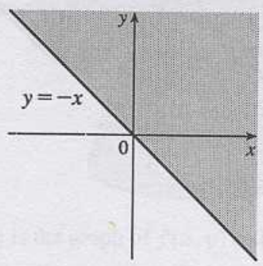
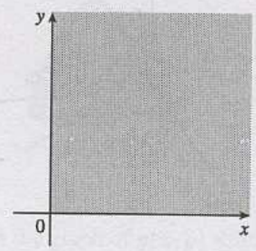
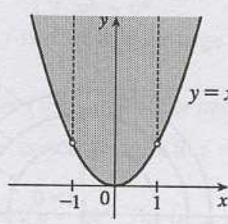
5. $\sqrt{x+y}$ is defined only when $x+y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x,y) \mid y \geq -x\}$.



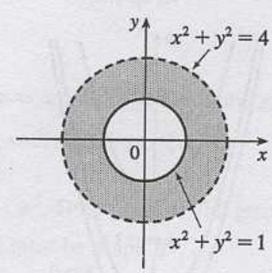
6. We need $x \geq 0$ and $y \geq 0$, so $D = \{(x,y) \mid x \geq 0 \text{ and } y \geq 0\}$, the first quadrant.



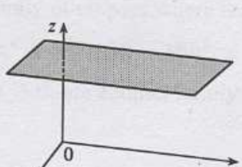
7. $\sqrt{y-x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0 \Rightarrow x = \pm 1$. Thus the domain of f is $\{(x,y) \mid y \geq x^2, x \neq \pm 1\}$.



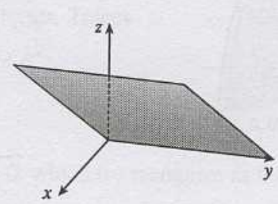
8. f is defined only when $x^2 + y^2 - 1 \geq 0 \Rightarrow x^2 + y^2 \geq 1$ and $4 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 4$. Thus $D = \{(x,y) \mid 1 \leq x^2 + y^2 < 4\}$.



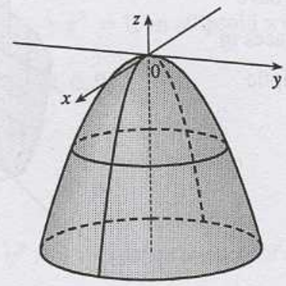
9. $z = 3$, a horizontal plane through the point $(0,0,3)$.



10. $z = x$, a plane which intersects the xz-plane in the line $z = x$, $y = 0$. The portion of this plane that lies in the first octant is shown.

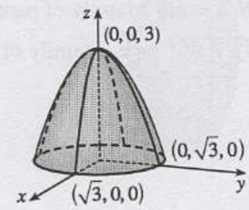


(b)



$g(x, y)$ is the graph of $f(x, y)$ reflected in the xy-plane. [Note that $g(x, y) = -f(x, y)$.]

(c)



$h(x, y)$ is the graph of $g(x, y)$ shifted upward 3 units.

15. All six graphs have different traces in the planes $x = 0$ and $y = 0$, so we investigate these for each function.

(a) $f(x, y) = |x| + |y|$. The trace in $x = 0$ is $z = |y|$, and in $y = 0$ is $z = |x|$, so it must be graph VI.

(b) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and in $y = 0$ is $z = 0$, so it must be graph V.

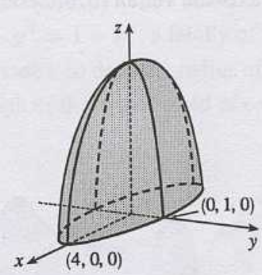
(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. In addition, we can see that f is close to 0 for large values of x and y, so this is graph I.

(d) $f(x, y) = (x^2 - y^2)^2$. The trace in $x = 0$ is $z = y^4$, and in $y = 0$ is $z = x^4$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x^2 - y^2)^2 \;\Rightarrow\; y = \pm x$, so it must be graph IV.

(e) $f(x, y) = (x - y)^2$. The trace in $x = 0$ is $z = y^2$, and in $y = 0$ is $z = x^2$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x - y)^2 \;\Rightarrow\; y = x$, so it must be graph II.

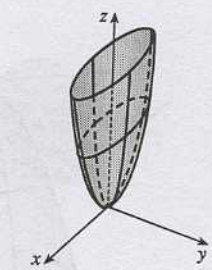
(f) $f(x, y) = \sin(|x| + |y|)$. The trace in $x = 0$ is $z = \sin|y|$, and in $y = 0$ is $z = \sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

16. The equation of the graph is $z = \sqrt{16 - x^2 - 16y^2}$ or equivalently $x^2 + 16y^2 + z^2 = 16$, $z \geq 0$. Traces in $x = k$ are $16y^2 + z^2 = 16 - k^2$, $z \geq 0$, a family of ellipses where here we have only the upper halves. Traces in $y = k$ are $x^2 + z^2 = 16 - 16k^2$, $z \geq 0$, again a family of half-ellipses. Traces in $z = k$, $k \geq 0$, are another family of ellipses, $x^2 + 16y^2 = 16 - k^2$.

