EXERCISES FOR SECTION 2.1

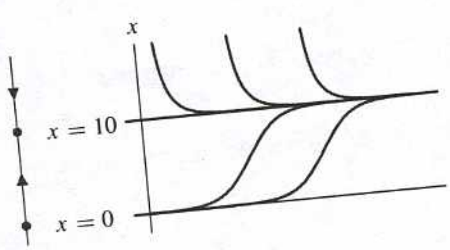
1. In the case where it takes many predators to eat one prey, the constant in the negative effect term of predators on the prey is small. Therefore, (ii) corresponds the system of large prey and small predators. On the other hand, one predator eats many prey for the system of large predators and small prey, and, therefore, the coefficient of negative effect term on predator-prey interaction on the prey is large. Hence, (i) corresponds to the system of small prey and large predators.

2. For (i), the equilibrium points are $x = y = 0$ and $x = 10, y = 0$. For the latter equilibrium point prey alone exist; there are no predators. For (ii), the equilibrium points are $(0, 0)$, $(0, 15)$, and $(3/5, 30)$. For the latter equilibrium point, both species coexist. For $(0, 15)$, the prey are extinct but the predators survive.

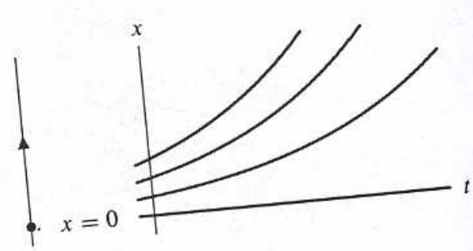
3. Substitution of $y = 0$ into the equation for dy/dt yields $dy/dt = 0$ for all t. Therefore, $y(t)$ is constant, and since $y(0) = 0$, $y(t) = 0$ for all t.

Note that to verify this assertion rigorously, we need a uniqueness theorem (see Section 2.4).

4. For (i), the prey obey a logistic model. The population tends to the equilibrium point at $x = 10$. For (ii), the prey obey an exponential growth model, so the population grows unchecked.



Phase line and graph for (i).

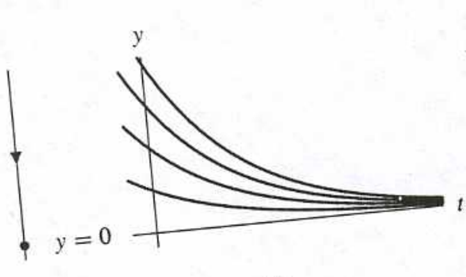


Phase line and graph for (ii).

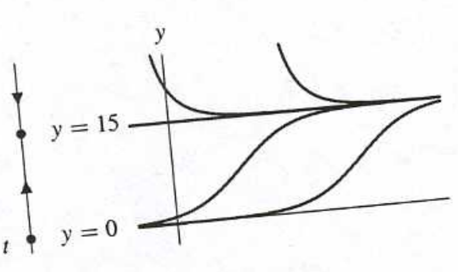
5. Substitution of $x = 0$ into the equation for dx/dt yields $dx/dt = 0$ for all t. Therefore, $x(t)$ is constant, and since $x(0) = 0$, $x(t) = 0$ for all t.

Note that to verify this assertion rigorously, we need a uniqueness theorem (see Section 2.4).

6. For (i), the predators obey an exponential decay model, so the population tends to 0. For (ii), the predators obey a logistic model. The population tends to the equilibrium point at $y = 15$.



