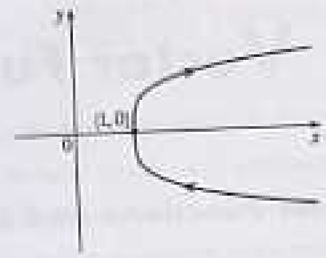
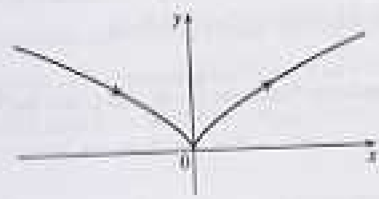
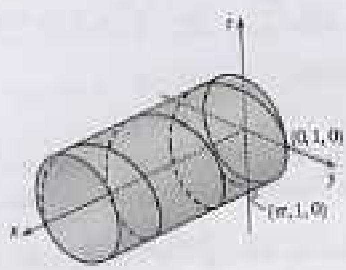
11. The corresponding parametric equations for this curve are $x = t^4 + 1$, $y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \Rightarrow x = y^4 + 1$, with $y \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



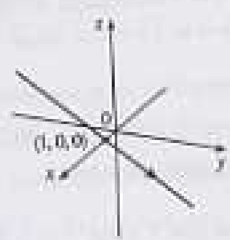
12. The corresponding parametric equations for this curve are $x = t^3$, $y = t^2$. We can make a table of values, or we can eliminate the parameter: $x = t^3 \Rightarrow t = \sqrt[3]{x} \Rightarrow y = t^2 = (\sqrt[3]{x})^2 = x^{2/3}$, with $t \in \mathbb{R} \Rightarrow x \in \mathbb{R}$. By comparing different values of t, we find the direction in which t increases as indicated in the graph.



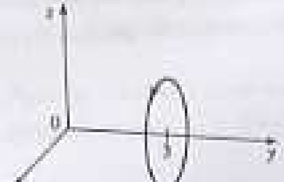
13. The corresponding parametric equations are $x = t$, $y = \cos 2t$, $z = \sin 2t$. Note that $y^2 + z^2 = \cos^2 2t + \sin^2 2t = 1$, so the curve lies on the circular cylinder $y^2 + z^2 = 1$. Since $x = t$, the curve is a helix.



14. The corresponding parametric equations are $x = 1 + t$, $y = 3t$, $z = -t$, which are parametric equations of a line through the point $(1, 0, 0)$ and with direction vector $\langle 1, 3, -1 \rangle$.



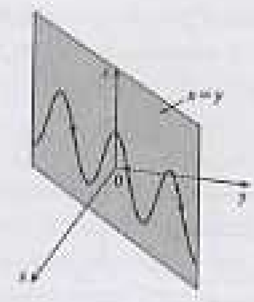
15. The parametric equations give $x^2 + z^2 = \sin^2 t + \cos^2 t = 1$, $y = 3$, which is a circle of radius 1, center $(0, 3, 0)$ in the plane $y = 3$.



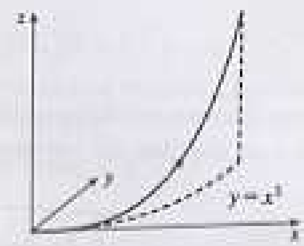
e curve is a

of a line

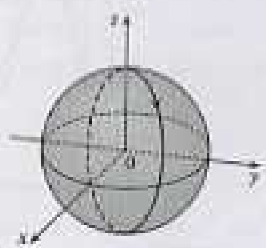
16. The parametric equations are $x = t$, $y = t$, $z = \cos t$. Thus $x = y$, so the curve must lie in the plane $x = y$. Combine this with $z = \cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x = y$.



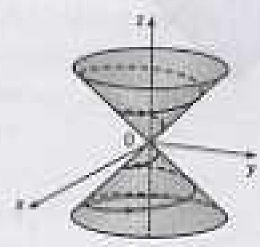
17. The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the xy-plane is $y = x^2$, $y > 0$, a half parabola. On the xz-plane $z = x^3$, $z > 0$, a half cubic, and the yz-plane, $y^3 = z^2$.



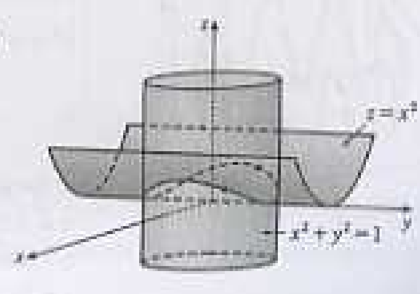
18. The parametric equations give $x^2 + y^2 + z^2 = 2\sin^2 t + 2\cos^2 t = 2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0, 0, 0)$. Furthermore $x = y = \sin t$, so the curve is the intersection of this sphere with the plane $x = y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0, 0, 0)$ in the plane $x = y$.



19. If $x = t\cos t$, $y = t\sin t$, and $z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



20. Here $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$.



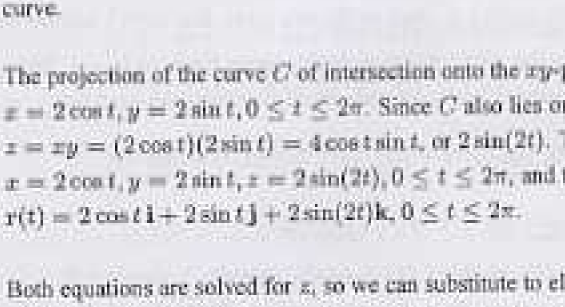
27. If $t = -1$, then $x = 1$, $y = 4$, $z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9$, $y = -8$, $z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \;\Rightarrow\; t = -2$. But then $z = 1 + (-2)^3 = -7 \neq -6$, so $(4, 7, -6)$ is not on the curve.

28. The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for C are $x = 2\cos t$, $y = 2\sin t$, $z = 2\sin(2t)$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}$, $0 \le t \le 2\pi$.