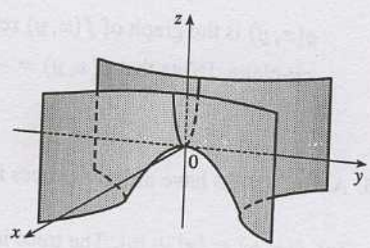


Note that the equation can be written as $\dfrac{x^2}{16} + y^2 + \dfrac{z^2}{16} = 1$, $z \geq 0$, which we recognize as the top half of an ellipsoid with intercepts ± 4, ± 1, and 4.

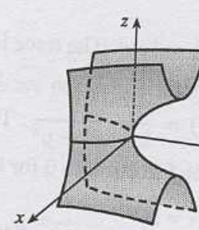
17. The equation of the graph is $z = x^2 + 9y^2$. The traces in $x = k$ are $z = 9y^2 + k$, a family of parabolas opening upward. In $y = k$, we have $z = x^2 + 9k^2$, again a family of parabolas opening upward. The traces in $z = k$ are $x^2 + 9y^2 = k$, a family of ellipses. The surface is an elliptic paraboloid.



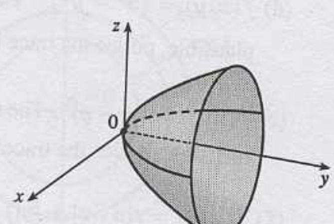
18. The equation of the graph is $z = x^2 - y^2$. The traces in $x = k$ are $z = -y^2 + k^2$, a family of parabolas opening downward. In $y = k$, we have $z = x^2 - k^2$, a family of parabolas opening upward. The traces in $z = k$ are $x^2 - y^2 = k$, a family of hyperbolas. The surface is a hyperbolic paraboloid with saddle point $(0, 0, 0)$.



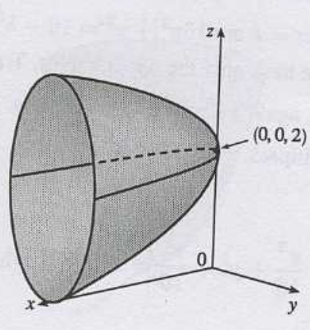
19. $y = z^2 - x^2$. The traces in $x = k$ are the parabolas $y = z^2 - k^2$; the traces in $y = k$ are $k = z^2 - x^2$, which are hyperbolas (note the hyperbolas are oriented differently for $k > 0$ than for $k < 0$); and the traces in $z = k$ are the parabolas $y = k^2 - x^2$. Thus, $\frac{y}{1} = \frac{z^2}{1^2} - \frac{x^2}{1^2}$ is a hyperbolic paraboloid.



20. For $y = x^2 + z^2$, the traces in $x = k$ are $y = z^2 + k^2$, a family of parabolas opening in the positive y-direction. The traces in $y = k$ are $x^2 + z^2 = k$, $k \geq 0$, a family of circles. The traces in $z = k$ are $y = x^2 + k^2$, a family of parabolas opening in the positive y-direction. We recognize the graph as a circular paraboloid with axis the y-axis.

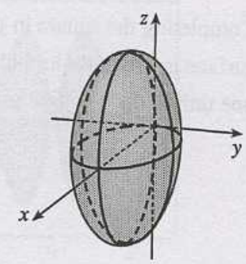


21. Factoring the equation gives $x = 4y^2 + (z - 2)^2$. This corresponds to an elliptic paraboloid with axis parallel to the x-axis and vertex $(0, 0, 2)$ that opens in the positive x-direction.



22. Completing the square in x gives $(x-1)^2 + 4y^2 + z^2 = 1$ or

$(x-1)^2 + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, an ellipsoid with center (1, 0, 0) and intercepts (0, 0, 0), (2, 0, 0).



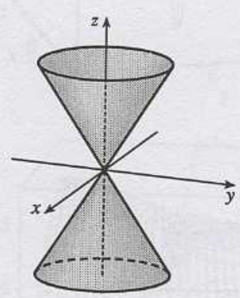
23. (a) In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents a circle of radius 1 centered at the origin.

(b) In $\mathbb{R}^3$, the equation doesn't involve z, which means that any horizontal plane $z = k$ intersects the surface in a circle $x^2 + y^2 = 1$, $z = k$. Thus the surface is a circular cylinder, made up of infinitely many shifted copies of the circle $x^2 + y^2 = 1$, with axis the z-axis.