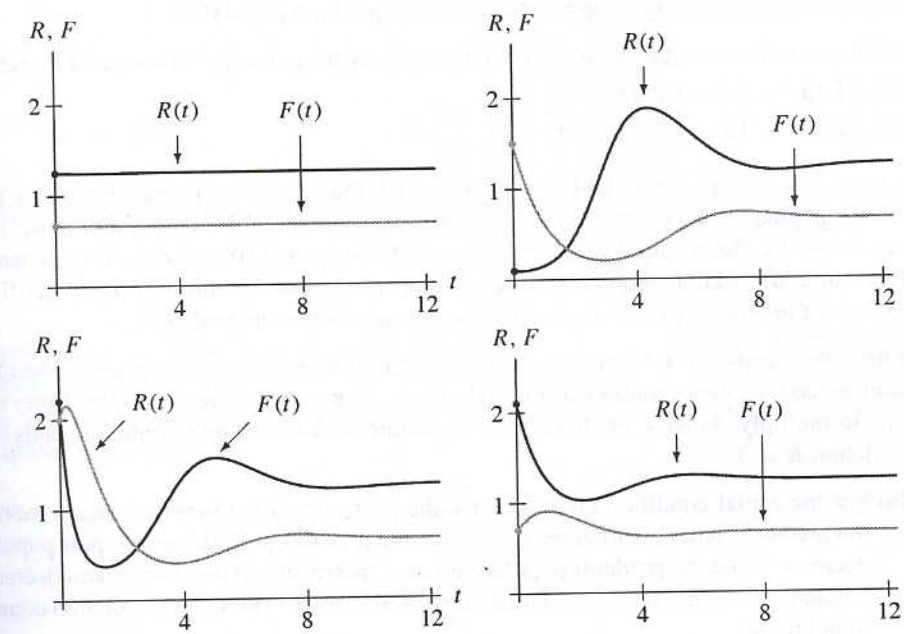
Phase line and graph for (i).



Phase line and graph for (ii).

7. The population starts with a relatively large rabbit (R) and a relatively small fox (F) population. The rabbit population grows, then the fox population grows while the rabbit population decreases. Next the fox population decreases until both populations are close to zero. Then the rabbit population grows again and the cycle starts over. Each repeat of the cycle is less dramatic (smaller total oscillation) and both populations oscillate toward an equilibrium which is approximately $(R, F) = (1/2, 3/2)$.

8. (a)



(b) Each of the solutions tends to the equilibrium point at $(R, F) = (5/4, 2/3)$. The populations of both species tend to a limit and the species coexist. For curve B, note that the F-population initially decreases while R increases. Eventually F bottoms out and begins to rise. Then R peaks and begins to fall. Then both populations tend to the limit.

9. By hunting, the number of prey decreases α units per unit of time. Therefore, the rate of change dR/dt of the number of prey has the term $-\alpha$. Only the equation for dR/dt needs modification.

(i) $dR/dt = 2R - 1.2RF - \alpha$

(ii) $dR/dt = R(2 - R) - 1.2RF - \alpha$

10. Hunting decreases the number of predators by an amount proportional to the number of predators alive (that is, by a term of the form $-kF$), so we have $dF/dt = -F + 0.9RF - kF$ in each case.

11. Since the second food source is unlimited, if $R = 0$ and k is the growth parameter for the predator population, F obeys an exponential growth model, $dF/dt = kF$. The only change we have to make is in the rate of F, dF/dt. For both (i) and (ii), $dF/dt = kF + 0.9RF$.

12. In the absence of prey, the predators would obey a logistic growth law. So we could modify both systems by adding a term of the form $-kF/N$, where k is the growth-rate parameter and N is the carrying capacity of predators. That is, we have $dF/dt = kF(1 - F/N) + 0.9RF$.

(d) The difference in the two solution curves is in how they are parameterized. The solution in this problem is at (0, 1) at time $t = 0$ and hence it lags behind the solution in the section by $\pi/2$. This information cannot be observed solely by looking at the solution curve in the phase plane.

