$$\sim\begin{bmatrix}1&0&-2&0&0\\0&1&-3&0&0\\0&0&18&-3&0\\0&0&6&-1&0\\0&0&6&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-2&0&0\\0&1&-3&0&0\\0&0&1&-1/6&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-1/3&0\\0&1&0&-1/2&0\\0&0&1&-1/6&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

The general solution is $x_1 = (1/3)x_4$, $x_2 = (1/2)x_4$, $x_3 = (1/6)x_4$, with x_4 free. Take $x_4 = 6$. Then $x_1 = 2$, $x_2 = 3$, and $x_3 = 1$. The balanced equation is

$$2Na_3PO_4 + 3Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + 6NaNO_3$$

7. The following vectors list the numbers of atoms of sodium (Na), hydrogen (H), carbon (C), and oxygen (O):

$$NaHCO_3:\begin{bmatrix}1\\1\\1\\3\end{bmatrix},\ H_3C_6H_5O_7:\begin{bmatrix}0\\8\\6\\7\end{bmatrix},\ Na_3C_6H_5O_7:\begin{bmatrix}3\\5\\6\\7\end{bmatrix},\ H_2O:\begin{bmatrix}0\\2\\0\\1\end{bmatrix},\ CO_2:\begin{bmatrix}0\\0\\1\\2\end{bmatrix}\begin{matrix}\text{sodium}\\\text{hydrogen}\\\text{carbon}\\\text{oxygen}\end{matrix}$$

The order of the various atoms is not important. The list here was selected by writing the elements in the order in which they first appear in the chemical equation, reading left to right:

$$x_1 \cdot NaHCO_3 + x_2 \cdot H_3C_6H_5O_7 \rightarrow x_3 \cdot Na_3C_6H_5O_7 + x_4 \cdot H_2O + x_5 \cdot CO_2.$$

The coefficients $x_1, \ldots, x_5$ satisfy the vector equation

$$x_1\begin{bmatrix}1\\1\\1\\3\end{bmatrix}+x_2\begin{bmatrix}0\\8\\6\\7\end{bmatrix}=x_3\begin{bmatrix}3\\5\\6\\7\end{bmatrix}+x_4\begin{bmatrix}0\\2\\0\\1\end{bmatrix}+x_5\begin{bmatrix}0\\0\\1\\2\end{bmatrix}$$

Move all the terms to the left side (changing the sign of each entry in the third, fourth, and fifth vectors) and reduce the augmented matrix:

$$\begin{bmatrix}1&0&-3&0&0&0\\1&8&-5&-2&0&0\\1&6&-6&0&-1&0\\3&7&-7&-1&-2&0\end{bmatrix}\sim\cdots\sim\begin{bmatrix}1&0&0&0&-1&0\\0&1&0&0&-1/3&0\\0&0&1&0&-1/3&0\\0&0&0&1&-1&0\end{bmatrix}$$

The general solution is $x_1 = x_5$, $x_2 = (1/3)x_5$, $x_3 = (1/3)x_5$, $x_4 = x_5$, and x_5 is free. Take $x_5 = 3$. Then $x_1 = x_4 = 3$, and $x_2 = x_3 = 1$. The balanced equation is

$$3NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + 3H_2O + 3CO_2$$

8. The following vectors list the numbers of atoms of potassium (K), manganese (Mn), oxygen (O), sulfur (S), and hydrogen (H):

$$KMnO_4:\begin{bmatrix}1\\1\\4\\0\\0\end{bmatrix},\ MnSO_4:\begin{bmatrix}0\\1\\4\\1\\0\end{bmatrix},\ H_2O:\begin{bmatrix}0\\0\\1\\0\\2\end{bmatrix},\ MnO_2:\begin{bmatrix}0\\1\\2\\0\\0\end{bmatrix},\ K_2SO_4:\begin{bmatrix}2\\0\\4\\1\\0\end{bmatrix},\ H_2SO_4:\begin{bmatrix}0\\0\\4\\1\\2\end{bmatrix}\begin{matrix}\text{potassium}\\\text{manganese}\\\text{oxygen}\\\text{sulfur}\\\text{hydrogen}\end{matrix}$$