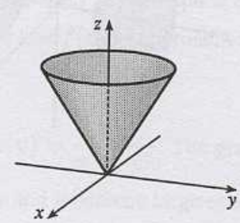
(c) In $\mathbb{R}^3$, $x^2 + z^2 = 1$ also represents a circular cylinder of radius 1, this time with axis the y-axis.

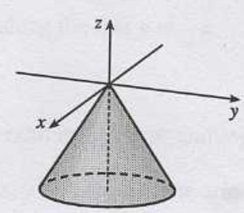
24. (a) The traces of $z^2 = x^2 + y^2$ in $x = k$ are $z^2 = y^2 + k^2$, a family of hyperbolas, as are traces in $y = k$, $z^2 = x^2 + k^2$. Traces in $z = k$ are $x^2 + y^2 = k^2$, a family of circles.

(b) The surface is a circular cone with axis the z-axis.



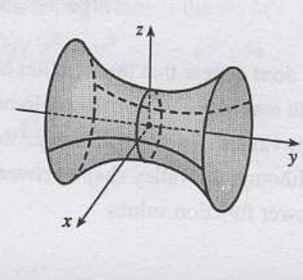
(c) The graph of $f(x, y) = \sqrt{x^2 + y^2}$ is the upper half of the cone in part (b), and the graph of $g(x, y) = -\sqrt{x^2 + y^2}$ is the lower half.



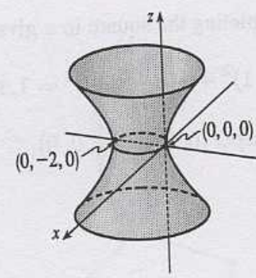


25. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy-plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 2.

(b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.

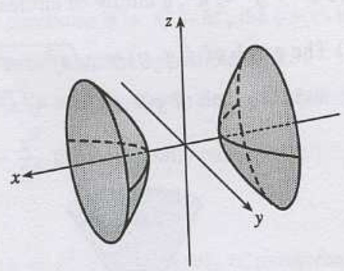


(c) Completing the square in y gives $x^2 + (y+1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.

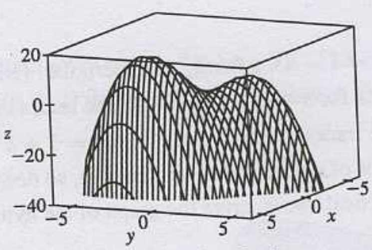


26. (a) The traces of $-x^2 - y^2 + z^2 = 1$ in $x = k$ are $-y^2 + z^2 = 1 + k^2$, a family of hyperbolas, as are the traces in $y = k$, $-x^2 + z^2 = 1 + k^2$. The traces in $z = k$ are $x^2 + y^2 = k^2 - 1$, a family of circles for $|k| > 1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k| < 1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 2.

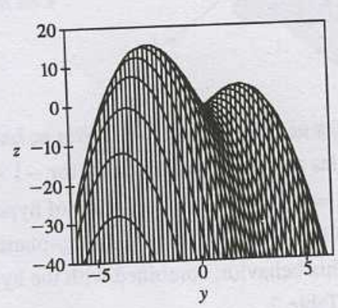
(b) The graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x-axis. Traces in $x = k$, $|k| > 1$, are circles, while traces in $y = k$ and $z = k$ are hyperbolas.



27. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



Three-dimensional view



Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

[redacted] **32.** Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

33. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \Rightarrow c + 2(b - a)t = (b + t)^2 - (a + t)^2 = b^2 - a^2 + 2(b - a)t \Rightarrow c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \Rightarrow c - 2(b + a)t = (b - t)^2 - (a + t)^2 = b^2 - a^2 - 2(b + a)t \Rightarrow c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

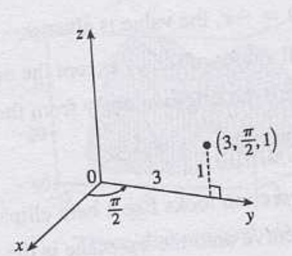
34. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x-axis is twice its distance from the yz-plane. The distance from P to the x-axis is $\sqrt{(x - x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz-plane $(x = 0)$ is $|x|/1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \Leftrightarrow y^2 + z^2 = 4x^2 \Leftrightarrow x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x-axis.

9.7 Cylindrical and Spherical Coordinates

1. See Figure 1 and the accompanying discussion on page 694; see the paragraph preceding Figure 3 on page 695.

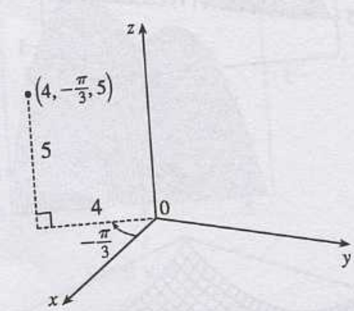
2. See Figure 5 and the accompanying discussion on page 696.

