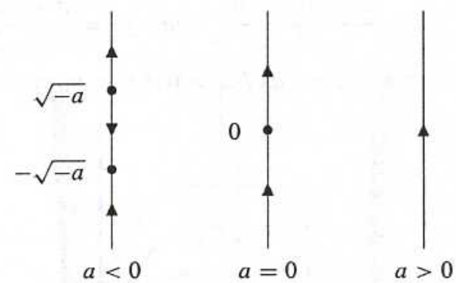
- For $a < 0$, $dy/dt < 0$ for $-\sqrt{-a} < y < \sqrt{-a}$, and $dy/dt > 0$ for $y < -\sqrt{-a}$ and for $y > \sqrt{-a}$.

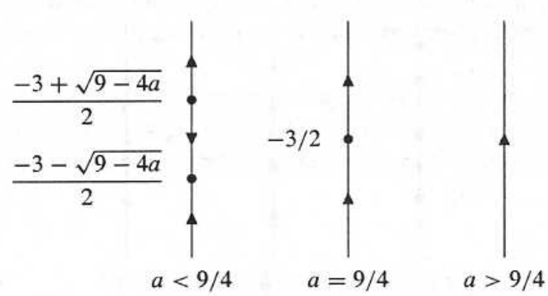


Phase lines for $a < 0$, $a = 0$, and $a > 0$.

2. The equilibrium points occur at solutions of $dy/dt = y^2 + 3y + a = 0$. From the quadratic formula, we have

$$y = \frac{-3 \pm \sqrt{9 - 4a}}{2}.$$

Hence, the bifurcation value of a is $9/4$. For $a < 9/4$, there are two equilibria, one source and one sink. For $a = 9/4$, there is one equilibrium which is a node, and for $a > 9/4$, there are no equilibria.



Phase lines for $a < 9/4$, $a = 9/4$, and $a > 9/4$.

3. The equilibrium points occur at solutions of $dy/dt = y^2 - ay + 1 = 0$. From the quadratic formula, we have

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

If $-2 < a < 2$, then $a^2 - 4 < 0$, and there are no equilibrium points. If $a > 2$ or $a < -2$, there are two equilibrium points. For $a = \pm 2$, there is one equilibrium point at $y = a/2$. The bifurcations occur at $a = \pm 2$.

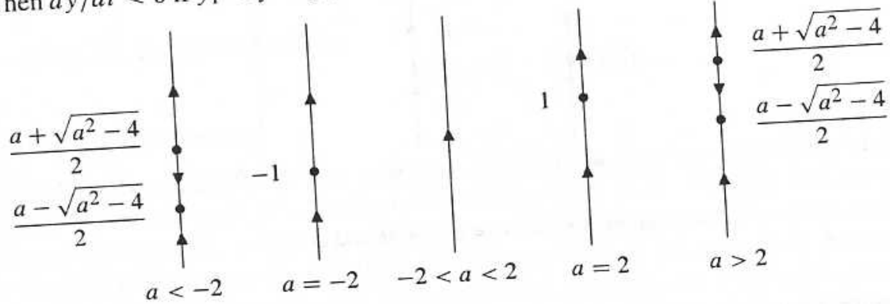
To draw the phase lines, note that:

- For $-2 < a < 2$, $dy/dt = y^2 - ay + 1 > 0$, so the solutions are always increasing.
- For $a = 2$, $dy/dt = (y - 1)^2 \geq 0$, and $y = 1$ is a node.

- For $a = -2$, $dy/dt = (y+1)^2 \geq 0$, and $y = -1$ is a node.
- For $a < -2$ or $a > 2$, let

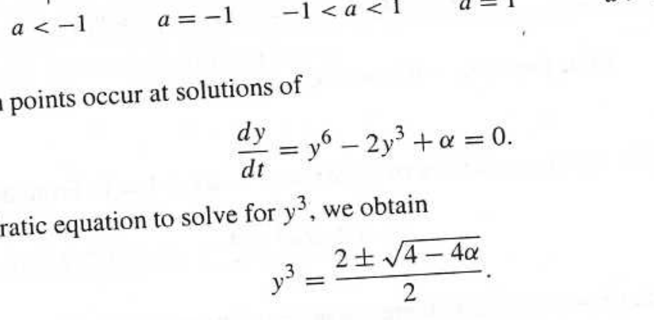
$$y_1 = \frac{a - \sqrt{a^2 - 4}}{2} \quad \text{and} \quad y_2 = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Then $dy/dt < 0$ if $y_1 < y < y_2$, and $dy/dt > 0$ if $y < y_1$ or $y > y_2$.



