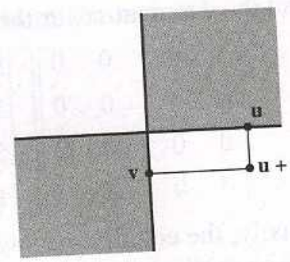
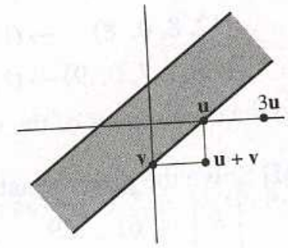
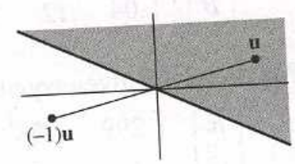
2. The set is closed under scalar multiples but not sums. For example, the sum of the vectors **u** and **v** shown here is not in H.



3. No. The set is not closed under sums or scalar multiples. The subset consisting of the points on the line $x_2 = x_1$ is a subspace, so any "counterexample" must use at least one point not on this line. Here are two counterexamples to the subspace conditions:



4. No. The set is closed under sums, but not under multiplication by a negative scalar.



5. The vector **w** is in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$ if and only if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{w}$ is consistent. The row operations below show that **w** is *not* in the subspace generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{w}] \sim \begin{bmatrix} 2 & -4 & 8 \\ 3 & -5 & 2 \\ -5 & 8 & -9 \end{bmatrix} \sim \begin{bmatrix} 2 & -4 & 8 \\ 0 & 1 & -10 \\ 0 & -2 & 11 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -4 & 8 \\ 0 & \textcircled{1} & -10 \\ 0 & 0 & \textcircled{-9} \end{bmatrix}$$

6. The vector **u** is in the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{u}$ is consistent. The row operations below show that **u** is *not* in the subspace generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

$$[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \mathbf{u}] \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ -2 & -7 & -8 & 10 \\ 4 & 9 & 6 & -7 \\ 3 & 7 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & -4 \\ 0 & 1 & 2 & 2 \\ 0 & -7 & -14 & 9 \\ 0 & -5 & -10 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 4 & 5 & -4 \\ 0 & \textcircled{1} & 2 & 2 \\ 0 & 0 & 0 & \textcircled{23} \\ 0 & 0 & 0 & 17 \end{bmatrix}$$

Note: For a quiz, you could use $\mathbf{w} = (1, -3, 11, 8)$, which is *in* Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. **a.** There are three vectors: $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. There are infinitely many vectors in Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ = Col A.

c. Deciding whether **p** is in Col A requires calculation:

$$[A\ \ \mathbf{p}] \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{2} & -3 & -4 & 6 \\ 0 & \textcircled{-4} & -10 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

...solution, so **p** is in Col A.

8. $[A\ \ \mathbf{p}] = \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 6 & 3 & 3 & -9 \end{bmatrix} \sim \begin{bmatrix} -3 & -2 & 0 & 1 \\ 0 & 2 & -6 & 14 \\ 0 & -1 & 3 & -7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{-3} & -2 & 0 & 1 \\ 0 & \textcircled{2} & -6 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Yes, the augmented matrix $[A\ \ \mathbf{p}]$ corresponds to a consistent system, so $\mathbf{p}$ is in Col A.

9. To determine whether $\mathbf{p}$ is in Nul A, simply compute $A\mathbf{p}$. Using A and $\mathbf{p}$ as in Exercise 7,

$$A\mathbf{p} = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix}\begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix}.$$ Since $A\mathbf{p} \neq \mathbf{0}$, $\mathbf{p}$ is *not* in Nul A.

10. To determine whether $\mathbf{u}$ is in Nul A, simply compute $A\mathbf{u}$. Using A as in Exercise 7 and $\mathbf{u} = (-2, 3, 1)$,

$$A\mathbf{u} = \begin{bmatrix} -3 & -2 & 0 \\ 0 & 2 & -6 \\ 6 & 3 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ Yes, $\mathbf{u}$ is in Nul A.

11. $p = 4$ and $q = 3$. Nul A is a subspace of $\mathbf{R}^4$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 4 entries, to match the columns of A. Col A is a subspace of $\mathbf{R}^3$ because each column vector has 3 entries.