29. Both equations are solved for z, so we can substitute to eliminate z: $\sqrt{x^2 + y^2} = 1 + y \;\Rightarrow\; x^2 + y^2 = 1 + 2y + y^2 \;\Rightarrow\; x^2 = 1 + 2y \;\Rightarrow\; y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve C of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$. Thus a vector function representing C is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$.

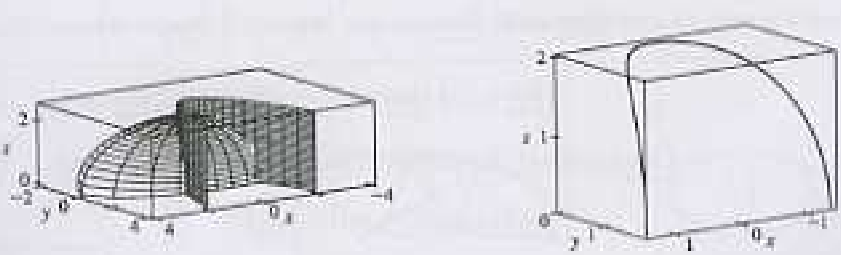
30. The projection of the curve C of intersection onto the xy-plane is the parabola $y = x^2$, $z = 0$. Then we can choose the parameter $x = t \;\Rightarrow\; y = t^2$. Since C also lies on the surface $z = 4x^2 + y^2$, we have $z = 4x^2 + y^2 = 4t^2 + (t^2)^2$. Then parametric equations for C are $x = t$, $y = t^2$, $z = 4t^2 + t^4$, and the corresponding vector function is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2 + t^4)\,\mathbf{k}$.

31.



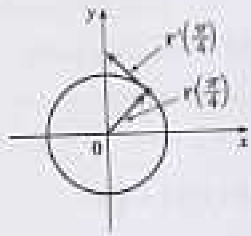
The projection of the curve C of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$. Since C also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for C are $x = 2\cos t$, $y = 2\sin t$, $z = 4\cos^2 t$, $0 \le t \le 2\pi$.

32.



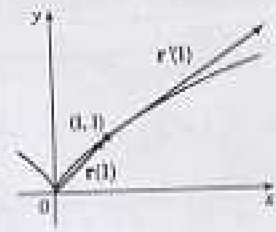
$x = t \;\Rightarrow\; y = t^2 \;\Rightarrow\; 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \;\Rightarrow\; z = \sqrt{4 - \left(\frac{1}{2}t\right)^2 - t^4}$. Note that z is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t$, $y = t^2$, $z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

3. (a), (c)



(b) $\mathbf{r}'(t) = \langle -\sin t, \cos t \rangle$

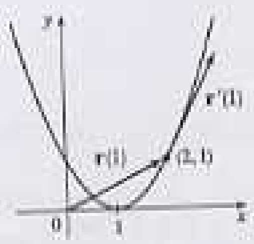
4. (a), (c)



(b) $\mathbf{r}'(t) = \langle 3t^2, 2t \rangle$

5. Since $(x-1)^2 = t^2 = y$, the curve is a parabola.

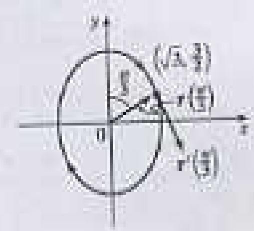
(a), (c)



(b) $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$

6. $x = 2\sin t$, $y = 3\cos t$, so $(x/2)^2 + (y/3)^2 = \sin^2 t + \cos^2 t = 1$ and the curve is an ellipse.

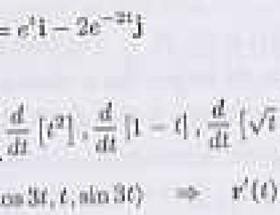
(a), (c)



(b) $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

7. $x^{-2} = e^{-2t} = y$, so $y = 1/x^2$, $x > 0$.

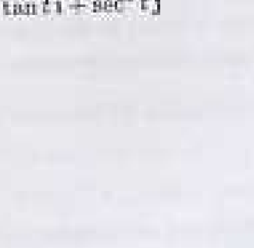
(a), (c)



(b) $\mathbf{r}'(t) = e^t\mathbf{i} - 2e^{-2t}\mathbf{j}$

8. $x^2 - y^2 = \sec^2 t - \tan^2 t = 1$, so the curve is a hyperbola.

(a), (c)



(b) $\mathbf{r}'(t) = \sec t \tan t\,\mathbf{i} + \sec^2 t\,\mathbf{j}$

9. $\mathbf{r}'(t) = \left\langle \frac{d}{dt}\left[t^2\right], \frac{d}{dt}\left[1-t\right], \frac{d}{dt}\left[\sqrt{t}\right] \right\rangle = \left\langle 2t, -1, \frac{1}{2\sqrt{t}} \right\rangle$

10. $\mathbf{r}(t) = \langle \cos 3t, t, \sin 3t \rangle \Rightarrow \mathbf{r}'(t) = \langle -3\sin 3t, 1, 3\cos 3t \rangle$

11. $\mathbf{r}(t) = e^{t^2}\mathbf{i} - \mathbf{j} + \ln(1+3t)\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = 2te^{t^2}\mathbf{i} + \frac{3}{1+3t}\mathbf{k}$

12. $\mathbf{r}(t) = \sin^{-1} t\,\mathbf{i} + \sqrt{1-t^2}\,\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}'(t) = \frac{1}{\sqrt{1-t^2}}\mathbf{i} - \frac{t}{\sqrt{1-t^2}}\mathbf{j}$

13. $\mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

14. To find $\mathbf{r}'(t)$, we first expand $\mathbf{r}(t) = t\,\mathbf{a} \times (\mathbf{b} + t\,\mathbf{c}) = t(\mathbf{a} \times \mathbf{b}) + t^2(\mathbf{a} \times \mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a} \times \mathbf{b} + 2t(\mathbf{a} \times \mathbf{c})$.