The five possible phase lines.

4. The bifurcation values are $a = \pm 1$. For $a < -1$, $dy/dt < 0$ for all y. For $a = -1$, there are equilibria at $y = n\pi$, where $n = 0, \pm 2, \pm 4, \ldots$. All of these equilibria are nodes. For $-1 < a < 1$, there are source-sink pairs of equilibria occurring periodically in y. For $a = 1$, there are equilibria at $y = n\pi$, where $n = \pm 1, \pm 3, \ldots$. All of these equilibria are nodes. For $a > 1$, $dy/dt > 0$ for all y.



5. The equilibrium points occur at solutions of

$$\frac{dy}{dt} = y^6 - 2y^3 + \alpha = 0.$$

Using the quadratic equation to solve for y^3, we obtain

$$y^3 = \frac{2 \pm \sqrt{4 - 4\alpha}}{2}.$$

So the equilibrium points are at

$$y = \left(1 \pm \sqrt{1 - \alpha}\right)^{1/3}.$$

If $\alpha > 1$, there are no equilibrium points because this equation has no real solutions. If $\alpha < 1$, the differential equation has two equilibrium points. A bifurcation occurs at $\alpha = 1$ where the differential equation has one equilibrium point at $y = 1$.

8. First we set $C = 0$ so that the population can grow. Once the population reaches the desired level ($P = 100{,}000$), then we set $C = 100{,}000k = 200{,}000$. With this value of C, the population $P = 100{,}000$ is an equilibrium point.

We must be diligent in our management of the population since the equilibrium point is a source. A small change in P due to random fluctuations will eventually cause extinction or explosion of the population.

9. **(a)** A model of the fish population that includes fishing is

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L,$$

where L is the number of licenses issued. The coefficient of 3 represents the average catch of 3 fish per year. As L is increased, the two equilibrium points for $L = 0$ (at $P = 0$ and $P = 100$) will move together. If L is sufficiently large, there are no equilibrium points. Hence we wish to pick L as large as possible so that there is still an equilibrium point present. In other words, we want the bifurcation value of L. The bifurcation value of L occurs if the equation

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L = 0$$

has just one solution for P in terms of L. Using the quadratic formula, we see that there is exactly one equilibrium point if $L = 50/3$. Since this value of L is not an integer, the largest number of licenses that should be allowed is 16.

(b) If we allow the fish population to come to equilibrium then the population will be at the carrying capacity, which is $P = 100$ if $L = 0$. If we then allow 16 licenses to be issued, we expect that the population is a solution to the new model with $L = 16$ and initial population $P = 100$. The model becomes

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 48,$$

which has a source at $P = 40$ and a sink at $P = 60$.
Thus, any initial population greater than 40 when fishing begins tends to the equilibrium level $P = 60$. If the initial population of fish was less than 40 when fishing begins, then the model predicts that the population will decrease to zero in a finite amount of time.

(c) The maximum "number" of licenses is $16\frac{2}{3}$. With $L = 16\frac{2}{3}$, there is an equilibrium at $P = 50$. This equilibrium is a node, and if $P(0) > 50$, the population will approach 50 as t increases. However, it is dangerous to allow this many licenses since an unforeseen event might cause the death of a few extra fish. That event would push the number of fish below the equilibrium value of $P = 50$. In this case, $dP/dt < 0$, and the population decreases to extinction.
If, however, we restrict to $L = 16$ licenses, then there are two equilibria, a sink at $P = 60$ and source at $P = 40$. As long as $P(0) > 40$, the population will tend to 60 as t increases. In this case, we have a small margin of safety. If $P \approx 60$, then it would have to drop to less than 40 before the fish are in danger of extinction.