12. $p = 3$ and $q = 4$. Nul A is a subspace of $\mathbf{R}^3$ because solutions of $A\mathbf{x} = \mathbf{0}$ must have 3 entries, to match the columns of A. Col A is a subspace of $\mathbf{R}^4$ because each column vector has 4 entries.

13. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$. (Include an augmented column of zeros, to avoid errors.)

$$\begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ -9 & -4 & 1 & 7 & 0 \\ 9 & 2 & -5 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & -4 & -8 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 2 & 4 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 2 & 1 & -5 & 0 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1 & 1 & 0 \\ 0 & \textcircled{1} & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} \textcircled{x_1} \quad - x_3 + x_4 &= 0 \\ \textcircled{x_2} + 2x_3 - 4x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = x_3 - x_4$, and $x_2 = -2x_3 + 4x_4$, with x_3 and x_4 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero vector. So choose any values for x_3 and x_4 (not both zero). For instance, set $x_3 = 1$ and $x_4 = 0$ to obtain the vector $(1, -2, 1, 0)$ in Nul A.

Note: Section 2.8 of *Study Guide* introduces the `ref` command (or `rref`, depending on the technology), which produces the reduced echelon form of a matrix. This will greatly speed up homework for students who have a matrix program available.

14. To produce a vector in Col A, select any column of A. For Nul A, solve the equation $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 7 & 0 \\ -5 & -1 & 0 & 0 \\ 2 & 7 & 11 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -3 & -5 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 3 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -1/3 & 0 \\ 0 & \textcircled{1} & 5/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = (1/3)x_3$ and $x_2 = (-5/3)\,x_3$, with x_3 free. The general solution in parametric vector form is not needed. All that is required here is one nonzero

15. Yes. Let A be the matrix whose columns are the vectors given. Then A is invertible because its determinant is nonzero, and so its columns form a basis for $\mathbf{R}^2$, by the Invertible Matrix Theorem (or by Example 5). (Other reasons for the invertibility of A could be given.)

16. No. One vector is a multiple of the other, so they are linearly dependent and hence cannot be a basis for any subspace.

17. No. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A=\begin{bmatrix}0&5&6\\1&-7&3\\-2&4&5\end{bmatrix}\sim\begin{bmatrix}1&-7&3\\0&5&6\\-2&4&5\end{bmatrix}\sim\begin{bmatrix}1&-7&3\\0&5&6\\0&-10&11\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-7&3\\0&\textcircled{5}&6\\0&0&\textcircled{23}\end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbf{R}^3$ (as pointed out in Example 5).

18. Yes. Place the three vectors into a 3×3 matrix A and determine whether A is invertible:

$$A=\begin{bmatrix}1&-5&7\\1&-1&0\\-2&2&-5\end{bmatrix}\sim\begin{bmatrix}1&-5&7\\0&4&-7\\0&-8&9\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-5&7\\0&\textcircled{4}&-7\\0&0&\textcircled{-5}\end{bmatrix}$$

The matrix A has three pivots, so A is invertible by the IMT and its columns form a basis for $\mathbf{R}^3$ (as pointed out in Example 5).

19. No. The vectors cannot be a basis for $\mathbf{R}^3$ because they only span a plan in $\mathbf{R}^3$. Or, point out that the columns of the matrix $\begin{bmatrix}1&-5\\1&-1\\-2&2\end{bmatrix}$ cannot possibly span $\mathbf{R}^3$ because the matrix cannot have a pivot in every row. So the columns are not a basis for $\mathbf{R}^3$.

Note: The *Study Guide* warns students not to say that the two vectors here are a basis for $\mathbf{R}^2$.

20. No. The vectors are linearly dependent because there are more vectors in the set than entries in each vector. (Theorem 8 in Section 1.7.) So the vectors cannot be a basis for any subspace.

21. **a.** False. See the definition at the beginning of the section. The critical phrases "for each" are missing.

b. True. See the paragraph before Example 4.

c. False. See Theorem 12. The null space is a subspace of $\mathbf{R}^n$, not $\mathbf{R}^m$.

d. True. See Example 5.

e. True. See the first part of the solution of Example 8.

22. **a.** False. See the definition at the beginning of the section. The condition about the zero vector is only one of the conditions for a subspace.

b. True. See Example 3.

c. True. See Theorem 12.

d. False. See the paragraph after Example 4.

e. False. See the Warning that follows Theorem 13.