The coefficients in the chemical equation

Rearrange the equations:

$$\begin{aligned} x_1 \quad\quad - x_3 - x_4 \quad\quad &= 40 \\ x_1 + x_2 \quad\quad\quad\quad\quad\quad &= 200 \\ x_2 + x_3 \quad\quad - x_5 &= 100 \\ x_4 + x_5 &= 60 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 0 & 1 & 100 \\ 0 & \textcircled{1} & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & \textcircled{1} & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution (written in the style of Section 1.2) is

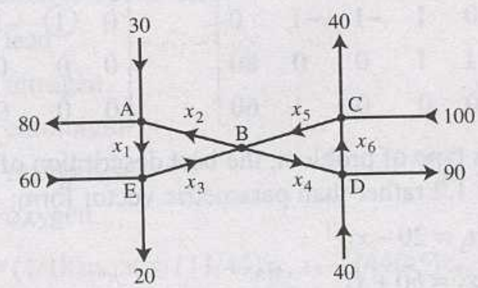
$$\begin{cases} x_1 = 100 + x_3 - x_5 \\ x_2 = 100 - x_3 + x_5 \\ x_3 \text{ is free} \\ x_4 = 60 - x_5 \\ x_5 \text{ is free} \end{cases}$$

b. When $x_4 = 0$, x_5 must be 60, and
$$\begin{cases} x_1 = 40 + x_3 \\ x_2 = 160 - x_3 \\ x_3 \text{ is free} \\ x_4 = 0 \\ x_5 = 60 \end{cases}$$

c. The minimum value of x_1 is 40 cars/minute, because x_3 cannot be negative.

13. Write the equations for each intersection:

Intersection	Flow in		Flow out
A	$x_2 + 30$	=	$x_1 + 80$
B	$x_3 + x_5$	=	$x_2 + x_4$
C	$x_6 + 100$	=	$x_5 + 40$
D	$x_4 + 40$	=	$x_6 + 90$
E	$x_1 + 60$	=	$x_3 + 20$
Total flow:	230	=	230



Rearrange the equations:

$$\begin{aligned} x_1 - x_2 \quad\quad\quad\quad\quad\quad\quad\quad &= -50 \\ x_2 - x_3 + x_4 - x_5 \quad\quad &= 0 \\ x_5 - x_6 &= 60 \\ x_4 \quad\quad - x_6 &= 50 \\ x_1 \quad\quad - x_3 \quad\quad\quad\quad\quad\quad &= -40 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 1 & 0 & -1 & 0 & 0 & 0 & -40 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

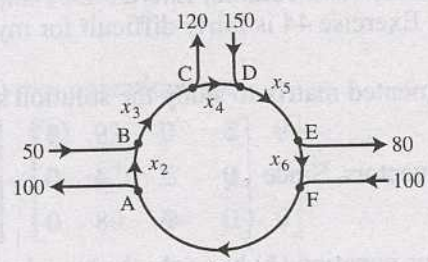
$$\sim\cdots\sim\begin{bmatrix} \textcircled{1} & 0 & -1 & 0 & 0 & 0 & -40 \\ 0 & \textcircled{1} & -1 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & \textcircled{1} & 0 & -1 & 50 \\ 0 & 0 & 0 & 0 & \textcircled{1} & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. The general solution is $\begin{cases} x_1 = x_3 - 40 \\ x_2 = x_3 + 10 \\ x_3 \text{ is free} \\ x_4 = x_6 + 50 \\ x_5 = x_6 + 60 \\ x_6 \text{ is free} \end{cases}$

b. To find minimum flows, note that since x_1 cannot be negative, $x_3 \geq 40$. This implies that $x_2 \geq 50$. Also, since x_6 cannot be negative, $x_4 \geq 50$ and $x_5 \geq 60$. The minimum flows are $x_2 = 50$, $x_3 = 40$, $x_4 = 50$, $x_5 = 60$ (when $x_1 = 0$ and $x_6 = 0$).

14. Write the equations for each intersection.

Intersection	Flow in		Flow out
A	x_1	=	$x_2 + 100$
B	$x_2 + 50$	=	x_3
C	x_3	=	$x_4 + 120$
D	$x_4 + 150$	=	x_5
E	x_5	=	$x_6 + 80$
F	$x_6 + 100$	=	x_1



Rearrange the equations:

$$\begin{aligned} x_1 - x_2 &= 100 \\ x_2 - x_3 &= -50 \\ x_3 - x_4 &= 120 \\ x_4 - x_5 &= -150 \\ x_5 - x_6 &= 80 \\ -x_1 + x_6 &= -100 \end{aligned}$$

Reduce the augmented matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix} \sim\cdots\sim \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7. Since A has 3 columns, B must match with 3 rows. Otherwise, AB is undefined. Since AB has 7 columns, so does B. Thus, B is 3×7.