3. (a)



$x = 3\cos\frac{\pi}{2} = 0$, $y = 3\sin\frac{\pi}{2} = 3$, and $z = 1$, so the point is $(0, 3, 1)$ in rectangular

(b)



$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$, $y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$, and $z = 5$, so the point is $(2, -2\sqrt{3}, 5)$ in rectangular

42. To find the osculating plane, we first calculate the tangent and normal vectors.

In Maple, we set `x:=t^3; y:=3*t;` and `z:=t^4;` and then calculate the components of the tangent vector $\mathbf{T}(t)$ using the `diff` command. We find that $\mathbf{T}(t) = \dfrac{\langle 3t^2, 3, 4t^3\rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. Differentiating the components of $\mathbf{T}(t)$, we find that $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\langle -6t\,(8t^6 - 9)\,, 3\,(48t^5 + 18t^3)\,, 36t^2\,(t^4 + 3)\rangle}{\sqrt{144t^2(8t^6 - 9)^2 + 9(96t^5 + 36t^3)^2 + 5{,}184t^{12} + 31{,}104t^8 + 46{,}656t^4}}$.

In Maple, we can calculate $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ using the `linalg` package. First we define $\mathbf{T}$ and $\mathbf{N}$ using `T:=array([f,g,h]);` and `N:=array([F,G,H]);` where `f`, `g`, `h`, `F`, `G`, and `H` are the components of $\mathbf{T}$ and $\mathbf{N}$. Then we use the command `B:=crossprod(T,N);`. After normalization and simplification, we find that $\mathbf{B}(t) = b\,\langle 6t, -2t^3, -3\rangle$, where

$$b = \frac{t\sqrt{16t^6 + 9t^4 + 9}}{\sqrt{16t^{2(8t^6-9)2} + (96t^5 + 36t^3)^2 + 576t^{12} + 3456t^8 + 5184t^4}}$$

In Mathematica, we use the command `Dt` to differentiate the components of $\mathbf{r}(t)$ and subsequently $\mathbf{T}(t)$, and then load the vector analysis package with the command `<<Calculus 'VectorAnalysis'`. After setting `T={f,g,h}` and `N={F,G,H}`, we use `CrossProduct[T,N]` to find $\mathbf{B}$ (before normalization).

Now $\mathbf{B}(t)$ is parallel to $\langle 6t, -2t^3, -3\rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1, 1, 1\rangle$ for some t, then $6t = 1 \;\Rightarrow\; t = \frac{1}{6}$, but $-2\left(\frac{1}{6}\right)^3 \neq 1$. So there is no such osculating plane.

43. $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N} = \dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{ds}{dt}} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

44. For a plane curve, $\mathbf{T} = |\mathbf{T}|\cos\phi\,\mathbf{i} + |\mathbf{T}|\sin\phi\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Then $\dfrac{d\mathbf{T}}{ds} = \left(\dfrac{d\mathbf{T}}{d\phi}\right)\left(\dfrac{d\phi}{ds}\right) = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\left(\dfrac{d\phi}{ds}\right)$ and $\left|\dfrac{d\mathbf{T}}{ds}\right| = |-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}|\left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa = |d\phi/ds|$.

45. (a) $|\mathbf{B}| = 1 \;\Rightarrow\; \mathbf{B}\cdot\mathbf{B} = 1 \;\Rightarrow\; \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 0 \;\Rightarrow\; 2\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0 \;\Rightarrow\; \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \;\Rightarrow$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\,\frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\,\frac{1}{|\mathbf{r}'(t)|}$$

$$= [(\mathbf{T}'\times\mathbf{N}) + (\mathbf{T}\times\mathbf{N}')]\,\frac{1}{|\mathbf{r}'(t)|} = \left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right) + (\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|}$$

$$\Rightarrow\; \frac{d\mathbf{B}}{ds} \perp \mathbf{T}$$

(c) $\mathbf{B} = \mathbf{T} \times \mathbf{N} \Rightarrow \mathbf{T} \perp \mathbf{N}, \mathbf{B} \perp \mathbf{T}$ and $\mathbf{B} \perp \mathbf{N}$. So $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Therefore, $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$, where $\tau(s)$ is a scalar.

(d) Since $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, $\mathbf{T} \perp \mathbf{N}$ and both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. For a plane curve, $\mathbf{T}$ and $\mathbf{N}$ always lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\mathbf{B}/ds = \mathbf{0}$, but $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ and $\mathbf{N} \neq \mathbf{0}$, so $\tau(s) = 0$.