20. **(a)** If we substitute $y(t) = \cos \beta t$ into the left-hand side of the equation, we obtain

$$\begin{aligned} \frac{d^2y}{dt^2} + \frac{k}{m}y &= \frac{d^2(\cos \beta t)}{dt^2} + \frac{k}{m}\cos \beta t \\ &= -\beta^2 \cos \beta t + \frac{k}{m}\cos \beta t \\ &= \left(\frac{k}{m} - \beta^2\right)\cos \beta t \end{aligned}$$

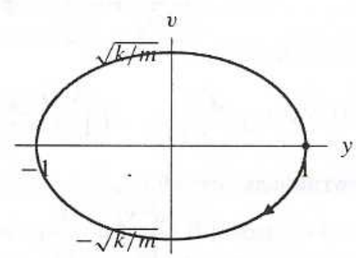
Hence, in order for $y(t) = \cos \beta t$ to be a solution we must have $k/m - \beta^2 = 0$. Thus,

$$\beta = \sqrt{\frac{k}{m}}.$$

(b) Substituting $t = 0$ into $y(t) = \cos \beta t$ and $v(t) = y'(t) = -\beta \sin \beta t$ we obtain the initial conditions $y(0) = 1$, $v(0) = 0$.

(c) The solution is $y(t) = \cos((\sqrt{k/m})t)$ and the period of this function is $2\pi/(\sqrt{k/m})$, which simplifies to $2\pi\sqrt{m}/\sqrt{k}$.

(d)



21. Hooke's law tells us that the restoring force exerted by a spring is linearly proportional to the spring's displacement from its rest position. In this case, the displacement is 3 in. while the restoring force is 12 lbs. Therefore, 12 lbs. $= k \cdot 3$ in. or $k = 4$ lbs. per in. $= 48$ lbs. per ft.

22. **(a)** First, we need to determine the spring constant k. Using Hooke's law, we have 4 lbs $= k \cdot 4$ in. Thus, $k = 1$ lbs/in $= 12$ lbs/ft. We will measure distance in feet since the mass is extended 1 foot.

To determine the mass of a 4 lb object, we use the fact that the force due to gravity is mg where $g = 32\text{ft/sec}^2$. Thus, $m = 4/32 = 1/8$.

Using the model

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0,$$

for the undamped harmonic oscillator, we obtain

$$\frac{d^2y}{dt^2} + 96y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

as our initial-value problem.

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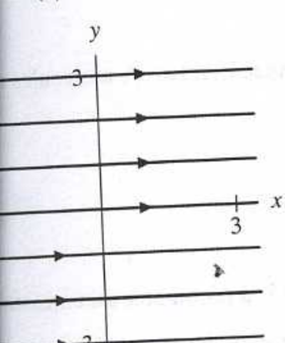
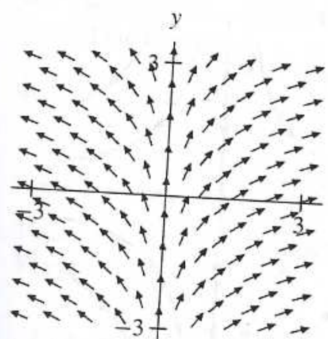


ward the right.

2. (a) $\mathbf{V}(x, y) = (x, 1)$

(b) See part (c).

(c)



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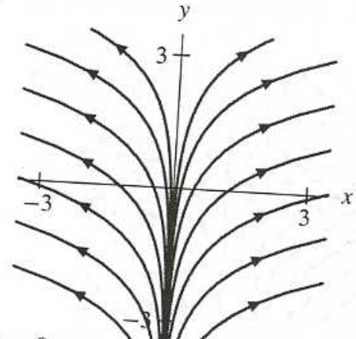
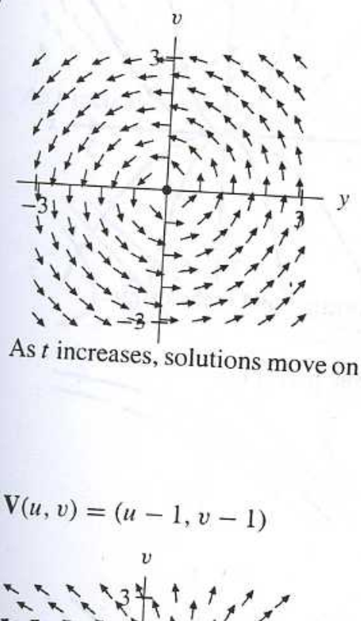


(e) As t increases, solutions move up and right if $x(0) > 0$, up and left if $x(0) < 0$.