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if $\alpha \geq 3/4$. As α decreases from $\alpha = 1$, a source and sink approach one another. At $\alpha = 3/4$, they coalesce and form a node. For $\alpha < 3/4$, there are no positive equilibria, and all solutions approach zero as $t \to \infty$.

(b) The bifurcation value is $\alpha = 3/4$.

15. **(a)** If $a = 0$, there is a single equilibrium point at $y = 0$. For $a \neq 0$, the equilibrium points occur at $y = 0$ and $y = a$. If $a < 0$, the equilibrium point at $y = 0$ is a sink and the equilibrium point at $y = a$ is a source. If $a > 0$, the equilibrium point at $y = 0$ is a source and the equilibrium point at $y = a$ is a sink.

Phase lines for $dy/dt = ay - y^2$.

(b) Given the results in part (a), there is one bifurcation value, $a = 0$.

(c) The equilibrium points satisfy the equation

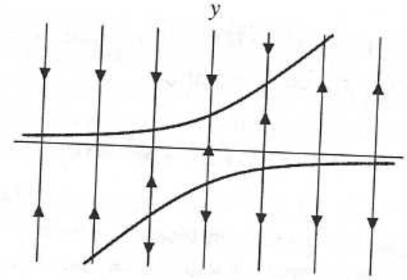
$$r + ay - y^2 = 0.$$

Solving it, we obtain

$$y = \frac{a \pm \sqrt{a^2 + 4r}}{2}.$$

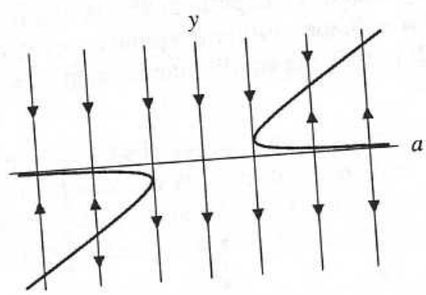
Hence, there are no equilibrium points if $a^2 + 4r < 0$, one equilibrium point if $a^2 + 4r = 0$, and two equilibrium points if $a^2 + 4r > 0$.

If $r > 0$, we always have two equilibrium points.



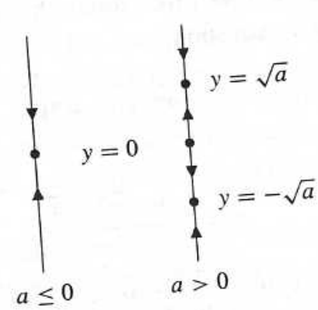
The bifurcation diagram for $r > 0$.

(d) If $r < 0$, there are no equilibrium points if $a^2+4r < 0$. In other words, there are no equilibrium points if $-2\sqrt{-r} < a < 2\sqrt{-r}$. If $a = \pm 2\sqrt{-r}$, there is a single equilibrium point, and if $|a| > 2\sqrt{-r}$, there are two equilibrium points.



The bifurcation diagram for $r < 0$.

16. **(a)** If $a \leq 0$, there is a single equilibrium point at $y = 0$, and it is a sink. For $a > 0$, there are equilibrium points at $y = 0$ and $y = \pm\sqrt{a}$. The equilibrium point at $y = 0$ is a source, and the other two are sinks.



Phase lines for $dy/dt = ay - y^3$.

(b) Given the results in part (a), there is one bifurcation value, $a = 0$.

(c) The equilibrium points satisfy the cubic equation

$$r + ay - y^3 = 0.$$

Rather than solving it explicitly, we rely on `PhaseLines`. If $r > 0$, there is a positive bifurcation value $a = a_0$. For $a < a_0$, the phase line has one equilibrium point, a positive sink. If $a > a_0$, there are two negative equilibria in addition to the positive sink. The larger of the two negative equilibria is a source and the smaller is a sink.

We have $f_\alpha(0) + r > 0$ and $f_\alpha(1) + r > 0$ for $0 \le r \le M$, so the index of $f_\alpha(y) + r$ must b
zero for $0 \le r \le M$.