15. $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. Thus

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{0^2+3^2+4^2}}(3\,\mathbf{j}+4\,\mathbf{k}) = \tfrac{1}{5}(3\,\mathbf{j}+4\,\mathbf{k}) = \tfrac{3}{5}\,\mathbf{j}+\tfrac{4}{5}\,\mathbf{k}.$$

16. $\mathbf{r}'(t) = \dfrac{2}{\sqrt{t}}\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; \mathbf{r}'(1) = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. Thus

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{2^2+2^2+1^2}}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}) = \tfrac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}) = \tfrac{2}{3}\,\mathbf{i}+\tfrac{2}{3}\,\mathbf{j}+\tfrac{1}{3}\,\mathbf{k}$$

17. $\mathbf{r}(t) = \langle t, t^2, t^3\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. Then $\mathbf{r}'(1) = \langle 1,2,3\rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, so

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \tfrac{1}{\sqrt{14}}\langle 1,2,3\rangle = \left\langle \tfrac{1}{\sqrt{14}}, \tfrac{2}{\sqrt{14}}, \tfrac{3}{\sqrt{14}}\right\rangle. \quad \mathbf{r}''(t) = \langle 0, 2, 6t\rangle, \text{ so}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\mathbf{k}$$

$$= (12t^2 - 6t^2)\,\mathbf{i} - (6t-0)\,\mathbf{j} + (2-0)\,\mathbf{k} = \langle 6t^2, -6t, 2\rangle.$$

18. $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t}\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \;\Rightarrow$

$\mathbf{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0\rangle = \langle 2,-2,1\rangle$ and $|\mathbf{r}'(0)| = \sqrt{2^2+(-2)^2+1^2} = 3$. Then

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \tfrac{1}{3}\langle 2,-2,1\rangle = \left\langle \tfrac{2}{3}, \tfrac{-2}{3}, \tfrac{1}{3}\right\rangle. \quad \mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle \;\Rightarrow$$

$\mathbf{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0\rangle = \langle 4,4,4\rangle$.

$$\begin{aligned}\mathbf{r}'(t)\cdot\mathbf{r}''(t) &= \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle\cdot\langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle \\ &= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t}) \\ &= 8e^{4t} - 8e^{-4t} + (8t^2+12t+4)e^{4t} = (8t^2+12t+12)e^{4t} - 8e^{-4t}\end{aligned}$$

19. The vector equation for the curve is $\mathbf{r}(t) = \langle t^5, t^4, t^3\rangle$, so $\mathbf{r}'(t) = \langle 5t^4, 4t^3, 3t^2\rangle$. The point $(1,1,1)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 5,4,3\rangle$. Thus, the tangent line goes through the point $(1,1,1)$ and is parallel to the vector $\langle 5,4,3\rangle$. Parametric equations are $x = 1+5t$, $y = 1+4t$, $z = 1+3t$.

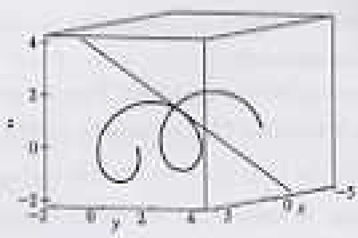
20. The vector equation for the curve is $\mathbf{r}(t) = \langle t^2-1, t^2+1, t+1\rangle$, so $\mathbf{r}'(t) = \langle 2t, 2t, 1\rangle$. The point $(-1,1,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 0,0,1\rangle$. Thus, the tangent line is parallel to the vector $\langle 0,0,1\rangle$ and parametric equations are $x = -1+0\cdot t = -1$, $y = 1+0\cdot t = 1$, $z = 1+1\cdot t = 1+t$.

21. The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, so

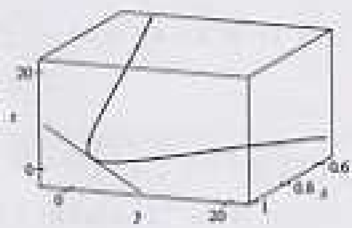
$\mathbf{r}'(t) = \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t})\rangle =$
$\langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t}\rangle$. The point $(1,0,1)$ corresponds to $t=0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0\rangle = \langle -1,1,-1\rangle$. Thus, the tangent line is parallel to the vector $\langle -1,1,-1\rangle$ and parametric equations are $x = 1+(-1)t = 1-t$, $y = 0+1\cdot t = t$, $z = 1+(-1)t = 1-t$.

22. $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2\rangle$, $\mathbf{r}'(t) = \langle 1/t, 1/\sqrt{t}, 2t\rangle$. At $(0,2,1)$, $t=1$ and $\mathbf{r}'(1) = \langle 1,1,2\rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 2+t$, $z = 1+2t$.

23. $\mathbf{r}(t) = \langle t, \sqrt{2}\cos t, \sqrt{2}\sin t\rangle \;\Rightarrow$
$\mathbf{r}'(t) = \langle 1, -\sqrt{2}\sin t, \sqrt{2}\cos t\rangle$. At $\left(\frac{\pi}{4}, 1, 1\right)$, $t = \frac{\pi}{4}$ and
$\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle 1, -1, 1\rangle$. Thus, parametric equations of the tangent line
are $x = \frac{\pi}{4} + t$, $y = 1 - t$, $z = 1 + t$.



24. $\mathbf{r}(t) = \langle \cos t, 3e^{2t}, 3e^{-2t}\rangle$, $\mathbf{r}'(t) = \langle -\sin t, 6e^{2t}, -6e^{-2t}\rangle$. At
$(1, 3, 3)$, $t = 0$ and $\mathbf{r}'(0) = \langle 0, 6, -6\rangle$. Thus, parametric equations of
the tangent line are $x = 1$, $y = 3 + 6t$, $z = 3 - 6t$.