8. The number of rows of B matches the number of rows of BC, so B has 3 rows.

9. $AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$, while $BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$.

Then $AB = BA$ if and only if $-10 + 5k = 15$ and $-9 = 6 - 3k$, which happens if and only if $k = 5$.

10. $AB = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$, $AC = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ -2 & 14 \end{bmatrix}$

11. $AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 6 & 15 \\ 2 & 12 & 25 \end{bmatrix}$

$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 6 & 9 \\ 5 & 20 & 25 \end{bmatrix}$

Right-multiplication (that is, multiplication on the right) by the diagonal matrix D multiplies each *column* of A by the corresponding diagonal entry of D. Left-multiplication by D multiplies each *row* of A by the corresponding diagonal entry of D. To make $AB = BA$, one can take B to be a multiple of I_3. For instance, if $B = 4I_3$, then AB and BA are both the same as $4A$.

12. Consider $B = [\mathbf{b}_1 \ \ \mathbf{b}_2]$. To make $AB = 0$, one needs $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$. By inspection of A, a suitable $\mathbf{b}_1$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Example: $B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$.

13. Use the definition of AB written in reverse order: $[A\mathbf{b}_1 \ \cdots \ A\mathbf{b}_p] = A[\mathbf{b}_1 \ \cdots \ \mathbf{b}_p]$. Thus

$[Q\mathbf{r}_1 \ \cdots \ Q\mathbf{r}_p] = QR$, when $R = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_p]$.

14. By definition, $UQ = U[\mathbf{q}_1 \ \cdots \ \mathbf{q}_4] = [U\mathbf{q}_1 \ \cdots \ U\mathbf{q}_4]$. From Example 6 of Section 1.8, the vector $U\mathbf{q}_1$ lists the total costs (material, labor, and overhead) corresponding to the amounts of products B and C specified in the vector $\mathbf{q}_1$. That is, the first column of UQ lists the total costs for materials, labor, and overhead used to manufacture products B and C during the first quarter of the year. Columns 2, 3, and 4 of UQ list the total amounts spent to manufacture B and C during the 2nd, 3rd, and 4th quarters, respectively.

15. a. False. See the definition of AB.

b. False. The roles of A and B should be reversed in the second half of the statement. See the box after Example 3.

c. True. See Theorem 2(b), read right to left.

d. True. See Theorem 3(b), read right to left.

e. False. The phrase "in the same order" should be "in the reverse order." See the box after Theorem 3.

16. a. False. AB must be a 3×3 matrix, but the formula for AB implies that it is 3×1. The plus signs should be just spaces (between columns). This is a common mistake.

b. True. See the box after Example 6.

c. False. The left-to-right order of B and C cannot be changed, in general.

d. False. See Theorem 3(d).

e. True. This general statement follows from Theorem 3(b).

17. Since $\begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} = AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad A\mathbf{b}_3]$, the first column of B satisfies the equation $A\mathbf{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Row reduction: $[A \quad A\mathbf{b}_1] \sim \begin{bmatrix} 1 & -2 & -1 \\ -2 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$. So $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$. Similarly, $[A \quad A\mathbf{b}_2] \sim \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & -5 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$.

Note: An alternative solution of Exercise 17 is to row reduce $[A \quad A\mathbf{b}_1 \quad A\mathbf{b}_2]$ with one sequence of row operations. This observation can prepare the way for the inversion algorithm in Section 2.2.

18. The first two columns of AB are $A\mathbf{b}_1$ and $A\mathbf{b}_2$. They are equal since $\mathbf{b}_1$ and $\mathbf{b}_2$ are equal.

19. (A solution is in the text). Write $B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3]$. By definition, the third column of AB is $A\mathbf{b}_3$. By hypothesis, $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$. So $A\mathbf{b}_3 = A(\mathbf{b}_1 + \mathbf{b}_2) = A\mathbf{b}_1 + A\mathbf{b}_2$, by a property of matrix-vector multiplication. Thus, the third column of AB is the sum of the first two columns of AB.