20. To compute the equilibria, we solve

$$\frac{dy}{dt} = y^3 + \alpha y^2 = 0.$$

Since $y^3 + \alpha y^2 = y^2(y + \alpha)$, we see that $y = 0$ and $y = -\alpha$ are equilibria. By considering the sig
of $y^3 + \alpha y^2$, we see that $y = 0$ is always a node and $y = -\alpha$ is always a source.

These equilibria coincide if $\alpha = 0$. As α passes through zero, the source "passes through" th
node at zero. The only change is in the direction that solutions travel for initial conditions near th
node.

21. We have

$$\frac{dy}{dt} = y^4 + \alpha y^2 = y^2(y^2 + \alpha).$$

If $\alpha > 0$, there is one equilibrium point at $y = 0$, and $dy/dt > 0$ otherwise. Hence $y = 0$ is a node

If $\alpha < 0$, $dy/dt = 0$ at $y = 0$ and $y = \pm\sqrt{-\alpha}$. From the sign of $y^4 + \alpha y^2$, we know that $y =$
is a node, $y = -\sqrt{-\alpha}$ is a sink, and $y = \sqrt{-\alpha}$ is a source.

22. The equilibria occur at $y = 0$ and

$$y = \pm\sqrt{\frac{1000\alpha \pm \sqrt{10^6\alpha^2 - 4\alpha}}{2}}.$$

If $\alpha < 0$, then $10^6\alpha^2 - 4\alpha > 10^6\alpha^2 > 0$. Hence

$$\frac{1000\alpha + \sqrt{10^6\alpha^2 - 4\alpha}}{2} > 0 \quad \text{but} \quad \frac{1000\alpha - \sqrt{10^6\alpha^2 - 4\alpha}}{2} < 0.$$

Therefore, there are three equilibria.

If $0 < \alpha < 4 \times 10^{-6}$, then $10^6\alpha^2 - 4\alpha < 0$. Consequently, there are no equilibria other than t
one at $y = 0$.

If $\alpha = 4 \times 10^{-6}$, then there are three equilibria. Finally, if $\alpha > 4 \times 10^{-6}$, then both

$$\frac{1000\alpha + \sqrt{10^6\alpha^2 - 4\alpha}}{2} \quad \text{and} \quad \frac{1000\alpha - \sqrt{10^6\alpha^2 - 4\alpha}}{2}$$

are positive, and there are five equilibria.

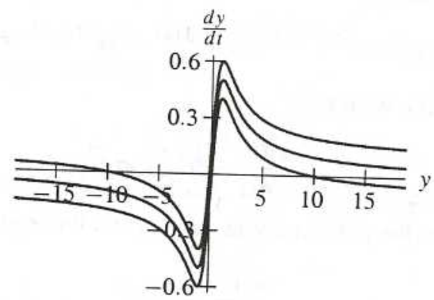
23. For $\alpha < 0$, both $y^2 + \alpha$ and $y^4 + \alpha$ have two roots. These four roots are distinct. Therefore, t
product, $y(y + \alpha)(y^2 + \alpha)(y^4 + \alpha)$ has six distinct roots for $\alpha < 0$. For $\alpha = 0$, the different
equation is

$$\frac{dy}{dt} = y^8,$$

which has only one equilibrium point, $y = 0$.

24. Note that $y/(y^2+1) \to 0$ as $y \to \pm\infty$. For $\alpha = 0$, the differential equation has only one equilibri
point, $y = 0$. Changing α (slightly) translates the graph up or down, and this translation mo
the source that was at 0 to a source that is near 0. It also introduces one new equilibrium po

far from 0. If $\alpha < 0$, then this new equilibrium point is positive and far from 0. If $\alpha > 0$, this new equilibrium point is negative and far from 0. This bifurcation is sometimes referred to as a "bifurcation at infinity."



Graphs of $y/(y^2+1)+\alpha$ for several values of α near zero.