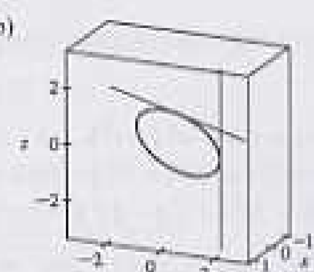
25. (a) $\mathbf{r}(t) = \langle t^3, t^4, t^5\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 3t^2, 4t^3, 5t^4\rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0\rangle = \mathbf{0}$, the curve is not smooth.

(b) $\mathbf{r}(t) = \langle t^3 + t, t^4, t^5\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 3t^2 + 1, 4t^3, 5t^4\rangle$. $\mathbf{r}'(t)$ is continuous since its component functions are continuous. Also, $\mathbf{r}'(t) \neq \mathbf{0}$, as the y- and z-components are 0 only for $t = 0$, but $\mathbf{r}'(0) = \langle 1, 0, 0\rangle \neq \mathbf{0}$. Thus, the curve is smooth.

(c) $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\rangle$. Since
$\mathbf{r}'(0) = \langle -3\cos^2 0\sin 0, 3\sin^2 0\cos 0\rangle = \langle 0, 0\rangle = \mathbf{0}$, the curve is not smooth.

26. (a) The tangent line at $t = 0$ is the line through the point with
position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0\rangle = \langle 0, 0, 1\rangle$,
and in the direction of the tangent vector,
$\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0\rangle = \langle \pi, 2\pi, 0\rangle$.
So an equation of the line is
$\langle x, y, z\rangle = \mathbf{r}(0) + u\,\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1\rangle = \langle \pi u, 2\pi u, 1\rangle$.

(b)



$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\right\rangle = \langle 1, 2, 0\rangle$, $\mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2}\right\rangle = \langle 0, 0, -\pi\rangle$.
So the equation of the second line is $\langle x, y, z\rangle = \langle 1, 2, 0\rangle + v\,\langle 0, 0, -\pi\rangle = \langle 1, 2, -\pi v\rangle$. The lines intersect where $\langle \pi u, 2\pi u, 1\rangle = \langle 1, 2, -\pi v\rangle$, so the point of intersection is $(1, 2, 1)$.

27. The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2\rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}_1'(0) = \langle 1, 0, 0\rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1\rangle$ and since $\mathbf{r}_2(0) = \langle 0, 0, 0\rangle$, $\mathbf{r}_2'(0) = \langle 1, 2, 1\rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0, 0, 0)$. If θ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}}\langle 1, 0, 0\rangle \cdot \langle 1, 2, 1\rangle = \frac{1}{\sqrt{6}}$ and

28. To find the point of intersection, we must find the values of t and s which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1, 0, 4)$. To find the angle θ of intersection, we proceed as in Exercise 27. The tangent vectors to the respective curves at $(1, 0, 4)$ are $\mathbf{r}_1'(1) = \langle 1, -1, 2\rangle$ and $\mathbf{r}_2'(2) = \langle -1, 1, 4\rangle$. So $\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1 - 1 + 8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

Note: In Exercise 27, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

29. $\int_0^1 (16t^3\,\mathbf{i} - 9t^2\,\mathbf{j} + 25t^4\,\mathbf{k})\,dt = \left(\int_0^1 16t^3\,dt\right)\mathbf{i} - \left(\int_0^1 9t^2\,dt\right)\mathbf{j} + \left(\int_0^1 25t^4\,dt\right)\mathbf{k}$

$$= \left[4t^4\right]_0^1\mathbf{i} - \left[3t^3\right]_0^1\mathbf{j} + \left[5t^5\right]_0^1\mathbf{k} = 4\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$$

30. $\displaystyle\int_0^1 \left(\frac{4}{1+t^2}\,\mathbf{j} + \frac{2t}{1+t^2}\,\mathbf{k}\right)dt = \left[4\tan^{-1}t\,\mathbf{j} + \ln(1+t^2)\,\mathbf{k}\right]_0^1$

$$= \left[4\tan^{-1}1\,\mathbf{j} + \ln 2\,\mathbf{k}\right] - \left[4\tan^{-1}0\,\mathbf{j} + \ln 1\,\mathbf{k}\right] = 4\left(\tfrac{\pi}{4}\right)\mathbf{j} + \ln 2\,\mathbf{k} - 0\,\mathbf{j} - 0\,\mathbf{k} = \pi\,\mathbf{j} + \ln 2\,\mathbf{k}$$

31. $\int_0^{\pi/4} (\cos 2t\,\mathbf{i} + \sin 2t\,\mathbf{j} + t\sin t\,\mathbf{k})\,dt = \left[\tfrac{1}{2}\sin 2t\,\mathbf{i} - \tfrac{1}{2}\cos 2t\,\mathbf{j}\right]_0^{\pi/4} + \left[\left[-t\cos t\right]_0^{\pi/4} + \int_0^{\pi/4}\cos t\,dt\right]\mathbf{k}$

$$= \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} + \left[-\tfrac{\pi}{4}\cos\tfrac{\pi}{4} + \sin\tfrac{\pi}{4}\right]\mathbf{k} = \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} + \tfrac{1}{\sqrt{2}}\left(1 - \tfrac{\pi}{4}\right)\mathbf{k} = \tfrac{1}{2}\,\mathbf{i} + \tfrac{1}{2}\,\mathbf{j} + \tfrac{4-\pi}{4\sqrt{2}}\,\mathbf{k}$$

32. $\int_1^4 (\sqrt{t}\,\mathbf{i} + te^{-t}\,\mathbf{j} + t^{-2}\,\mathbf{k})\,dt = \left[\tfrac{2}{3}t^{3/2}\,\mathbf{i} - t^{-1}\,\mathbf{k}\right]_1^4 + \left(\left[-te^{-t}\right]_1^4 + \int_1^4 e^{-t}\,dt\right)\mathbf{j}$

$$= \left(\tfrac{16}{3} - \tfrac{2}{3}\right)\mathbf{i} - \left(\tfrac{1}{4} - 1\right)\mathbf{k} + \left(-4e^{-4} + e^{-1} - e^{-4} + e^{-1}\right)\mathbf{j} = \tfrac{14}{3}\,\mathbf{i} + e^{-1}\left(2 - 5e^{-3}\right)\mathbf{j} + \tfrac{3}{4}\,\mathbf{k}$$

33. $\int (e^t\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k})\,dt = \left(\int e^t\,dt\right)\mathbf{i} + \left(\int 2t\,dt\right)\mathbf{j} + \left(\int \ln t\,dt\right)\mathbf{k}$

$= e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a vector constant of integration.