3. (a) $\mathbf{V}(y, v) = (-v, y)$

(b) See part (c).

(c)



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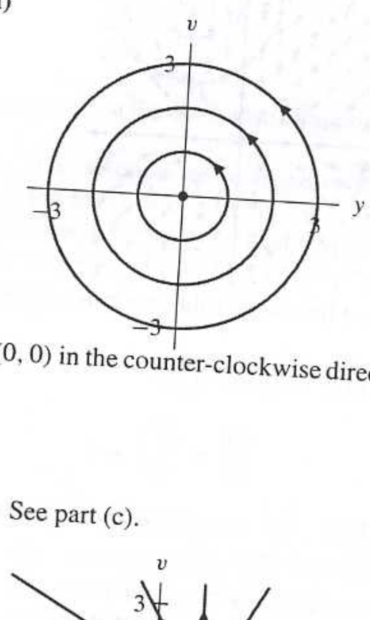
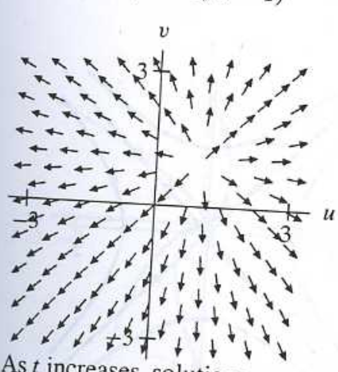


(e) As t increases, solutions move on circles around $(0, 0)$ in the counter-clockwise direction.

4. (a) $\mathbf{V}(u, v) = (u - 1, v - 1)$

(b) See part (c).

(c)



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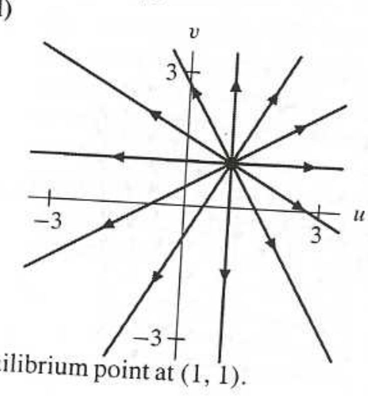


(e) As t increases, solutions move away from the equilibrium point at $(1, 1)$.

(e) As t increases, solutions in the 2nd and 4th quadrants move toward the origin and away from the line $y = -v$. Solutions in the 1st and 3rd quadrants move away from the origin and toward the line $y = v$.

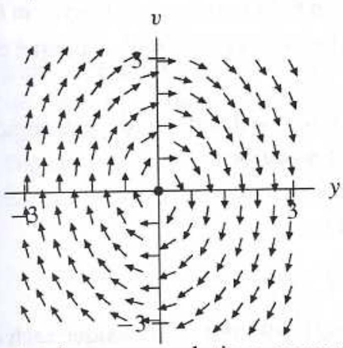
8. (a) Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -2y.$$

Thus the associated vector field is $\mathbf{V}(y, v) = (v, -2y)$.

(b) See part (c).