20. The second column of AB is also all zeros because $A\mathbf{b}_2 = A\mathbf{0} = \mathbf{0}$.

21. Let $\mathbf{b}_p$ be the last column of B. By hypothesis, the last column of AB is zero. Thus, $A\mathbf{b}_p = \mathbf{0}$. However, $\mathbf{b}_p$ is not the zero vector, because B has no column of zeros. Thus, the equation $A\mathbf{b}_p = \mathbf{0}$ is a linear dependence relation among the columns of A, and so the columns of A are linearly dependent.

Note: The text answer for Exercise 21 is, "The columns of A are linearly dependent. Why?" The *Study Guide* supplies the argument above, in case a student needs help.

22. If the columns of B are linearly dependent, then there exists a nonzero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. From this, $A(B\mathbf{x}) = A\mathbf{0}$ and $(AB)\mathbf{x} = \mathbf{0}$ (by associativity). Since $\mathbf{x}$ is nonzero, the columns of AB must be linearly dependent.

23. If $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{0}$, then $CA\mathbf{x} = C\mathbf{0} = \mathbf{0}$ and so $I_n\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$. This shows that the equation $A\mathbf{x} = \mathbf{0}$ has no free variables. So every variable is a basic variable and every column of A is a pivot column. (A variation of this argument could be made using linear independence and Exercise 30 in Section 1.7.) Since each pivot is in a different row, A must have at least as many rows as columns.

24. Take any $\mathbf{b}$ in $\mathbf{R}^m$. By hypothesis, $AD\mathbf{b} = I_m\mathbf{b} = \mathbf{b}$. Rewrite this equation as $A(D\mathbf{b}) = \mathbf{b}$. Thus, the vector $\mathbf{x} = D\mathbf{b}$ satisfies $A\mathbf{x} = \mathbf{b}$. This proves that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$. By Theorem 4 in Section 1.4, A has a pivot position in each row. Since each pivot is in a different column, A must have at least as many columns as rows.

25. By Exercise 23, the equation $CA = I_n$ implies that (number of rows in A) $\geq$ (number of columns), that is, $m \geq n$. By Exercise 24, the equation $AD = I_m$ implies that (number of rows in A) $\leq$ (number of columns), that is, $m \leq n$. Thus $m = n$. To prove the second statement, observe that $DAC = (DA)C = I_nC = C$, and also $DAC = D(AC) = DI_m = D$. Thus $C = D$. A shorter calculation is

$$C = I_nC = (DA)C = D(AC) = DI_n = D$$

26. Write $I_3 = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]$ and $D = [\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3]$. By definition of AD, the equation $AD = I_3$ is equivalent to the three equations $A\mathbf{d}_1 = \mathbf{e}_1$, $A\mathbf{d}_2 = \mathbf{e}_2$, and $A\mathbf{d}_3 = \mathbf{e}_3$. Each of these equations has at least one solution because the columns of A span $\mathbf{R}^3$. (See Theorem 4 in Section 1.4.) Select one solution of each equation

27. The product $\mathbf{u}^T\mathbf{v}$ is a 1×1 matrix, which usually is identified with a real number and is written without the matrix brackets.

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -2a+3b-4c\,,\ \mathbf{v}^T\mathbf{u} = \begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = -2a+3b-4c$$

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}\begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$\mathbf{v}\mathbf{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} -2 & 3 & -4 \end{bmatrix} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

28. Since the inner product $\mathbf{u}^T\mathbf{v}$ is a real number, it equals its transpose. That is, $\mathbf{u}^T\mathbf{v} = (\mathbf{u}^T\mathbf{v})^T = \mathbf{v}^T(\mathbf{u}^T)^T = \mathbf{v}^T\mathbf{u}$, by Theorem 3(d) regarding the transpose of a product of matrices and by Theorem 3(a). The outer product $\mathbf{u}\mathbf{v}^T$ is an $n\times n$ matrix. By Theorem 3, $(\mathbf{u}\mathbf{v}^T)^T = (\mathbf{v}^T)^T\mathbf{u}^T = \mathbf{v}\mathbf{u}^T$.