34. $\int (\cos\pi t\,\mathbf{i} + \sin\pi t\,\mathbf{j} + t\,\mathbf{k})\,dt = \left(\int \cos\pi t\,dt\right)\mathbf{i} + \left(\int \sin\pi t\,dt\right)\mathbf{j} + \left(\int t\,dt\right)\mathbf{k}$

$$= \tfrac{1}{\pi}\sin\pi t\,\mathbf{i} - \tfrac{1}{\pi}\cos\pi t\,\mathbf{j} + \tfrac{1}{2}t^2\,\mathbf{k} + \mathbf{C}$$

35. $\mathbf{r}'(t) = t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(t) = \tfrac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \tfrac{1}{3}t^3\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector. But $\mathbf{j} = \mathbf{r}(0) = (0)\,\mathbf{i} + (0)\,\mathbf{j} - (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = \mathbf{j}$ and $\mathbf{r}(t) = \tfrac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \tfrac{1}{3}t^3\,\mathbf{k} + \mathbf{j} = \tfrac{1}{3}t^3\,\mathbf{i} + (t^4 + 1)\,\mathbf{j} - \tfrac{1}{3}t^3\,\mathbf{k}$.

36. $\mathbf{r}'(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}(t) = (-\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}$. But $\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \mathbf{r}(0) = -\mathbf{i} + (0)\,\mathbf{j} + (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = 2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{r}(t) = (2 - \cos t)\,\mathbf{i} + (1 - \sin t)\,\mathbf{j} + (2 + t^2)\,\mathbf{k}$

For Exercises 37–40, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

37. $\frac{d}{dt}\left[\mathbf{u}(t) + \mathbf{v}(t)\right] = \frac{d}{dt}\left\langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t)\right\rangle$

$$= \left\langle \frac{d}{dt}\left[u_1(t) + v_1(t)\right], \frac{d}{dt}\left[u_2(t) + v_2(t)\right], \frac{d}{dt}\left[u_3(t) + v_3(t)\right]\right\rangle$$

41. $D_t\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$$= (-4t\,\mathbf{j} + 9t^2\,\mathbf{k})\cdot(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) + (\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k})\cdot(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k})$$

$$= -4t\cos t + 9t^2\sin t + 1 + 2t^2\sin t + 3t^3\cos t$$

$$= 1 - 4t\cos t + 11t^2\sin t + 3t^3\cos t$$

42. $D_t\,[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$$= (-4t\,\mathbf{j} + 9t^2\,\mathbf{k})\times(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) + (\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k})\times(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k})$$

$$= (-4t\sin t - 9t^2\cos t)\,\mathbf{i} + (9t^3 - 0)\,\mathbf{j} + (0 + 4t^2)\,\mathbf{k}$$
$$+ (-2t^2\cos t + 3t^3\sin t)\,\mathbf{i} + (3t^3 - \cos t)\,\mathbf{j} + (-\sin t + 2t^2)\,\mathbf{k}$$

$$= [(\sin t)(3t^3 - 4t) - 11t^2\cos t]\,\mathbf{i} + (12t^3 - \cos t)\,\mathbf{j} + (6t^2 - \sin t)\,\mathbf{k}$$

43. $\dfrac{d}{dt}\,[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (by the margin note on page 668). Thus, $\dfrac{d}{dt}\,[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

44. $\dfrac{d}{dt}\,(\mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]) = \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot\dfrac{d}{dt}\,[\mathbf{v}(t)\times\mathbf{w}(t)]$

$$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t) + \mathbf{v}(t)\times\mathbf{w}'(t)]$$

$$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}'(t)]$$

$$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] - \mathbf{v}'(t)\cdot[\mathbf{u}(t)\times\mathbf{w}(t)] + \mathbf{w}'(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)]$$

45. $\dfrac{d}{dt}\,|\mathbf{r}(t)| = \dfrac{d}{dt}\,[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{1/2} = \frac{1}{2}[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{-1/2}\,[2\mathbf{r}(t)\cdot\mathbf{r}'(t)] = \dfrac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{|\mathbf{r}(t)|}$

46. Since $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, we have $0 = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = \dfrac{d}{dt}\,[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \dfrac{d}{dt}\,|\mathbf{r}(t)|^2$. Thus $|\mathbf{r}(t)|^2$, and so $|\mathbf{r}(t)|$, is a constant, and hence the curve lies on a sphere with center the origin.

47. Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)]$,

$$\mathbf{u}'(t) = \mathbf{r}'(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)] + \mathbf{r}(t)\cdot\frac{d}{dt}\,[\mathbf{r}'(t)\times\mathbf{r}''(t)]$$

$$= 0 + \mathbf{r}(t)\cdot[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)] \qquad [\text{since } \mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)]$$

$$= \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}'''(t)] \qquad [\text{since } \mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}]$$

10.3 Arc Length and Curvature

1. $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{29}$. Then using Formula 3, we have $L = \int_{-10}^{10} |\mathbf{r}'(t)|\,dt = \int_{-10}^{10} \sqrt{29}\,dt = \sqrt{29}\,t\big]_{-10}^{10} = 20\sqrt{29}$.