(c)



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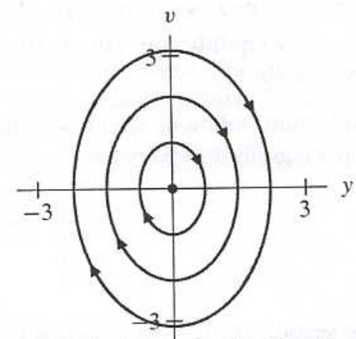
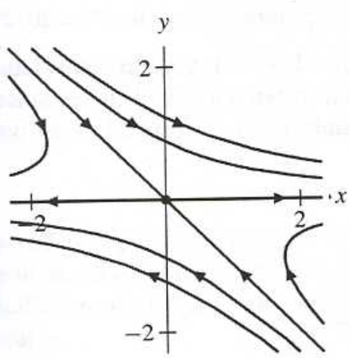


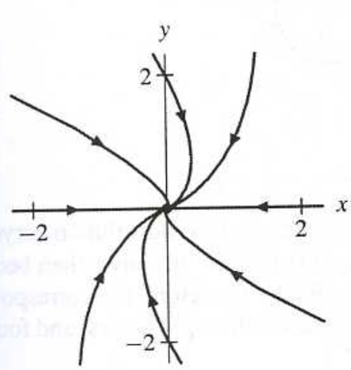
(e) As t increases, solutions move around the origin on ovals in the clockwise direction.

9. (a)



(b) The solution tends to the origin along the line $y = -x$ in the xy-phase plane. Therefore both $x(t)$ and $y(t)$ tend to zero as $t \to \infty$.

10. (a)



(b) The solution enters the first quadrant and tends to the origin tangent to the positive x-axis. Therefore $x(t)$ initially increases, reaches a maximum value, and then tends to zero as $t \to \infty$. It remains positive for all positive values of t. The function $y(t)$ decreases toward zero as $t \to 0$.

11. **(a)** There are equilibrium points at $(\pm 1, 0)$, so only systems (ii) and (vii) are possible. Since the direction field points away from the origin along the y-axis, the equation $dy/dt = -y$ does not match this field. Therefore, system (ii) is the system that generated this direction field.

(b) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). Vectors point directly away from the origin on the y-axis, so this direction field does not correspond to systems (iii) and (viii). Along the line $y = x$, the vectors are more vertical than horizontal. Consequently this direction field corresponds to system (v) rather than system (iv).

(c) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). The direction field is not tangent to the y-axis, so it does not match either system (iv) or (v). Vectors point toward the origin on the line $y = x$, so $dy/dt = dx/dt$ if $y = x$. This condition is not satisfied by system (iii). Therefore, this direction field corresponds to system (viii).

(d) The only equilibrium point is $(0, 1)$. System (i) is the only system with a unique equilibrium point at $(0, 1)$.

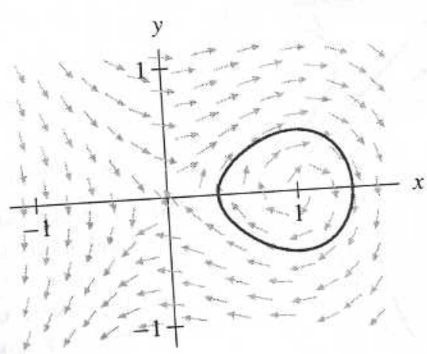
12. The equilibrium solutions are those solutions for which $dR/dt = 0$ and $dF/dt = 0$ simultaneously. To find the equilibrium points, we must solve the system of equations

$$\begin{cases} 2R\left(1 - \dfrac{R}{2}\right) - 1.2RF = 0 \\ -F + 0.9RF = 0. \end{cases}$$

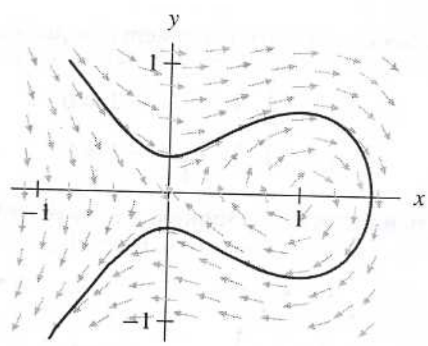
The second equation is satisfied if $F = 0$ or if $R = 10/9$, and we consider each case independently. If $F = 0$, then the first equation is satisfied if and only if $R = 0$ or $R = 2$. Thus two equilibrium solutions are $(R, F) = (0, 0)$ and $(R, F) = (2, 0)$.