28. The easiest construction is to write a 3×3 matrix in echelon form that has only 2 pivots, and let **b** be any vector in $\mathbf{R}^3$ whose third entry is nonzero.

29. (Solution in *Study Guide*) A simple construction is to write any nonzero 3×3 matrix whose columns are obviously linearly dependent, and then make **b** a vector of weights from a linear dependence relation among the columns. For instance, if the first two columns of A are equal, then **b** could be $(1, -1, 0)$.

30. Since Col A is the set of all linear combinations of $\mathbf{a}_1, \ldots, \mathbf{a}_p$, the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ spans Col A. Because $\{\mathbf{a}_1, \ldots, \mathbf{a}_p\}$ is also linearly independent, it is a basis for Col A. (There is no need to discuss pivot columns and Theorem 13, though a proof could be given using this information.)

31. If Col $F \neq \mathbf{R}^5$, then the columns of F do not span $\mathbf{R}^5$. Since F is square, the IMT shows that F is not invertible and the equation $F\mathbf{x} = \mathbf{0}$ has a nontrivial solution. That is, Nul F contains a nonzero vector. Another way to describe this is to write Nul $F \neq \{\mathbf{0}\}$.

32. If Nul R contains nonzero vectors, then the equation $R\mathbf{x} = \mathbf{0}$ has nontrivial solutions. Since R is square, the IMT shows that R is not invertible and the columns of R do not span $\mathbf{R}^6$. So Col R is a subspace of $\mathbf{R}^6$, but Col $R \neq \mathbf{R}^6$.

33. If Col $Q = \mathbf{R}^4$, then the columns of Q span $\mathbf{R}^4$. Since Q is square, the IMT shows that Q is invertible and the equation $Q\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^4$. Also, each solution is unique, by Theorem 5 in Section 2.2.

34. If Nul $P = \{\mathbf{0}\}$, then the equation $P\mathbf{x} = \mathbf{0}$ has only the trivial solution. Since P is square, the IMT shows that P is invertible and the equation $P\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^5$. Also, each solution is unique, by Theorem 5 in Section 2.2.

35. If the columns of B are linearly independent, then the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial (zero) solution. That is, Nul $B = \{\mathbf{0}\}$.

36. If the columns of A form a basis, they are linearly independent. This means that A cannot have more columns than rows. Since the columns also span $\mathbf{R}^m$, A must have a pivot in each row, which means that A cannot have more rows than columns. As a result, A must be a square matrix.

37. **[M]** Use the command that produces the reduced echelon form in one step (`ref` or `rref` depending on the program). See the Section 2.8 in the *Study Guide* for details. By Theorem 13, the pivot columns of A form a basis for Col A.

$$A = \begin{bmatrix} 3 & -5 & 0 & -1 & 3 \\ -7 & 9 & -4 & 9 & -11 \\ -5 & 7 & -2 & 5 & -7 \\ 3 & -7 & -3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 2.5 & -4.5 & 3.5 \\ 0 & \textcircled{1} & 1.5 & -2.5 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Basis for Col } A: \begin{bmatrix} 3 \\ -7 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 9 \\ 7 \\ -7 \end{bmatrix}$$

For Nul A, obtain the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$\begin{aligned} \textcircled{x_1} \quad + 2.5x_3 - 4.5x_4 + 3.5x_5 &= 0 \\ \textcircled{x_2} + 1.5x_3 - 2.5x_4 + 1.5x_5 &= 0 \end{aligned}$$

$$\text{Solution: } \begin{cases} x_1 = -2.5x_3 + 4.5x_4 - 3.5x_5 \\ x_2 = -1.5x_3 + 2.5x_4 - 1.5x_5 \\ x_3, x_4, \text{ and } x_5 \text{ are free} \end{cases}$$

6. $[\mathbf{b}_1\ \ \mathbf{b}_2\ \ \mathbf{x}] = \begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 22 & 11 \\ 0 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}.$$

7. Fig. 1 suggests that $\mathbf{w} = 2\mathbf{b}_1 - \mathbf{b}_2$ and $\mathbf{x} = 1.5\mathbf{b}_1 + .5\mathbf{b}_2$, in which case,