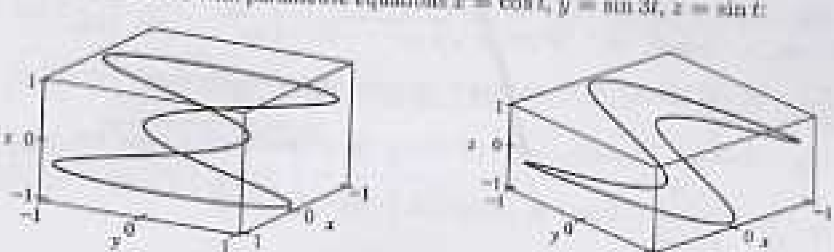
2. $\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \quad\Rightarrow$
$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}\,|t| = \sqrt{5}\,t$ for $0 \le t \le \pi$. Then using Formula 3, we have $L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{5}\,t\,dt = \sqrt{5}\,\frac{t^2}{2}\Big]_0^\pi = \frac{\sqrt{5}}{2}\pi^2$.

3. $\mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k} \quad\Rightarrow$
$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ (since $e^t + e^{-t} > 0$).
Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

4. $\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$, $|\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + (1/t)^2} = \dfrac{1 + 2t^2}{|t|} = \dfrac{1 + 2t^2}{t}$ for $1 \le t \le e$.
$$L = \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt = \left[\ln t + t^2\right]_1^e = e^2$$

5. The point $(2, 4, 8)$ corresponds to $t = 2$, so by Equation 2, $L = \int_0^2 \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\,dt$.
If $f(t) = \sqrt{1 + 4t^2 + 9t^4}$, then Simpson's Rule gives
$L \approx \dfrac{2-0}{10\cdot 3}\left[f(0) + 4f(0.2) + 2f(0.4) + \cdots + 4f(1.8) + f(2)\right] \approx 9.5706$.

6. Here are two views of the curve with parametric equations $x = \cos t$, $y = \sin 3t$, $z = \sin t$:



The complete curve is given by the parameter interval $[0, 2\pi]$, so
$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (3\cos 3t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{1 + 9\cos^2 3t}\,dt \approx 13.9744$.

7. $\mathbf{r}'(t) = e^t(\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j}$,
$ds/dt = |\mathbf{r}'(t)| = e^t\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} = e^t\sqrt{2\cos^2 t + 2\sin^2 t} = \sqrt{2}e^t$.
$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{2}e^u\,du = \sqrt{2}(e^t - 1) \quad\Rightarrow\quad \frac{1}{\sqrt{2}}s + 1 = e^t \quad\Rightarrow\quad t(s) = \ln\left(\frac{1}{\sqrt{2}}s + 1\right)$.
Therefore, $\mathbf{r}(t(s)) = \left(\frac{1}{\sqrt{2}}s + 1\right)\left[\sin\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right)\mathbf{i} + \cos\left(\ln\left(\frac{1}{\sqrt{2}}s + 1\right)\right)\mathbf{j}\right]$.

8. $\mathbf{r}'(t) = 2\,\mathbf{i} + \mathbf{j} - 5\,\mathbf{k}$, $ds/dt = |\mathbf{r}'(t)| = \sqrt{4 + 1 + 25} = \sqrt{30}$ and $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{30}\,du = \sqrt{30}\,t$
$\Rightarrow\quad t(s) = \frac{1}{\sqrt{30}}s$. Therefore, $\mathbf{r}(t(s)) = \left(1 + \frac{2}{\sqrt{30}}s\right)\mathbf{i} + \left(3 + \frac{1}{\sqrt{30}}s\right)\mathbf{j} - \frac{5}{\sqrt{30}}s\,\mathbf{k}$.

9. $|\mathbf{r}'(t)| = \sqrt{(3\cos t)^2 + 16 + (-3\sin t)^2} = \sqrt{9 + 16} = 5$ and $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t \quad\Rightarrow$
$t(s) = \frac{1}{5}s$. Therefore, $\mathbf{r}(t(s)) = 3\sin\left(\frac{1}{5}s\right)\mathbf{i} + \frac{4}{5}s\,\mathbf{j} + 3\cos\left(\frac{1}{5}s\right)\mathbf{k}$.

10. $\mathbf{r}'(t) = \dfrac{-4t}{(t^2+1)^2}\,\mathbf{i} + \dfrac{-2t^2+2}{(t^2+1)^2}\,\mathbf{j}$.

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}$$

$$= \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}$$

Since the initial point $(1,0)$ corresponds to $t=0$, the arc length function $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \;\Rightarrow\; t = \tan\frac{1}{2}s$. Substituting, we have

$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2(\frac{1}{2}s)+1} - 1\right]\mathbf{i} + \frac{2\tan(\frac{1}{2}s)}{\tan^2(\frac{1}{2}s)+1}\,\mathbf{j} = \frac{1-\tan^2(\frac{1}{2}s)}{1+\tan^2(\frac{1}{2}s)}\,\mathbf{i} + \frac{2\tan(\frac{1}{2}s)}{\sec^2(\frac{1}{2}s)}\,\mathbf{j}$$

$$= \frac{1-\tan^2(\frac{1}{2}s)}{\sec^2(\frac{1}{2}s)}\,\mathbf{i} + 2\tan(\tfrac{1}{2}s)\cos^2(\tfrac{1}{2}s)\,\mathbf{j}$$

$$= \left[\cos^2(\tfrac{1}{2}s) - \sin^2(\tfrac{1}{2}s)\right]\mathbf{i} + 2\sin(\tfrac{1}{2}s)\cos(\tfrac{1}{2}s)\,\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ (k an integer) but then $t = \tan(\frac{1}{2}s)$ is undefined.