46. $\mathbf{N} = \mathbf{B} \times \mathbf{T} \Rightarrow$

$$\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} \quad \text{[by Theorem 10.2.3 \#5]}$$
$$= -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa\mathbf{N} \quad \text{[by Formulas 3 and 1]}$$
$$= -\tau(\mathbf{N} \times \mathbf{T}) + \kappa(\mathbf{B} \times \mathbf{N}) \quad \text{[by Property 2 of the cross product]}$$

But $\mathbf{B} \times \mathbf{N} = \mathbf{B} \times (\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \cdot \mathbf{T})\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\mathbf{T}$ [by Equation 9.4.8] $= -\mathbf{T} \Rightarrow$

$d\mathbf{N}/ds = \tau(\mathbf{T} \times \mathbf{N}) - \kappa\mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

47. (a) $\mathbf{r}' = s'\mathbf{T} \Rightarrow \mathbf{r}'' = s''\mathbf{T} + s'\mathbf{T}' = s''\mathbf{T} + s'\frac{d\mathbf{T}}{ds}s' = s''\mathbf{T} + \kappa(s')^2\mathbf{N}$ by the first Serret-Frenet formula.

(b) Using part (a), we have

$$\mathbf{r}' \times \mathbf{r}'' = (s'\mathbf{T}) \times \left[s''\mathbf{T} + \kappa(s')^2\mathbf{N}\right]$$
$$= \left[(s'\mathbf{T}) \times (s''\mathbf{T})\right] + \left[(s'\mathbf{T}) \times \left(\kappa(s')^2\mathbf{N}\right)\right] \quad \text{[By Property 3 of the cross product]}$$
$$= (s's'')(\mathbf{T} \times \mathbf{T}) + \kappa(s')^3(\mathbf{T} \times \mathbf{N}) = \mathbf{0} + \kappa(s')^3\mathbf{B} = \kappa(s')^3\mathbf{B}$$

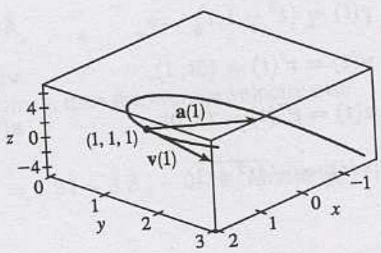
(c) Using part (a), we have

$$\mathbf{r}''' = \left[s''\mathbf{T} + \kappa(s')^2\mathbf{N}\right]' = s'''\mathbf{T} + s''\mathbf{T}' + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^2\mathbf{N}'$$
$$= s'''\mathbf{T} + s''\frac{d\mathbf{T}}{ds}s' + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^2\frac{d\mathbf{N}}{ds}s'$$
$$= s'''\mathbf{T} + s''s'\kappa\mathbf{N} + \kappa'(s')^2\mathbf{N} + 2\kappa s's''\mathbf{N} + \kappa(s')^3(-\kappa\mathbf{T} + \tau\mathbf{B}) \quad \text{[by the second formula]}$$
$$= \left[s''' - \kappa^2(s')^3\right]\mathbf{T} + \left[3\kappa s's'' + \kappa'(s')^2\right]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B} \cdot \mathbf{T} = 0$, $\mathbf{B} \cdot \mathbf{N} = 0$, and $\mathbf{B} \cdot \mathbf{B} = 1$, we get

$$\frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\kappa(s')^3\mathbf{B} \cdot \left\{\left[s''' - \kappa^2(s')^3\right]\mathbf{T} + \left[3\kappa s's'' + \kappa'(s')^2\right]\mathbf{N} + \kappa\tau(s')^3\mathbf{B}\right\}}{|\kappa(s')^3\mathbf{B}|^2}$$
$$= \frac{\kappa(s')^3\kappa\tau(s')^3}{[\kappa(s')^3]^2} = \tau$$

8. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \quad \Rightarrow$
$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$, $\mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$
$\mathbf{a}(t) = 2\,\mathbf{j} + 6t\,\mathbf{k}$, $\mathbf{a}(1) = 2\,\mathbf{j} + 6\,\mathbf{k}$
$|\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$
The path is a "twisted cubic" (see Example 10.1.6).



9. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle 0, 2, 6t \rangle$,
$|\mathbf{v}(t)| = \sqrt{1^2 + (2t)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}$.

10. $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t \rangle$, $\mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t \rangle$,
$|\mathbf{v}(t)| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}$.

11. $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$.