$[\mathbf{w}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $[\mathbf{x}]_B = \begin{bmatrix} 1.5 \\ .5 \end{bmatrix}$. To confirm $[\mathbf{x}]_B$, compute

$$1.5\mathbf{b}_1 + .5\mathbf{b}_2 = 1.5\begin{bmatrix} 3 \\ 0 \end{bmatrix} + .5\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \mathbf{x}$$

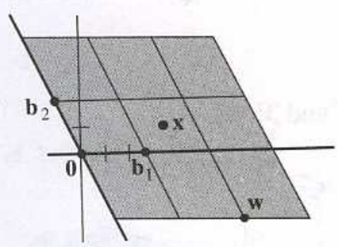


Figure 1

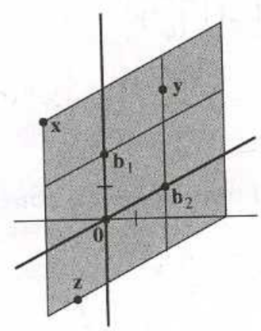


Figure 2

Note: Figures 1 and 2 display what Section 4.4 calls *B*-graph paper.

8. Fig. 2 suggests that $\mathbf{x} = 2\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{y} = 1.5\mathbf{b}_1 + \mathbf{b}_2$, and $\mathbf{z} = -\mathbf{b}_1 - .5\mathbf{b}_2$. If so, then

$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $[\mathbf{y}]_B = \begin{bmatrix} 1.5 \\ 1.0 \end{bmatrix}$, and $[\mathbf{z}]_B = \begin{bmatrix} -1 \\ -.5 \end{bmatrix}$. To confirm $[\mathbf{y}]_B$ and $[\mathbf{z}]_B$, compute

$$1.5\mathbf{b}_1 + \mathbf{b}_2 = 1.5\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \mathbf{y} \text{ and } -\mathbf{b}_1 - .5\mathbf{b}_2 = -1\begin{bmatrix} 0 \\ 2 \end{bmatrix} - .5\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2.5 \end{bmatrix} = \mathbf{z}.$$

9. The information $A = \begin{bmatrix} 1 & -3 & 2 & -4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -3 & 2 & -4 \\ 0 & 0 & \textcircled{5} & -7 \\ 0 & 0 & 0 & \textcircled{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is enough to see that columns 1, 3, and 4 of

A form a basis for Col *A*: $\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ -3 \\ 7 \end{bmatrix}.$

Columns 1, 2 and 4, of the echelon form certain cannot span Col *A* since those vectors all have zero in their fourth entries. For Nul *A*, use the reduced echelon form, augmented with a zero column to insure that the equation $A\mathbf{x} = \mathbf{0}$ is kept in mind:

$$\begin{bmatrix} \textcircled{1} & -3 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} \textcircled{x_1} - 3x_2 &= 0 \\ \textcircled{x_3} &= 0 \\ \textcircled{x_4} &= 0 \end{aligned}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \text{ So } \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ is}$$

x_2 is the free variable

a basis for Nul A. From this information, dim Col $A = 3$ (because A has three pivot columns) and dim Nul $A = 1$ (because the equation $A\mathbf{x} = \mathbf{0}$ has only one free variable).

10. The information $A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -2 & 9 & 5 & 4 \\ 0 & \textcircled{1} & -3 & 0 & -7 \\ 0 & 0 & 0 & \textcircled{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix}$. For Nul A,

$$[A \ \ \mathbf{0}] \sim \begin{bmatrix} \textcircled{1} & 0 & 3 & 0 & 0 & 0 \\ 0 & \textcircled{1} & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & \textcircled{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} \textcircled{x_1} \quad + 3x_3 \quad\quad &= 0 \\ \textcircled{x_2} - 3x_3 \quad - 7x_5 &= 0 \\ \textcircled{x_4} - 2x_5 &= 0 \end{aligned}$$

x_3 and x_5 are free variables

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ 3x_3 + 7x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}. \text{ Basis for Nul } A\text{: } \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

From this, dim Col $A = 3$ and dim Nul $A = 2$.