11. (a) $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}$. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle$ or $\left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\sin t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle \;\Rightarrow\; |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 0 + 4\cos^2 t} = \frac{2}{\sqrt{29}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/\sqrt{29}}{\sqrt{29}} = \dfrac{2}{29}$.

12. (a) $\mathbf{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5t^2} = \sqrt{5}\,t$ (since $t>0$). Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \;\Rightarrow\; |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$

13. (a) $\mathbf{r}'(t) = \langle t^2, 2t, 2\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$. Then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{t^2+2}\langle t^2, 2t, 2\rangle.$$

$$\mathbf{T}'(t) = \frac{-2t}{(t^2+2)^2}\langle t^2, 2t, 2\rangle + \frac{1}{t^2+2}\langle 2t, 2, 0\rangle \qquad \text{[by Theorem 10.2.3 \#3]}$$

$$= \frac{1}{(t^2+2)^2}\langle -2t^3, -4t^2, -4t\rangle + \frac{1}{(t^2+2)^2}\langle 2t^3 + 4t, 2t^2 + 4, 0\rangle = \frac{1}{(t^2+2)^2}\langle 4t, 4 - 2t^2, -4t\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(t^2+2)^2}\sqrt{16t^2 + (16 - 16t^2 + 4t^4) + 16t^2} = \frac{1}{(t^2+2)^2}\sqrt{4t^4 + 16t^2 + 16}$$

$$= \frac{1}{(t^2+2)^2}\sqrt{4(t^2+2)^2} = \frac{2(t^2+2)}{(t^2+2)^2} = \frac{2}{t^2+2}$$

Thus $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/(t^2+2)^2}{2/(t^2+2)}\langle 4t, 4 - 2t^2, -4t\rangle = \dfrac{1}{t^2+2}\langle 2t, 2 - t^2, -2t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(t^2+2)}{t^2+2} = \dfrac{2}{(t^2+2)^2}$

14. (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2 + 4 + (1/t)^2}}\langle 2t, 2, 1/t\rangle = \dfrac{|t|}{2t^2+1}\langle 2t, 2, 1/t\rangle$. But since the $\mathbf{k}$-component is $\ln t$, t is positive, $|t| = t$ and $\mathbf{T}(t) = \dfrac{1}{2t^2+1}\langle 2t^2, 2t, 1\rangle$. Then

$$\mathbf{T}'(t) = \frac{1}{2t^2+1}\langle 4t, 2, 0\rangle - (2t^2+1)^{-2}(4t)\langle 2t^2, 2t, 1\rangle = \frac{1}{(2t^2+1)^2}\langle 4t, 2 - 4t^2, -4t\rangle, \text{ so}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle 4t, 2 - 4t^2, -4t\rangle}{\sqrt{(4t)^2 + (2-4t^2)^2 + (-4t)^2}} = \frac{1}{2t^2+1}\langle 2t, 1 - 2t^2, -2t\rangle.$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{2t^2+1}\left(\dfrac{t}{2t^2+1}\right) = \dfrac{2t}{(2t^2+1)^2}$

15. $\mathbf{r}'(t) = 2t\,\mathbf{i} + \mathbf{k}$, $\mathbf{r}''(t) = 2\mathbf{i}$, $|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 0^2 + 1^2} = \sqrt{4t^2+1}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2$.
Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2}{\left(\sqrt{4t^2+1}\right)^3} = \dfrac{2}{(4t^2+1)^{3/2}}$.

16. $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + 1^2 + (2t)^2} = \sqrt{4t^2+2}$,
$\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\mathbf{i} - 2\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2^2 + 2^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$. Then
$\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{2}\sqrt{2t^2+1}\right)^3} = \dfrac{1}{(2t^2+1)^{3/2}}$.

17. $\mathbf{r}'(t) = \langle \cos t, -\sin t, \cos t\rangle$, $\mathbf{r}''(t) = \langle -\sin t, -\cos t, -\sin t\rangle$, $|\mathbf{r}'(t)|^3 = \left(\sqrt{\cos^2 t + 1}\right)^3$,
$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 1, 0, -1\rangle| = \sqrt{2}$, $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{2}}{(1 + \cos^2 t)^{3/2}}$

18. $\mathbf{r}'(t) = \langle e^t \cos t - e^t \sin t, e^t \cos t + e^t \sin t, 1\rangle$. The point $(1, 0, 0)$ corresponds to $t = 0$, and $\mathbf{r}'(0) = \langle 1, 1, 1\rangle$ $\Rightarrow$ $|\mathbf{r}'(0)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

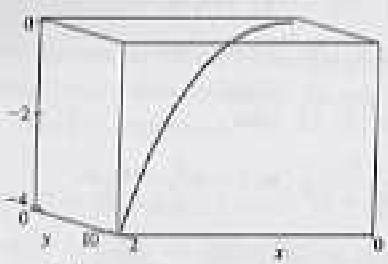
$\mathbf{r}''(t) = \langle e^t \cos t - e^t \sin t - e^t \cos t - e^t \sin t, e^t \cos t - e^t \sin t + e^t \cos t + e^t \sin t, 0\rangle$
$= \langle -2e^t \sin t, 2e^t \cos t, 0\rangle$ $\Rightarrow$ $\mathbf{r}''(0) = \langle 0, 2, 0\rangle$. $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2, 0, 2\rangle$.

$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$. Then $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{(\sqrt{3})^3} = \dfrac{2\sqrt{2}}{3\sqrt{3}}$

or $\dfrac{2\sqrt{6}}{9}$.

19. $\mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t}\rangle$. The point $(0, 1, 1)$ corresponds to $t = 0$, and $\mathbf{r}'(0) = \langle \sqrt{2}, 1, -1\rangle$ $\Rightarrow$
$|\mathbf{r}'(0)| = \sqrt{(\sqrt{2})^2 + 1^2 + (-1)^2} = 2$. $\mathbf{r}''(t) = \langle 0, e^t, e^{-t}\rangle$ $\Rightarrow$ $\mathbf{r}''(0) = \langle 0, 1, 1\rangle$.
$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 2, -\sqrt{2}, \sqrt{2}\rangle$, $|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{2^2 + (-\sqrt{2})^2 + (\sqrt{2})^2} = \sqrt{8} = 2\sqrt{2}$. Then
$\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{2^3} = \dfrac{\sqrt{2}}{4}$.