29. The (i, j)-entry of $A(B + C)$ equals the (i, j)-entry of $AB + AC$, because

$$\sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

The (i, j)-entry of $(B + C)A$ equals the (i, j)-entry of $BA + CA$, because

$$\sum_{k=1}^{n} (b_{ik}+c_{ik})a_{kj} = \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

30. The (i, j))-entries of $r(AB)$, $(rA)B$, and $A(rB)$ are all equal, because

$$r\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} (ra_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(rb_{kj})$$

31. Use the definition of the product I_mA and the fact that $I_m\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$ in $\mathbf{R}^m$.

$$I_mA = I_m[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = [I_m\mathbf{a}_1 \ \cdots \ I_m\mathbf{a}_n] = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = A$$

32. Let $\mathbf{e}_j$ and $\mathbf{a}_j$ denote the jth columns of I_n and A, respectively. By definition, the jth column of AI_n *is* $A\mathbf{e}_j$, which is simply $\mathbf{a}_j$ because $\mathbf{e}_j$ has 1 in the jth position and zeros elsewhere. Thus corresponding columns of AI_n and A are equal. Hence $AI_n = A$.

33. The (i, j)-entry of $(AB)^T$ is the (j, i)-entry of AB, which is

$$a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$$

The entries in row i of B^T are $b_{1i}, \ldots, b_{ni}$, because they come from column i of B. Likewise, the entries in column j of A^T are $a_{j1}, \ldots, a_{jn}$, because they come from row j of A. Thus the (i, j)-entry in B^TA^T is $a_{j1}b_{1i} + \cdots + a_{jn}b_{ni}$, as above.

34. Use Theorem 3(d), treating $\mathbf{x}$ as an $n\times 1$ matrix: $(AB\mathbf{x})^T = \mathbf{x}^T(AB)^T = \mathbf{x}^TB^TA^T$.

35. [M] The answer here depends on the choice of matrix program. For MATLAB, use the `help` command to read about `zeros`, `ones`, `eye`, and `diag`. For other programs see the appendices in the *Study Guide*. (The TI calculators have fewer single commands that produce special matrices.)

proof of the IMT rather than simply watch an instructor carry out the proof. Also, this activity will help students understand *why* the theorem is true.

1. $\begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}^{-1} = \frac{1}{32-30}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}^{-1} = \frac{1}{12-14}\begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 7/2 & -3/2 \end{bmatrix}$

3. $\begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}^{-1} = \frac{1}{-40-(-35)}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ -1.4 & -1.6 \end{bmatrix}$

4. $\begin{bmatrix} 3 & -4 \\ 7 & -8 \end{bmatrix}^{-1} = \frac{1}{-24-(-28)}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -8 & 4 \\ -7 & 3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -7/4 & 3/4 \end{bmatrix}$

5. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and the solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}.$$ Thus $x_1 = 7$ and $x_2 = -9$.

6. The system is equivalent to $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 9 \\ 11 \end{bmatrix}$, and the solution is $\mathbf{x} = A^{-1}\mathbf{b}$. To compute this by hand, the arithmetic is simplified by keeping the fraction 1/det(A) in front of the matrix for A^{-1}. (The *Study Guide* comments on this in its discussion of Exercise 7.) From Exercise 3,

$$\mathbf{x} = A^{-1}\mathbf{b} = -\frac{1}{5}\begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -9 \\ 11 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -10 \\ 25 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}.$$ Thus $x_1 = 2$ and $x_2 = -5$.

7. **a.** $\begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}^{-1} = \frac{1}{1 \cdot 12 - 2 \cdot 5}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}$ or $\begin{bmatrix} 6 & -1 \\ -2.5 & .5 \end{bmatrix}$

$$\mathbf{x} = A^{-1}\mathbf{b}_1 = \frac{1}{2}\begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -18 \\ 8 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}.$$ Similar calculations give

$$A^{-1}\mathbf{b}_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}, A^{-1}\mathbf{b}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, A^{-1}\mathbf{b}_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}.$$

b. $[A \;\; \mathbf{b}_1 \;\; \mathbf{b}_2 \;\; \mathbf{b}_3 \;\; \mathbf{b}_4] = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 2 & 8 & -10 & -4 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

The solutions are $\begin{bmatrix} -9 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 11 \\ -5 \end{bmatrix}$, $\begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 13 \\ -5 \end{bmatrix}$, the same as in part (a).