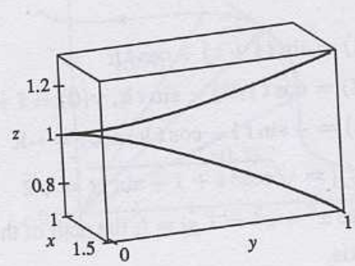
12. $\mathbf{r}(t) = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = (\sin t + t\cos t)\,\mathbf{i} + (\cos t - t\sin t)\,\mathbf{j} + 2t\,\mathbf{k}$,
$\mathbf{a}(t) = \mathbf{v}'(t) = (2\cos t - t\sin t)\,\mathbf{i} + (-2\sin t - t\cos t)\,\mathbf{j} + 2\mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{(\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + 4t^2} = \sqrt{5t^2 + 1}$.

13. $\mathbf{a}(t) = \mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \mathbf{k}\,dt = t\,\mathbf{k} + \mathbf{c}_1$ and $\mathbf{i} - \mathbf{j} = \mathbf{v}(0) = 0\,\mathbf{k} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ and
$\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\,\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} - \mathbf{j} + t\,\mathbf{k})\,dt = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k} + \mathbf{c}_2$. But $\mathbf{0} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so
$\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k}$.

14. $\mathbf{a}(t) = -10\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int (-10\mathbf{k})\,dt = -10t\,\mathbf{k} + \mathbf{c}_1$, and $\mathbf{i} + \mathbf{j} - \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and
$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - (10t + 1)\,\mathbf{k}$.
$\mathbf{r}(t) = \int [\mathbf{i} + \mathbf{j} - (10t + 1)\,\mathbf{k}]\,dt = t\,\mathbf{i} + t\,\mathbf{j} - (5t^2 + t)\,\mathbf{k} + \mathbf{c}_2$. But $2\,\mathbf{i} + 3\,\mathbf{j} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = 2\,\mathbf{i} + 3\,\mathbf{j}$
and $\mathbf{r}(t) = (t + 2)\,\mathbf{i} + (t + 3)\,\mathbf{j} - (5t^2 + t)\,\mathbf{k}$.

15. (a) $\mathbf{a}(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k} \quad \Rightarrow$
$\mathbf{v}(t) = \int (\mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k})\,dt = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{c}_1$, and
$\mathbf{0} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{0}$ and $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$.
$\mathbf{r}(t) = \int (t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k})\,dt = \frac{1}{2}t^2\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{c}_2$.
But $\mathbf{i} + \mathbf{k} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{i} + \mathbf{k}$ and
$\mathbf{r}(t) = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + t^2\,\mathbf{j} + \left(1 + \frac{1}{3}t^3\right)\mathbf{k}$.

(b)



16. (a) $\mathbf{a}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k} \;\Rightarrow$

$$\mathbf{v}(t) = \int \left(t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k}\right) dt = \frac{t^2}{2}\,\mathbf{i} + \frac{t^3}{3}\,\mathbf{j} + \frac{\sin 2t}{2}\,\mathbf{k} + \mathbf{c}_1$$

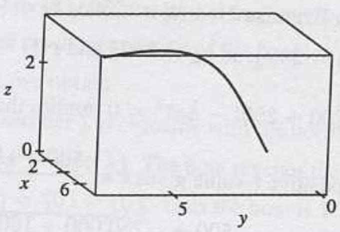
and $\mathbf{i} + \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{k}$ and

$$\mathbf{v}(t) = \left(\tfrac{1}{2}t^2 + 1\right)\mathbf{i} + \tfrac{1}{3}t^3\,\mathbf{j} + \left(1 + \tfrac{1}{2}\sin 2t\right)\mathbf{k}.$$

$$\mathbf{r}(t) = \int \left[\left(\tfrac{1}{2}t^2 + 1\right)\mathbf{i} + \tfrac{1}{3}t^3\,\mathbf{j} + \left(1 + \tfrac{1}{2}\sin 2t\right)\mathbf{k}\right] dt = \left(\tfrac{1}{6}t^3 + t\right)\mathbf{i} + \tfrac{1}{12}t^4\,\mathbf{j} + \left(t - \tfrac{1}{4}\cos 2t\right)\mathbf{k} + \mathbf{c}_2$$

But $\mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \left(1 + \frac{1}{12}t^4\right)\mathbf{j} + \left(\frac{1}{4} + t - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

(b)



17. $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t\rangle \;\Rightarrow\; \mathbf{v}(t) = \langle 2t, 5, 2t - 16\rangle$,
$|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$ and
$\dfrac{d}{dt}\,|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is, $16t - 64 = 0$ or $t = 4$. Since $\dfrac{d}{dt}\,|\mathbf{v}(t)| < 0$ for $t < 4$ and $\dfrac{d}{dt}\,|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

18. Since $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$. By Newton's Second Law, $\mathbf{F}(t) = m\,\mathbf{a}(t) = 6mt\,\mathbf{i} + 2m\,\mathbf{j} + 6mt\,\mathbf{k}$ is the required force.