20.



$\mathbf{r}(t) = \left\langle t, 4t^{3/2}, -t^2\right\rangle$ $\Rightarrow$ $\mathbf{r}'(t) = \left\langle 1, 6t^{1/2}, -2t\right\rangle$,
$\mathbf{r}''(t) = \left\langle 0, 3t^{-1/2}, -2\right\rangle$, $|\mathbf{r}'(t)|^3 = (1 + 36t + 4t^2)^{3/2}$,
$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -12t^{1/2} + 6t^{1/2}, 2, 3t^{-1/2}\right\rangle$ $\Rightarrow$
$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t + 4 + 9t^{-1}} = \left[\dfrac{36t^2 + 4t + 9}{t}\right]^{1/2}$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \left(\frac{36t^2 + 4t + 9}{t}\right)^{1/2} \frac{1}{(1 + 36t + 4t^2)^{3/2}} = \frac{\sqrt{36t^2 + 4t + 9}}{t^{1/2}(1 + 36t + 4t^2)^{3/2}}.$$

The point $(1, 4, -1)$ corresponds to $t = 1$, so the curvature at this point is $\kappa(1) = \dfrac{\sqrt{36 + 4 + 9}}{(1 + 36 + 4)^{3/2}} = \dfrac{7}{41\sqrt{41}}$.

21. $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $\kappa(x) = \dfrac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \dfrac{6|x|}{(1 + 9x^4)^{3/2}}$

22. $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$,

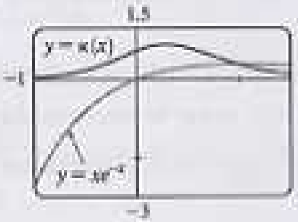
$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \frac{|-\cos x|}{\left[1 + (-\sin x)^2\right]^{3/2}} = \frac{|\cos x|}{(1 + \sin^2 x)^{3/2}}$$

23. $f(x) = 4x^{5/2}$, $f'(x) = 10x^{3/2}$, $f''(x) = 15x^{1/2}$,

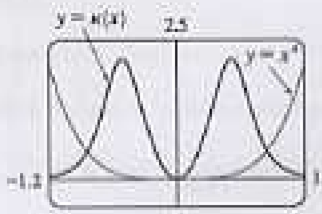
$$\kappa(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \frac{\left|15x^{1/2}\right|}{\left[1 + (10x^{3/2})^2\right]^{3/2}} = \frac{15\sqrt{x}}{(1 + 100x^3)^{3/2}}$$

28. $y = xe^{-x} \Rightarrow y' = e^{-x}(1-x), y'' = e^{-x}(x-2)$, and $\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{e^{-x}|x-2|}{[1+e^{-2x}(1-x)^2]^{3/2}}$.

The graph of the curvature here is what we would expect. The graph of xe^{-x} is bending most sharply slightly to the right of the origin. As $x \to \infty$, the graph of xe^{-x} is asymptotic to the x-axis, and so the curvature approaches zero.



29. $y = x^4 \Rightarrow y' = 4x^3, y'' = 12x^2$, and $\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{12x^2}{(1+16x^6)^{3/2}}$. The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^4$ is very flat around the origin, and so here the curvature is zero.



30. Notice that the curve a is highest for the same x-values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y = \kappa(x)$, and b is the graph of $y = f(x)$.

31. Notice that the curve b has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve a isn't near 0 there. Thus, a must be the graph of $y = f(x)$ rather than the graph of curvature, and b is the graph of $y = \kappa(x)$.

32. Here $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$, $\mathbf{r}''(t) = \langle f''(t), g''(t) \rangle$,

$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2}\right]^3 = \left[(f'(t))^2 + (g'(t))^2\right]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)g''(t) - f''(t)g'(t) \rangle| = \left[(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2\right]^{1/2} = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$.

Thus $\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

48. First we find the quantities required to compute κ:

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \Rightarrow \mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle \Rightarrow \mathbf{r}'''(t) = \langle a\sin t, -a\cos t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\,\mathbf{i} - ab\cos t\,\mathbf{j} + a^2\,\mathbf{k}$$

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4}$$

$$(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t) = (ab\sin t)(a\sin t) + (-ab\cos t)(-a\cos t) + (a^2)(0) = a^2b$$

Then by Theorem 10,

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{a^2b^2+a^4}}{\left(\sqrt{a^2+b^2}\right)^3} = \frac{a\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}\right)^3} = \frac{a}{a^2+b^2}$$

which is a constant.

From Exercise 47(d), the torsion τ is given by

$$\tau = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}'\times\mathbf{r}''|^2} = \frac{a^2b}{\left(\sqrt{a^2b^2+a^4}\right)^2} = \frac{b}{a^2+b^2}$$

which is also a constant.

49. For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi)\rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and z increases by 34 angstroms for each increase of 2π in t. Using the arc length formula, letting t go from 0 to $2.9\times10^8\times2\pi$, we find the approximate length of each helix to be

$$\begin{aligned} L &= \int_0^{2.9\times10^8\times2\pi} |\mathbf{r}'(t)|\,dt \\ &= \int_0^{2.9\times10^8\times2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\,dt \\ &= \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\,t\,\Big]_0^{2.9\times10^8\times2\pi} \\ &= 2.9\times10^8\times2\pi\sqrt{100+\left(\tfrac{34}{2\pi}\right)^2} \\ &\approx 2.07\times10^{10}\text{ Å — more than two meters!} \end{aligned}$$