11. The information $A = \begin{bmatrix} 1 & 2 & -5 & 0 & -1 \\ 2 & 5 & -8 & 4 & 3 \\ -3 & -9 & 9 & -7 & -2 \\ 3 & 10 & -7 & 11 & 7 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 2 & -5 & 0 & -1 \\ 0 & \textcircled{1} & 2 & 4 & 5 \\ 0 & 0 & 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ shows that columns 1, 2, and 4 of A form a basis for Col A: $\begin{bmatrix} 1 \\ 2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -9 \\ 10 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -7 \\ 11 \end{bmatrix}$. For Nul A,

$$[A \ \ \mathbf{0}] \sim \begin{bmatrix} \textcircled{1} & 0 & -9 & 0 & 5 & 0 \\ 0 & \textcircled{1} & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \begin{aligned} \textcircled{x_1} \quad - 9x_3 \quad + 5x_5 &= 0 \\ \textcircled{x_2} + 2x_3 \quad - 3x_5 &= 0 \\ \textcircled{x_4} + 2x_5 &= 0 \end{aligned}$$

x_3 and x_5 are free variables

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}9x_3-5x_5\\-2x_3+3x_5\\x_3\\-2x_5\\x_5\end{bmatrix}=x_3\begin{bmatrix}9\\-2\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-5\\3\\0\\-2\\1\end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix}9\\-2\\1\\0\\0\end{bmatrix},\begin{bmatrix}-5\\3\\0\\-2\\1\end{bmatrix}.$$

From this, dim Col $A = 3$ and dim Nul $A = 2$.

12. The information $A=\begin{bmatrix}1&2&-4&3&3\\5&10&-9&-7&8\\4&8&-9&-2&7\\-2&-4&5&0&-6\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&2&-4&3&3\\0&0&\textcircled{1}&-2&0\\0&0&0&0&\textcircled{-5}\\0&0&0&0&0\end{bmatrix}$ shows that columns 1, 3, and 5 of A form a basis for Col A: $\begin{bmatrix}1\\5\\4\\-2\end{bmatrix},\begin{bmatrix}-4\\-9\\-9\\5\end{bmatrix},\begin{bmatrix}3\\8\\7\\-6\end{bmatrix}$. For Nul A

$$[A\ \ \mathbf{0}]\sim\begin{bmatrix}\textcircled{1}&2&0&-5&0&0\\0&0&\textcircled{1}&-2&0&0\\0&0&0&0&\textcircled{1}&0\\0&0&0&0&0&0\end{bmatrix}. \qquad \begin{aligned}\textcircled{x_1}+2x_2\quad-5x_4&=0\\ \textcircled{x_3}-2x_4&=0\\ \textcircled{x_5}&=0\end{aligned}$$

x_2 and x_4 are free variables

$$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-2x_2+5x_4\\x_2\\2x_4\\x_4\\0\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}5\\0\\2\\1\\0\end{bmatrix}. \text{ Basis for Nul } A: \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}5\\0\\2\\1\\0\end{bmatrix}.$$

From this, dim Col $A = 3$ and dim Nul $A = 2$.

13. The four vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix}1&-3&2&-4\\-3&9&-1&5\\2&-6&4&-3\\-4&12&2&7\end{bmatrix}\sim\begin{bmatrix}1&-3&2&-4\\0&0&5&-7\\0&0&0&5\\0&0&10&-9\end{bmatrix}\sim\begin{bmatrix}1&-3&2&-4\\0&0&5&-7\\0&0&0&5\\0&0&0&5\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&-3&2&-4\\0&0&\textcircled{5}&-7\\0&0&0&\textcircled{5}\\0&0&0&0\end{bmatrix}$$

Columns 1, 3, and 4 of the original matrix form a basis for H, so dim $H = 3$.

Note: Either Exercise 13 or 14 should be assigned because there are always one or two students who confuse Col A with Nul A. Or, they wrongly connect "set of linear combinations" with "parametric vector form" (of the general solution of $A\mathbf{x} = \mathbf{0}$).

14. The five vectors span the column space H of a matrix that can be reduced to echelon form:

$$\begin{bmatrix}1&2&0&-1&3\\-1&-3&2&4&-8\\-2&-1&-6&-7&9\\5&6&8&7&-5\end{bmatrix}\sim\begin{bmatrix}1&2&0&-1&3\\0&-1&2&3&-5\\0&3&-6&-9&15\\0&-4&8&12&-20\end{bmatrix}\sim\begin{bmatrix}\textcircled{1}&2&0&-1&3\\0&\textcircled{-1}&2&3&-5\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

Columns 1 and 2 of the original matrix form a basis for H, so dim $H = 2$.