19. $|\mathbf{F}(t)| = 20$ N in the direction of the positive z-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

20. The argument here is the same as that in Example 10.2.5 with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

21. $|\mathbf{v}(0)| = 500$ m/s and since the angle of elevation is $30\,^\circ$, the direction of the velocity is $\frac{1}{2}(\sqrt{3}\,\mathbf{i} + \mathbf{j})$. Thus $\mathbf{v}(0) = 250(\sqrt{3}\,\mathbf{i} + \mathbf{j})$ and if we set up the axes so the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so $\mathbf{F}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8\text{ m/s}^2$. Thus $\mathbf{a}(t) = -g\,\mathbf{j}$ and $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{c}_1$. But $250(\sqrt{3}\,\mathbf{i} + \mathbf{j}) = \mathbf{v}(0) = \mathbf{c}_1$, so $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{c}_2$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2$. Thus $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + \left(250t - \frac{1}{2}gt^2\right)\mathbf{j}$.

(a) Setting $250t - \frac{1}{2}gt^2 = 0$ gives $t = 0$ or $t = \frac{500}{g} \approx 51.0$ s. So the range is $250\sqrt{3} \cdot \frac{500}{g} \approx 22$ km.

(b) $0 = \dfrac{d}{dt}\left(250t - \frac{1}{2}gt^2\right) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s. Thus, the maximum height is $(250)(250/g) - g(250/g)^2\,\frac{1}{2} = (250)^2/(2g) \approx 3.2$ km.

(c) From part (a), impact occurs at $t = 500/g \approx 51.0$. Thus, the velocity at impact is $\mathbf{v}(500/g) = 250\sqrt{3}\,\mathbf{i} + [250 - g(500/g)]\,\mathbf{j} = 250\sqrt{3}\,\mathbf{i} - 250\,\mathbf{j}$ and the speed is $|\mathbf{v}(500/g)| = 250\sqrt{3 + 1} = 500$ m/s.

28. As in Exercise 27(b), let α be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\mathbf{i} + 5(\sin\alpha)\mathbf{j}$. At t seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $3\sin(\pi x/40)\mathbf{j} = 3\sin[\pi \cdot 5(\cos\alpha)t/40]\,\mathbf{j} = 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\mathbf{j}$. The resultant velocity of the boat then is given by $\mathbf{v}(t) = 5(\cos\alpha)\mathbf{i} + \left[5\sin\alpha + 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}$. Integrating,

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j} + \mathbf{C}.$$

If we place the origin at A then $\mathbf{r}(0) = \mathbf{0} \;\Rightarrow\; -\dfrac{24}{\pi\cos\alpha}\mathbf{j} + \mathbf{C} = \mathbf{0} \;\Rightarrow\; \mathbf{C} = \dfrac{24}{\pi\cos\alpha}\mathbf{j}$ and

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha}\right]\mathbf{j}.$$

The boat will reach the east bank when $5t\cos\alpha = 40 \;\Rightarrow\; t = \dfrac{8}{\cos\alpha}$. In order to land at point $B\,(40,0)$ we need $5t\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right) + \dfrac{24}{\pi\cos\alpha} = 0$

$$\Rightarrow\; 5\left(\frac{8}{\cos\alpha}\right)\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left[\frac{\pi}{8}\left(\frac{8}{\cos\alpha}\right)\cos\alpha\right] + \frac{24}{\pi\cos\alpha} = 0 \;\Rightarrow$$

$$\frac{1}{\cos\alpha}\left(40\sin\alpha - \frac{24}{\pi}\cos\pi + \frac{24}{\pi}\right) = 0 \;\Rightarrow\; 40\sin\alpha + \frac{48}{\pi} = 0 \;\Rightarrow\; \sin\alpha = -\frac{6}{5\pi}.$$ Thus

$\alpha = \sin^{-1}\left(-\dfrac{6}{5\pi}\right) \approx -22.5^\circ$, so the boat should head 22.5° south of east.

29. $\mathbf{r}(t) = (3t - t^3)\,\mathbf{i} + 3t^2\,\mathbf{j} \;\Rightarrow\; \mathbf{r}'(t) = (3 - 3t^2)\,\mathbf{i} + 6t\,\mathbf{j}$,
$|\mathbf{r}'(t)| = \sqrt{(3-3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3 - 3t^2)^2} = 3 + 3t^2$,
$\mathbf{r}''(t) = -6t\,\mathbf{i} + 6\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = (18 + 18t^2)\,\mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(3-3t^2)(-6t) + (6t)(6)}{3 + 3t^2} = \frac{18t + 18t^3}{3 + 3t^2} = \frac{18t(1+t^2)}{3(1+t^2)} = 6t \quad \text{[or by Equation 8,}$$

$a_T = v' = \dfrac{d}{dt}\left[3 + 3t^2\right] = 6t]$ and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{18 + 18t^2}{3 + 3t^2} = \dfrac{18(1+t^2)}{3(1+t^2)} = 6.$

30. $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + 3^2} = \sqrt{4t^2 + 10}$, $\mathbf{r}''(t) = 2\mathbf{j}$,
$\mathbf{r}'(t)\times\mathbf{r}''(t) = -6\mathbf{i} + 2\mathbf{k}$. Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2 + 10}}$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2\sqrt{10}}{\sqrt{4t^2+10}}$.

31. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,
$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

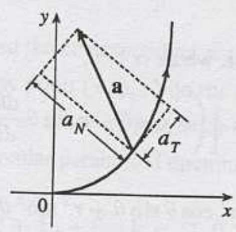
Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t\cos t - \sin t\cos t}{\sqrt{2}} = 0$ and

$$a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

32. $\mathbf{r}(t) = t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \sin^2 t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} - 2\cos t\sin t\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k} = \mathbf{i} - \sin 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$,
$|\mathbf{r}'(t)| = \sqrt{1 + 2\sin^2 2t}$, $\mathbf{r}''(t) = 2(\sin^2 t - \cos^2 t)\,\mathbf{j} + 2(\cos^2 t - \sin^2 t)\,\mathbf{k} = -2\cos 2t\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$. So

$$a_T = \frac{2\sin 2t\cos 2t + 2\sin 2t\cos 2t}{\sqrt{1 + 2\sin^2 2t}} = \frac{4\sin 2t\cos 2t}{\sqrt{1 + 2\sin^2 2t}}$$ and $$a_N = \frac{|-2\cos 2t\,\mathbf{j} - 2\cos 2t\,\mathbf{k}|}{\sqrt{1 + 2\sin^2 2t}} = \frac{2\sqrt{2}\,|\cos 2t|}{\sqrt{1 + 2\sin^2 t}}.$$

33. The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide).



Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5\ \text{cm/s}^2$ and $a_N \approx 9.0\ \text{cm/s}^2$.

34. $\mathbf{L}(t) = m\,\mathbf{r}(t) \times \mathbf{v}(t) \;\Rightarrow$

$\mathbf{L}'(t) = m[\mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)]$ [by Theorem 10.2.3 #5]

$\qquad = m[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)] = m[\mathbf{0} + \mathbf{r}(t) \times \mathbf{a}(t)] = \boldsymbol{\tau}(t)$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all t, and so $\mathbf{L}(t)$ is constant.

35. If the engines are turned off at time t, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a t such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9\rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{8t}{(t^2+1)^2}\mathbf{k} \;\Rightarrow$

$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s, 2 + \ln t + \frac{s}{t}, 7 - \frac{4}{t^2+1} + \frac{8st}{(t^2+1)^2}\right\rangle \;\Rightarrow\; 3 + t + s = 6 \;\Rightarrow\; s = 3 - t$, so

$7 - \frac{4}{t^2+1} + \frac{8(3-t)t}{(t^2+1)^2} = 9 \;\Leftrightarrow\; \frac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \;\Leftrightarrow\; t^4 + 8t^2 - 12t + 3 = 0$. It is easily seen that $t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \frac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

36. (a) $m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{v}_e \;\Leftrightarrow\; \frac{d\mathbf{v}}{dt} = \frac{1}{m}\frac{dm}{dt}\mathbf{v}_e$. Integrating both sides of this equation with respect to t gives

$\int_0^t \frac{d\mathbf{v}}{du}\,du = \mathbf{v}_e \int_0^t \frac{1}{m}\frac{dm}{du}\,du \;\Rightarrow\; \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e \int_{m(0)}^{m(t)} \frac{dm}{m}$ [Substitution Rule] $\Rightarrow$

$\mathbf{v}(t) - \mathbf{v}(0) = \ln\left(\frac{m(t)}{m(0)}\right)\mathbf{v}_e \;\Rightarrow\; \mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e$.

(b) $|\mathbf{v}(t)| = 2\,|\mathbf{v}_e|$, and $|\mathbf{v}(0)| = 0$. Therefore, by part (a), $2\,|\mathbf{v}_e| = \left|-\ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e\right| \;\Rightarrow$

$2\,|\mathbf{v}_e| = \ln\left(\frac{m(0)}{m(t)}\right)|\mathbf{v}_e|$. [*Note:* $m(0) > m(t)$ so that $\ln(m(0)/m(t)) > 0$] $\Rightarrow\; m(t) = e^{-2}m(0)$.

Thus $\frac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2}$ is the fraction of the initial mass that is burned as fuel.