Note: The *Study Guide* also discusses the number of arithmetic calculations for this Exercise 7, stating that when A is large, the method used in (b) is much faster than using A^{-1}.

8. Left-multiply each side of the equation $AD = I$ by A^{-1} to obtain

$$A^{-1}AD = A^{-1}I,\ ID = A^{-1}, \text{ and } D = A^{-1}.$$

Parentheses are routinely suppressed because of the associative property of matrix multiplication.

9. **a.** True, by definition of *invertible*. **b.** False. See Theorem 6(b).

c. False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $ab - cd = 1 - 0 \neq 0$, but Theorem 4 shows that this matrix is not invertible, because $ad - bc = 0$.

d. True. This follows from Theorem 5, which also says that the solution of $A\mathbf{x} = \mathbf{b}$ is unique, for each $\mathbf{b}$.

e. True, by the box just before Example 6.

10. **a.** False. The product matrix is invertible, but the product of inverses should be in the *reverse* order. See Theorem 6(b).

b. True, by Theorem 6(a). **c.** True, by Theorem 4.

d. True, by Theorem 7. **e.** False. The last part of Theorem 7 is misstated here.

11. (The proof can be modeled after the proof of Theorem 5.) The $n\times p$ matrix B is given (but is arbitrary). Since A is invertible, the matrix $A^{-1}B$ satisfies $AX = B$, because $A(A^{-1}B) = A\,A^{-1}B = IB = B$. To show this solution is unique, let X be any solution of $AX = B$. Then, left-multiplication of each side by A^{-1} shows that X must be $A^{-1}B$:

$$A^{-1}(AX) = A^{-1}B, \quad IX = A^{-1}B, \quad \text{and} \quad X = A^{-1}B.$$

12. If you assign this exercise, consider giving the following *Hint*: Use elementary matrices and imitate the proof of Theorem 7. The solution in the Instructor's Edition follows this hint. Here is another solution, based on the idea at the end of Section 2.2.

Write $B = [\mathbf{b}_1 \cdots \mathbf{b}_p]$ and $X = [\mathbf{u}_1 \cdots \mathbf{u}_p]$. By definition of matrix multiplication, $AX = [A\mathbf{u}_1 \cdots A\mathbf{u}_p]$. Thus, the equation $AX = B$ is equivalent to the p systems:

$$A\mathbf{u}_1 = \mathbf{b}_1, \quad \ldots \quad A\mathbf{u}_p = \mathbf{b}_p$$

Since A is the coefficient matrix in each system, these systems may be solved simultaneously, placing the augmented columns of these systems next to A to form $[A \ \mathbf{b}_1 \cdots \mathbf{b}_p] = [A \ \ B]$. Since A is invertible, the solutions $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are uniquely determined, and $[A \ \mathbf{b}_1 \cdots \mathbf{b}_p]$ must row reduce to $[I \ \mathbf{u}_1 \cdots \mathbf{u}_p] = [I \ \ X]$. By Exercise 11, X is the unique solution $A^{-1}B$ of $AX = B$.

13. Left-multiply each side of the equation $AB = AC$ by A^{-1} to obtain

$$A^{-1}AB = A^{-1}AC, \quad IB = IC, \quad \text{and} \quad B = C.$$

This conclusion does not always follow when A is singular. Exercise 10 of Section 2.1 provides a counterexample.

14. Right-multiply each side of the equation $(B - C)D = 0$ by D^{-1} to obtain

$$(B - C)DD^{-1} = 0D^{-1}, \quad (B - C)I = 0, \quad B - C = 0, \quad \text{and} \quad B = C.$$

15. The box following Theorem 6 suggests what the inverse of ABC should be, namely, $C^{-1}B^{-1}A^{-1}$. To verify that this is correct, compute:

$$(ABC)\,C^{-1}B^{-1}A^{-1} = ABCC^{-1}B^{-1}A^{-1} = ABIB^{-1}A^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and

$$C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC = C^{-1}B^{-1}BC = C^{-1}IC = C^{-1}C = I$$

16. Let $C = AB$. Then $CB^{-1} = ABB^{-1}$, so $CB^{-1} = AI = A$. This shows that A is the product of invertible matrices and hence is invertible, by Theorem 6.