EXERCISES FOR SECTION 1.8

1. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{t} = 2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/t)\,dt} = e^{\ln t} = t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t\frac{dy}{dt} + y = 2t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(ty)}{dt} = 2t,$$

and integrating both sides with respect to t, we obtain

$$ty = t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{1}{t}(t^2 + c) = t + \frac{c}{t}.$$

4. We rewrite the equation in the form

$$\frac{dy}{dt} + 2ty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 4.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{t^2}y)}{dt} = 4,$$

and integrating both sides with respect to t, we obtain

$$e^{t^2}y = 4t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

5. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{2t}{1+t^2}\,y = \frac{2}{1+t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-2t/(1+t^2))\,dt} = e^{-\ln(1+t^2)} = \left(e^{\ln(1+t^2)}\right)^{-1} = \frac{1}{1+t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{1+t^2}\frac{dy}{dt} - \frac{2t}{(1+t^2)^2}\,y = \frac{2}{(1+t^2)^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{1+t^2}\right) = \frac{2}{(1+t^2)^2}.$$

An antiderivative of $2/(1+t^2)^2$ is

$$\arctan t + \frac{t}{1+t^2},$$

so integrating both sides with respect to t, we obtain

$$\frac{y}{1+t^2} = \arctan t + \frac{t}{1+t^2} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t + (1+t^2)(\arctan(t) + c).$$

11. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{t} = 2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/t)\,dt} = e^{\ln t} = t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t\frac{dy}{dt} + y = 2t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(ty)}{dt} = 2t,$$

and integrating both sides with respect to t, we obtain

$$ty = t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t + \frac{c}{t}.$$

To find the solution that satisfies the initial condition $y(1) = 3$, we evaluate the general solution at $t = 1$ and obtain

$$1 + c = 3.$$

Hence, $c = 2$, and the desired solution is

$$y(t) = t + \frac{2}{t}.$$

12. In Exercise 5, we derived the general solution

$$y(t) = t + (1 + t^2)(\arctan(t) + c).$$

To find the solution that satisfies the initial condition $y(0) = -2$, we evaluate the general solution at $t = 0$ and obtain $c = -2$. The desired solution is

$$y(t) = t + (1 + t^2)(\arctan(t) - 2).$$

13. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{2y}{t} = 2t^2$$

and note that the integrating factor is

$$\mu(t) = e^{\int -(2/t)\,dt} \quad 1$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2y}{t^3} = 2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 2,$$

and integrating both sides with respect to t, we obtain

$$\frac{y}{t^2} = 2t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 2t^3 + ct^2.$$

To find the solution that satisfies the initial condition $y(-2) = 4$, we evaluate the general solution at $t = -2$ and obtain

$$-16 + 4c = 4.$$

Hence, $c = 5$, and the desired solution is

$$y(t) = 2t^3 + 5t^2.$$

14. We rewrite the equation in the form

$$\frac{dy}{dt} + 5y = \sin t$$

and note that the integrating factor is

$$\mu(t) = e^{\int 5\,dt} = e^{5t}.$$

Multiplying both sides of the differential equation by $\mu(t)$, we obtain

$$e^{5t}\frac{dy}{dt} + 5e^{5t}y = e^{5t}\sin t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{5t}y)}{dt} = e^{5t}\sin t,$$

and integrating both sides with respect to t, we obtain

$$e^{5t}y = \int e^{5t}\sin t\,dt.$$

We can calculate the integral on the right-hand side using integration by parts twice. We have

$$e^{5t}y = \tfrac{5}{26}e^{5t}\sin t - \tfrac{1}{26}e^{5t}\cos t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \tfrac{5}{26}\sin t - \tfrac{1}{26}\cos t + ce^{-5t}.$$