If $R = 10/9$, we substitute this value into the first equation and obtain $F = 20/27$.

13. The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively large, these graphs must correspond to the outermost closed solution curve.

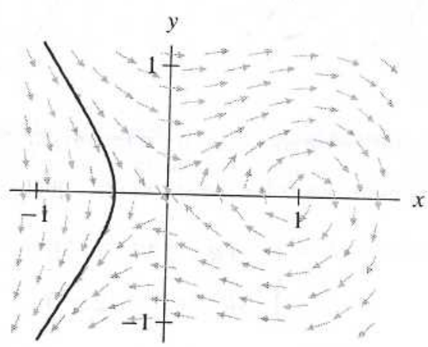


14. The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Both graphs cross the t-axis. The value of $x(t)$ is initially negative, then becomes positive and reaches a maximum, and finally becomes negative again. Therefore, the corresponding solution curve is the one that starts in the second quadrant, then travels through the first and fourth quadrants and finally enters the third quadrant.

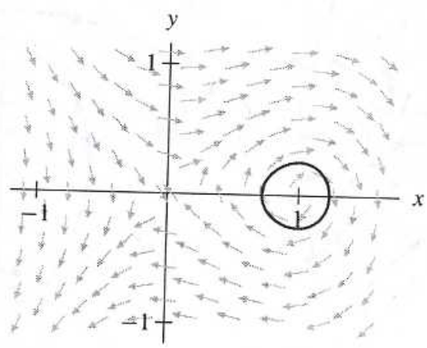


15. The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Only one graph crosses the t-axis. The other graph remains negative for all time. Note that the two graphs cross.

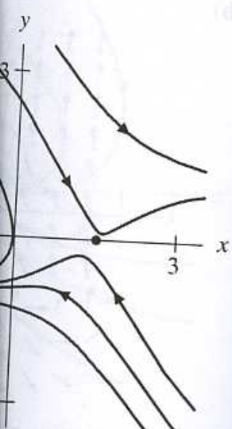
The corresponding solution curve is the one that starts in the second quadrant and crosses the x-axis and the line $y = x$ as it moves through the third quadrant.



16. The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively small, these graphs must correspond to the intermost closed solution curve.



nts are $(\pi/2 + k\pi, 0)$ for any



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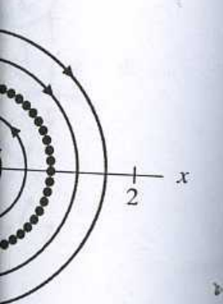
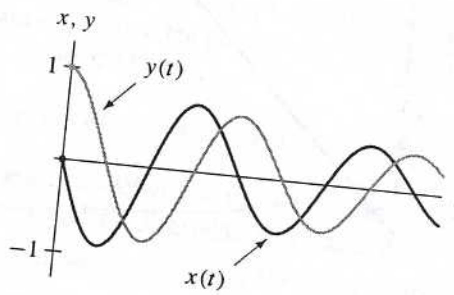
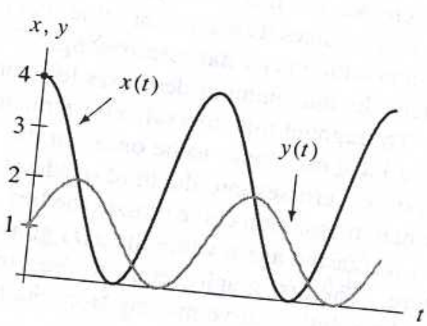


(c) As t increases, typical solutions move on a circle around the origin, either counter-clockwise inside the unit circle, which consists entirely of equilibrium points, or clockwise outside the unit circle.