Note: The *Study Guide* warns against using the formula $(AB)^{-1} = B^{-1}A^{-1}$ here, because this formula can be used only when both A and B are already known to be invertible.

17. Right-multiply each side of $AB = BC$ by B^{-1}:
$$ABB^{-1} = BCB^{-1}, \quad AI = BCB^{-1}, \quad A = BCB^{-1}.$$

18. Left-multiply each side of $A = PBP^{-1}$ by P^{-1}:
$$P^{-1}A = P^{-1}PBP^{-1}, \quad P^{-1}A = IBP^{-1}, \quad P^{-1}A = BP^{-1}$$
Then right-multiply each side of the result by P:
$$P^{-1}AP = BP^{-1}P, \quad P^{-1}AP = BI, \quad P^{-1}AP = B$$

19. Unlike Exercise 17, this exercise asks two things, "Does a solution exist and what is it?" First, find what the solution must be, if it exists. That is, suppose X satisfies the equation $C^{-1}(A + X)B^{-1} = I$. Left-multiply each side by C, and then right-multiply each side by B:
$$CC^{-1}(A+X)B^{-1} = CI, \quad I(A+X)B^{-1} = C, \quad (A+X)B^{-1}B = CB, \quad (A+X)I = CB$$
Expand the left side and then subtract A from both sides:
$$AI + XI = CB, \quad A + X = CB, \quad X = CB - A$$
If a solution exists, it must be $CB - A$. To show that $CB - A$ really *is* a solution, substitute it for X:
$$C^{-1}[A + (CB - A)]B^{-1} = C^{-1}[CB]B^{-1} = C^{-1}CBB^{-1} = II = I.$$

Note: The *Study Guide* suggests that students ask their instructor about how many details to include in their proofs. After some practice with algebra, an expression such as $CC^{-1}(A + X)B^{-1}$ could be simplified directly to $(A + X)B^{-1}$ without first replacing CC^{-1} by I. However, you may wish this detail to be included in the homework for this section.

20. a. Left-multiply both sides of $(A - AX)^{-1} = X^{-1}B$ by X to see that B is invertible because it is the product of invertible matrices.

b. Invert both sides of the original equation and use Theorem 6 about the inverse of a product (which applies because X^{-1} and B are invertible):
$$A - AX = (X^{-1}B)^{-1} = B^{-1}(X^{-1})^{-1} = B^{-1}X$$
Then $A = AX + B^{-1}X = (A + B^{-1})X$. The product $(A + B^{-1})X$ is invertible because A is invertible. Since X is known to be invertible, so is the other factor, $A + B^{-1}$, by Exercise 16 or by an argument similar to part (a). Finally,
$$(A + B^{-1})^{-1}A = (A + B^{-1})^{-1}(A + B^{-1})X = X$$

Note: This exercise is difficult. The algebra is not trivial, and at this point in the course, most students will not recognize the need to verify that a matrix is invertible.

21. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{0}$ has only one solution, namely, the zero solution. This means that the columns of A are linearly independent, by a remark in Section 1.7.

22. Suppose A is invertible. By Theorem 5, the equation $A\mathbf{x} = \mathbf{b}$ has a solution (in fact, a unique solution) for each $\mathbf{b}$. By Theorem 4 in Section 1.4, the columns of A span $\mathbf{R}^n$.

23. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Then there are no free variables in this equation, and so A has n pivot columns. Since A is *square* and the n pivot positions must be in different rows, the pivots in an echelon form of A must be on the main diagonal. Hence A is row equivalent to the $n \times n$ identity matrix.

$$= \begin{bmatrix} \text{row}_1(I)\cdot A \\ \text{row}_2(I)\cdot A \\ [\text{row}_3(I) - 4\cdot\text{row}_1(I)]\cdot A \end{bmatrix} = \begin{bmatrix} \text{row}_1(I) \\ \text{row}_2(I) \\ \text{row}_3(I) - 4\cdot\text{row}_1(I) \end{bmatrix} A = EA$$

Here, E is obtained by replacing $\text{row}_3(I)$ by $\text{row}_3(I) - 4\cdot\text{row}_1(I)$.