To find the solution that satisfies the initial condition $y(0) = 1$, we evaluate the general solution at $t = 0$ and obtain

$$-\tfrac{1}{26} + c = 1.$$

Hence, $c = 27/26$, and the desired solution is

$$y(t) = \tfrac{5}{26}\sin t - \tfrac{1}{26}\cos t + \tfrac{27}{26}e^{-5t}.$$

15. We rewrite the equation in the form

$$\frac{dy}{dt} - (\sin t)y = 4$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-\sin t)\,dt} = e^{\cos t}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{\cos t}\frac{dy}{dt} - e^{\cos t}(\sin t)y = 4e^{\cos t}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{\cos t}y)}{dt} = 4e^{\cos t},$$

and integrating both sides with respect to t, we obtain

$$e^{\cos t}y = \int 4e^{\cos t}\,dt.$$

Since the integral on the right is impossible to express using elementary functions, we write the general solution as

$$y(t) = 4e^{-\cos t}\int e^{\cos t}\,dt.$$

16. We rewrite the equation in the form

$$\frac{dy}{dt} - t^2y = 4$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-t^2)\,dt} = e^{-t^3/3}.$$

Multiplying both sides of the equation by $\mu(t)$, we obtain

$$e^{-t^3/3}\frac{dy}{dt} - t^2e^{-t^3/3}y = 4e^{-t^3/3}.$$

20. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{y}{\sqrt{t^3-3}} = t$$

and note that the integrating factor is

$$\mu(t) = e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}.$$

This integral is impossible to express in terms of elementary functions. Multiplying both sides by $\mu(t)$, we obtain

$$e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\frac{dy}{dt} - e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\frac{y}{\sqrt{t^3-3}} = te^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d\left(e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\,y\right)}{dt} = te^{-\int \frac{1}{\sqrt{t^3-3}}\,dt},$$

and integrating both sides with respect to t, we obtain

$$e^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\,y = \int te^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\,dt.$$

These integrals are also impossible to compute, so we write the general solution in the form

$$y(t) = e^{\int \frac{1}{\sqrt{t^3-3}}\,dt}\int te^{-\int \frac{1}{\sqrt{t^3-3}}\,dt}\,dt.$$

21. We rewrite the equation in the form

$$\frac{dy}{dt} - aty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int(-at)\,dt} = e^{-at^2/2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{-at^2/2}\frac{dy}{dt} - ate^{-at^2/2}y = 4e^{-t^2}e^{-at^2/2}.$$

Applying the Product Rule to the left-hand side and simplifying the right-hand side, we see that this equation is the same as

$$\frac{d(e^{-at^2/2}y)}{dt} = 4e^{-(1+a/2)t^2}.$$

Integrating both sides with respect to t, we obtain

$$e^{-at^2/2}y = \int 4e^{-(1+a/2)t^2}\,dt.$$

The integral on the right-hand side can be expressed in terms of elementary functions only if $1 + a/2 = 0$ (that is, if the factor involving e^{t^2} really isn't there). Hence, the only value of a that [we c]an express in closed form is $a = -2$ (see Exercise 4).

The integrating factor is

$$\mu(t) = e^{\int 1/(15+t)\,dt} = e^{\ln(15+t)} = 15 + t.$$

Multiplying both sides of the equation by $\mu(t)$, we obtain

$$(15+t)\frac{dS}{dt} + S = 2(15+t),$$

which via the Product Rule is equivalent to

$$\frac{d((15+t)S)}{dt} = 30 + 2t.$$

Integration and simplification yields

$$S(t) = \frac{t^2 + 30t + c}{15 + t}.$$

Using the initial condition $S(0) = 6$, we have

$$\frac{c}{15} = 6,$$

which implies that $c = 90$ and

$$S(t) = \frac{t^2 + 30t + 90}{15 + t}.$$

The tank is full when $t = 15$, and the amount of salt at that time is $S(15) = 51/2$ pounds.

27. We will use the term "parts" as shorthand for the product of parts per billion of dioxin and the volume of water in the tank. Basically this product represents the total amount of dioxin in the tank. The tank initially contains 200 gallons at a concentration of 2 parts per billion, which results in 400 parts of dioxin.