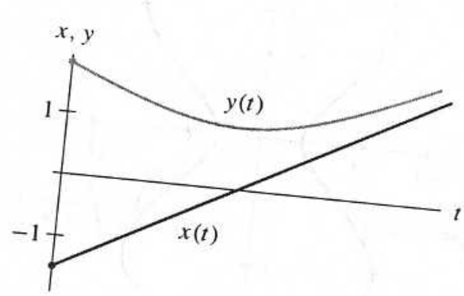
23. Since the solution curve spirals into the origin, the corresponding $x(t)$- and $y(t)$-graphs must oscillate about the t-axis with the decreasing amplitudes.



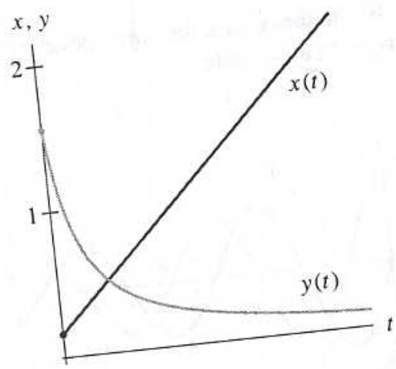
24. Since the solution curve is an ellipse that is centered at (2, 1), the $x(t)$- and $y(t)$-graphs are periodic. They oscillate about the lines $x = 2$ and $y = 1$.



25. The $x(t)$-graph satisfies $-2 < x(0) < -1$ and increases as t increases. The $y(t)$-graph satisfies $1 < y(0) < 2$. Initially it decreases until it reaches its minimum value of $y = 1$ when $x = 0$. Then it increases as t increases.



26. The $x(t)$-graph starts with a small positive value and increases as t increases. The $y(t)$-graph starts at approximately 1.6 and decreases as t increases. However, $y(t)$ remains positive for all t.

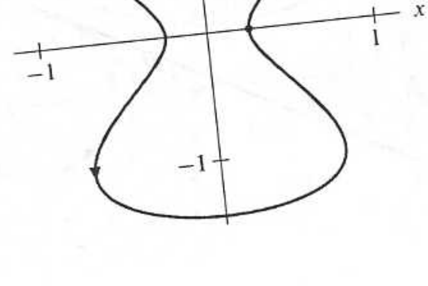


27. From the graphs, we see that $y(0) = 0$ and $x(0)$ is slightly positive. Initially both graphs increase. Then they cross, and slightly later $x(t)$ attains its maximum value. Continuing along we see that $y(t)$ attains its maximum at the same time as $x(t)$ crosses the t-axis.

In the xy-phase plane these graphs correspond to a solution curve that starts on the positive x-axis, enters the first quadrant, crosses the line $y = x$, and eventually crosses the y-axis into the second quadrant exactly when $y(t)$ assumes its maximum value. For this portion of the curve, $y(t)$ is increasing while $x(t)$ assumes a maximum and starts decreasing.

We see that once $y(t)$ attains its maximum, it decreases for a prolonged period of time until it assumes its minimum value. Throughout this interval, $x(t)$ remains negative although it assumes its minimum value twice and a local maximum value once. In the phase plane, the solution curve enters the second quadrant and then crosses into the third quadrant when $y(t) = 0$. The $x(t)$- and $y(t)$-graphs cross precisely when the solution curve crosses the line $y = x$ in the third quadrant.

Finally the $y(t)$-graph is increasing again while the $x(t)$-graph becomes positive and assumes its maximum value once more. The two graphs return to their initial values. In the phase plane this behavior corresponds to the solution curve moving from the third quadrant through the fourth quadrant and back to the original starting point.



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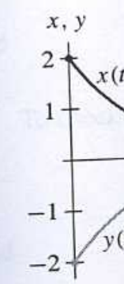

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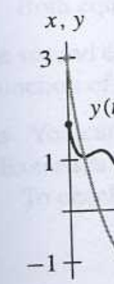

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