29. $[A \quad I] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$$

30. $[A \quad I] = \begin{bmatrix} 5 & 10 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & -1 & -4/5 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1/5 & 0 \\ 0 & 1 & 4/5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7/5 & 2 \\ 0 & 1 & 4/5 & -1 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} -7/5 & 2 \\ 4/5 & -1 \end{bmatrix}$$

31. $[A \quad I] = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}. \quad A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$$

32. $[A \quad I] = \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 10 & -2 & 1 \end{bmatrix}.$$ The matrix A is not invertible.

33. Let $B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & 0 \\ 0 & -1 & 1 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$, and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$ (because $\mathbf{a}_j$ and $\mathbf{a}_{j+1}$ have the same entries except for the jth row), $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$ and $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$,

$$A\mathbf{b}_j = A(\mathbf{e}_j - \mathbf{e}_{j+1}) = A\mathbf{e}_j - A\mathbf{e}_{j+1} = \mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$$

and $A\mathbf{b}_n = A\mathbf{e}_n = \mathbf{a}_n = \mathbf{e}_n$. Next, observe that $\mathbf{a}_j = \mathbf{e}_j + \cdots + \mathbf{e}_n$ for each j. Thus,

$$B\mathbf{a}_j = B(\mathbf{e}_j + \cdots + \mathbf{e}_n) = \mathbf{b}_j + \cdots + \mathbf{b}_n$$
$$= (\mathbf{e}_j - \mathbf{e}_{j+1}) + (\mathbf{e}_{j+1} - \mathbf{e}_{j+2}) + \cdots + (\mathbf{e}_{n-1} - \mathbf{e}_n) + \mathbf{e}_n = \mathbf{e}_j$$

This proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: Students who do this problem and then do the corresponding exercise in Section 2.4 will appreciate the Invertible Matrix Theorem, partitioned matrix notation, and the power of a proof by induction.

34. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 0 & & 0 \\ 1 & 2 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1/2 & 1/2 & 0 & & \\ 0 & -1/3 & 1/3 & & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & -1/n & 1/n \end{bmatrix}$$

and for $j = 1, \ldots, n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$ denote the jth columns of A, B, and I, respectively. Note that for $j = 1, \ldots, n-1$, $\mathbf{a}_j = j(\mathbf{e}_j + \cdots + \mathbf{e}_n)$, $\mathbf{b}_j = \frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}$, and $\mathbf{b}_n = \frac{1}{n}\mathbf{e}_n$.

To show that $AB = I$, it suffices to show that $A\mathbf{b}_j = \mathbf{e}_j$ for each j. For $j = 1, \ldots, n-1$,

$$A\mathbf{b}_j = A\left(\frac{1}{j}\mathbf{e}_j - \frac{1}{j+1}\mathbf{e}_{j+1}\right) = \frac{1}{j}\mathbf{a}_j - \frac{1}{j+1}\mathbf{a}_{j+1}$$
$$= (\mathbf{e}_j + \cdots + \mathbf{e}_n) - (\mathbf{e}_{j+1} + \cdots + \mathbf{e}_n) = \mathbf{e}_j$$

Also, $A\mathbf{b}_n = A\left(\frac{1}{n}\mathbf{e}_n\right) = \frac{1}{n}\mathbf{a}_n = \mathbf{e}_n$. Finally, for $j = 1, \ldots, n$, the sum $\mathbf{b}_j + \cdots + \mathbf{b}_n$ is a "telescoping sum" whose value is $\frac{1}{j}\mathbf{e}_j$. Thus,

$$B\mathbf{a}_j = j(B\mathbf{e}_j + \cdots + B\mathbf{e}_n) = j(\mathbf{b}_j + \cdots + \mathbf{b}_n) = j\left(\frac{1}{j}\mathbf{e}_j\right) = \mathbf{e}_j$$

which proves that $BA = I$. Combined with the first part, this proves that $B = A^{-1}$.

Note: If you assign Exercise 34, you may wish to supply a hint using the notation from Exercise 33: Express each column of A in terms of the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the identity matrix. Do the same for B.

35. Row reduce $[A \quad \mathbf{e}_3]$:

$$\begin{bmatrix} -2 & -7 & -9 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & 3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & 0 \\ -2 & -7 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -15 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Answer: The third column of A^{-1} is $\begin{bmatrix} 3 \\ -6 \\ 4 \end{bmatrix}$.