Let $y(t)$ be the amount of dioxin in the tank at time t. Since water with 4 parts per billion of dioxin flows in at the rate of 5 gallons per minute, 20 parts of dioxin enter the tank each minute. Also, the volume of water in the tank at time t is $200 + 2t$, so the concentration of dioxin in the tank is $y/(200 + 2t)$. Since well-mixed water leaves the tank at the rate of 2 gallons per minute, the differential equation that represents the change in the amount of dioxin in the tank is

$$\frac{dy}{dt} = 20 - 2\left(\frac{y}{200 + 2t}\right),$$

which simplifies to

$$\frac{dy}{dt} = 20 - \left(\frac{1}{100 + t}\right) y.$$

We can rewrite this equation as

$$\frac{dy}{dt} + \left(\frac{1}{100 + t}\right) y = 20,$$

and the integrating factor is

$$\mu(t) = e^{\int (1/(100+t))\,dt} = e^{\ln(100+t)} = 100 + t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(100+t)\frac{dy}{dt} + y = 20(100+t),$$

which is equivalent to

$$\frac{d((100+t)y)}{dt} = 20(100+t)$$

by the Product Rule. Integrating both sides with respect to t, we obtain

$$(100+t)y = 2000t + 10t^2 + c.$$

Since $y(0) = 400$, we see that $c = 40,000$. Therefore,

$$y(t) = \frac{10t^2 + 2000t + 40,000}{t+100}.$$

The tank fills up at $t = 100$, and $y(100) = 1,700$. To express our answer in terms of concentration, we calculate $y(100)/400 = 4.25$ parts per billion.

28. Let $S(t)$ denote the amount of sugar in the tank at time t. Sugar is added to the tank at the rate of p pounds per minute. The amount of sugar that leaves the tank is the product of the concentration of the sugar in the water and the rate that the water leaves the tank. At time t, there are $100 - t$ gallons of sugar water in the tank, so the concentration of sugar is $S(t)/(100-t)$. Since sugar water leaves the tank at the rate of 1 gallon per minute, the differential equation for S is

$$\frac{dS}{dt} = p - \frac{S}{100-t}.$$

Since this equation is linear, we rewrite it as

$$\frac{dS}{dt} + \frac{S}{100-t} = p,$$

and the integrating factor is

$$\mu(t) = e^{\int (1/(100-t))\,dt} = e^{-\ln(100-t)} = \frac{1}{100-t}.$$

Multiplying both sides of the differential equation by $\mu(t)$ yields

$$\left(\frac{1}{100-t}\right)\frac{dS}{dt} + \frac{S}{(100-t)^2} = \frac{p}{100-t},$$

which is equivalent to

$$\frac{d}{dt}\left(\frac{S}{100-t}\right) = \frac{p}{100-t}$$

by the Product Rule. We integrate both sides and obtain

$$\frac{S}{100-t} = -p\ln(100-t) + c,$$

EXERCISES FOR SECTION 1.9

1. We rewrite the equation as

$$\frac{dy}{dt} = (y - 4t) + (y - 4t)^2 + 4,$$

use that

$$\frac{dy}{dt} = \frac{du}{dt} + 4,$$

and substitute to obtain

$$\frac{du}{dt} + 4 = u + u^2 + 4.$$

This equation simplifies to

$$\frac{du}{dt} = u^2 + u,$$

which is nonlinear, autonomous, and separable.

2. Note that $y = tu$, so

$$\frac{dy}{dt} = t\frac{du}{dt} + u.$$

Substituting for y and dy/dt, we obtain

$$t\frac{du}{dt} + u = \frac{(tu)^2 + t(tu)}{(tu)^2 + 3t^2} = \frac{t^2(u^2 + u)}{t^2(u^2 + 3)}.$$

Simplifying we obtain

$$\frac{du}{dt} = \frac{1}{t}\left(\frac{u^2 + u}{u^2 + 3} - u\right).$$

This equation is nonlinear and nonautonomous.

3. Rewrite the equation as

$$\frac{dy}{dt} = ty + (ty)^2 + \cos(ty).$$

Using that $y = u/t$, we have

$$\frac{dy}{dt} = \frac{1}{t}\frac{du}{dt} - \frac{1}{t^2}u.$$

We substitute to obtain

$$\frac{1}{t}\frac{du}{dt} - \frac{1}{t^2}u = u + u^2 + \cos u,$$

which simplifies to

$$\frac{du}{dt} = \frac{u}{t} + t(u + u^2 + \cos u).$$

This equation is nonlinear and nonautonomous.

