1(d), following the solution of Example 1.

d. True. See the discussion following the definition of a linear transformation.

e. True. See the paragraph following equation (5).

23.



24. Given any $\mathbf{x}$ in $\mathbf{R}^n$, there are constants $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_p\mathbf{v}_p$, because $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbf{R}^n$. Then, from property (5) of a linear transformation,

$$T(\mathbf{x}) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) = c_1\mathbf{0} + \cdots + c_p\mathbf{0} = \mathbf{0}$$

25. Any point $\mathbf{x}$ on the line through $\mathbf{p}$ in the direction of $\mathbf{v}$ satisfies the parametric equation $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ for some value of t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{v}) = T(\mathbf{p}) + tT(\mathbf{v}) \qquad (*)$$

If $T(\mathbf{v}) = \mathbf{0}$, then $T(\mathbf{x}) = T(\mathbf{p})$ for all values of t, and the image of the original line is just a single point. Otherwise, (*) is the parametric equation of a line through $T(\mathbf{p})$ in the direction of $T(\mathbf{v})$.

26. Any point $\mathbf{x}$ on the plane P satisfies the parametric equation $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ for some values of s and t. By linearity, the image $T(\mathbf{x})$ satisfies the parametric equation

$$T(\mathbf{x}) = sT(\mathbf{u}) + tT(\mathbf{v}) \qquad (s, t \text{ in } \mathbf{R}) \qquad (*)$$

The set of images is just Span$\{T(\mathbf{u}), T(\mathbf{v})\}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly independent, Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a plane through $T(\mathbf{u})$, $T(\mathbf{v})$, and $\mathbf{0}$. If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent and not both zero, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is a line through $\mathbf{0}$. If $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{0}$, then Span$\{T(\mathbf{u}), T(\mathbf{v})\}$ is $\{\mathbf{0}\}$.

27. a. From Fig. 7 in the exercises for Section 1.5, the line through $T(\mathbf{p})$ and $T(\mathbf{q})$ is in the direction of $\mathbf{q} - \mathbf{p}$, and so the equation of the line is $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = \mathbf{p} + t\mathbf{q} - t\mathbf{p} = (1 - t)\mathbf{p} + t\mathbf{q}$.

b. Consider $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$ for t such that $0 \le t \le 1$. Then, by linearity of T,

$$T(\mathbf{x}) = T((1 - t)\mathbf{p} + t\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{q}) \qquad 0 \le t \le 1 \qquad (*)$$

If $T(\mathbf{p})$ and $T(\mathbf{q})$ are distinct, then (*) is the equation for the line segment between $T(\mathbf{p})$ and $T(\mathbf{q})$, as shown in part (a) Otherwise, the set of images is just the single point $T(\mathbf{p})$, because

$$(1 - t)T(\mathbf{p}) + tT(\mathbf{q}) = (1 - t)T(\mathbf{p}) + tT(\mathbf{p}) = T(\mathbf{p})$$

28. Consider a point $\mathbf{x}$ in the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, say $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ for $0 \le a \le 1$, $0 \le b \le 1$. By linearity of T, the image of $\mathbf{x}$ is

$$T(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v}), \text{ for } 0 \le a \le 1,\ 0 \le b \le 1 \qquad (*)$$

This image point lies in the parallelogram determined by $T(\mathbf{u})$ and $T(\mathbf{v})$.

Special "degenerate" cases arise when $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent. If one of the images is not zero, then the "parallelogram" is actually the line segment from $\mathbf{0}$ to $T(\mathbf{u}) + T(\mathbf{v})$. If both $T(\mathbf{u})$ and $T(\mathbf{v})$ are zero, then the parallelogram is just $\{\mathbf{0}\}$. Another possibility is that even $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, in which case the original parallelogram is degenerate (either a line segment or the zero vector). In this case, the set of images must be degenerate, too.

29. a. When $b = 0$, $f(x) = mx$. In this case, for all x, y in $\mathbf{R}$ and all scalars c and d,

$$f(cx + dy) = m(cx + dy) = mcx + mdy = c(mx) + d(my) = c \cdot f(x) + d \cdot f(y)$$

This shows that f is linear.