This is a good example of a change of variables that looks like it is going to greatly simplify the equation but does not because the term that replaces dy/dt is complicated.

4. We have $y = \ln u$, so

$$\frac{dy}{dt} = \frac{1}{u}\frac{du}{dt}.$$

Substituting we obtain

$$\frac{1}{u}\frac{du}{dt} = u + \frac{t^2}{u},$$

which simplifies to

$$\frac{du}{dt} = u^2 + t^2.$$

This equation is nonlinear and nonautonomous.

5. Let $u = y - t$. Then

$$\frac{du}{dt} = \frac{dy}{dt} - 1,$$

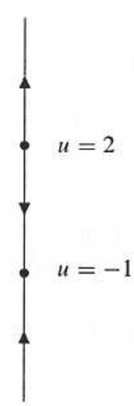
and the differential equation becomes

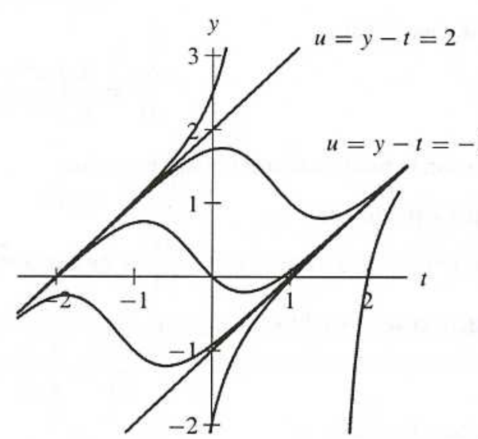
$$\frac{du}{dt} + 1 = u^2 - u - 1,$$

which simplifies to

$$\frac{du}{dt} = u^2 - u - 2 = (u-2)(u+1).$$

The equilibrium points are $u = 2$ (a source) and $u = -1$ (a sink). These equilibria correspond to the solutions $y_1(t) = 2 + t$ and $y_2(t) = -1 + t$.





6. We let $u = y/t$. Then $y = tu$, so

$$\frac{dy}{dt} = u + t\frac{du}{dt}.$$

Replacing y by tu, we obtain

$$u + t\frac{du}{dt} = \frac{(tu)^2}{t} + 2(tu) - 4t + u,$$

In terms of the original dependent variable y, we have

$$y(t) = \pm\sqrt{(t+c)e^{t^2/2}}.$$

The choice of sign depends on the initial condition.

9. If $u = y/(1+t)$, we have

$$\frac{du}{dt} = \frac{1}{1+t}\frac{dy}{dt} - \frac{y}{(1+t)^2}.$$

Then

$$\frac{dy}{dt} = (1+t)\frac{du}{dt} + u,$$

and the differential equation becomes

$$(1+t)\frac{du}{dt} + u = u - \frac{u(1+t)}{t} + t^2(t+1),$$

which reduces to

$$\frac{du}{dt} = -\frac{u}{t} + t^2.$$

This differential equation is linear, and we rewrite it as

$$\frac{du}{dt} + \frac{u}{t} = t^2.$$

Its integrating factor is

$$\mu(t) = e^{\int (1/t)\,dt} = e^{\ln t} = t.$$

Multiplying both sides of the differential equation by $\mu(t)$, we obtain

$$t\frac{du}{dt} + u = t^3,$$

which is equivalent to

$$\frac{d(tu)}{dt} = t^3.$$

Integrating both sides, we have

$$tu = \frac{t^4}{4} + c,$$

where c is an arbitrary constant. Therefore,

$$u(t) = \frac{t^3}{4} + \frac{c}{t}.$$

To determine the general solution for $y(t)$, we have $y = u(1+t)$, and therefore

$$y(t) = \frac{t^3(1+t)}{4} + \frac{c(1+t)}{t}$$

$$= \frac{t^3(1+t)}{4} + \frac{c}{t} + c.$$