15. Col $A = \mathbf{R}^3$, because A has a pivot in each row and so the columns of A span $\mathbf{R}^3$. Nul A *cannot* equal $\mathbf{R}^2$, because Nul A is a subspace of $\mathbf{R}^5$. It is true, however, that Nul A is two-dimensional. Reason: the equation $A\mathbf{x} = \mathbf{0}$ has two free variables, because A has five columns and only three of them are pivot columns.

16. Col A *cannot* be $\mathbf{R}^3$ because the columns of A have four entries. (In fact, Col A is a 3-dimensional subspace of $\mathbf{R}^4$, because the 3 pivot columns of A form a basis for Col A.) Since A has 7 columns and 3 pivot columns, the equation $A\mathbf{x} = \mathbf{0}$ has 4 free variables. So, dim Nul $A = 4$.

17. **a.** True. This is the definition of a B-coordinate vector.
   b. False. Dimension is defined only for a subspace. A line must be through the origin in $\mathbf{R}^n$ to be a subspace of $\mathbf{R}^n$.
   c. True. The sentence before Example 1 concludes that the number of pivot columns of A is the rank of A, which is the dimension of Col A by definition.
   d. True. This is equivalent to the Rank Theorem because rank A *is* the dimension of Col A.
   e. True, by the Basis Theorem. In this case, the spanning set is automatically a linearly independent set.

18. **a.** True. This fact is justified in the second paragraph of this section.
   b. True. See the second paragraph after Fig. 1.
   c. False. The dimension of Nul A is the number of *free* variables in the equation $A\mathbf{x} = \mathbf{0}$. See Example 2.
   d. True, by the definition of *rank*.
   e. True, by the Basis Theorem. In this case, the linearly independent set is automatically a spanning set.

19. The fact that the solution space of $A\mathbf{x} = \mathbf{0}$ has a basis of three vectors means that dim Nul $A = 3$. Since a 5×7 matrix A has 7 columns, the Rank Theorem shows that rank $A = 7 - \dim \text{Nul } A = 4$.

Note: One can solve Exercises 19–22 without explicit reference to the Rank Theorem. For instance, in Exercise 19, if the null space of a matrix A is three-dimensional, then the equation $A\mathbf{x} = \mathbf{0}$ has three free variables, and three of the columns of A are nonpivot columns. Since a 5×7 matrix has seven columns, A must have four pivot columns (which form a basis of Col A). So rank A = dim Col $A = 4$.

20. A 4×5 matrix A has 5 columns. By the Rank Theorem, rank $A = 5 - \dim \text{Nul } A$. Since the null space is three-dimensional, rank $A = 2$.

21. A 7×6 matrix has 6 columns. By the Rank Theorem, dim Nul $A = 6 - \text{rank } A$. Since the rank is four, dim Nul $A = 2$. That is, the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is two.

22. The wording of this problem was poor in the first printing, because the phrase "it spans a four-dimensional subspace" was never defined. Here is a revision that I will put in later printings of the third edition:

   Show that a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ in $\mathbf{R}^n$ is linearly dependent if dim Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_5\} = 4$.

   Solution: Suppose that the subspace $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ is four-dimensional. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ were linearly independent, it would be a basis for H. This is impossible, by the statement just before the definition of *dimension* in Section 2.9, which essentially says that *every* basis of a p-dimensional subspace consists of p vectors. Thus, $\{\mathbf{v}_1, \ldots, \mathbf{v}_5\}$ must be linearly dependent.

23. A 3×4 matrix A with a two-dimensional column space has two pivot columns. The remaining two columns will correspond to free variables in the equation $A\mathbf{x} = \mathbf{0}$. So the desired construction is possible.

4. We calculate that

$$\mathbf{x} = (-4)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} + (-7)\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}.$$

5. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

6. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$.

7. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

8. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$.

9. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is
$P_B = [\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$.

10. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^3$ is

$$P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3] = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}.$$

11. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

12. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix} 4 & 6 \\ 5 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7/2 & 3 \\ 5/2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.$$

13. We must find c_1, c_2, and c_3 such that

$$c_1(1+t^2) + c_2(t+t^2) + c_3(1+2t+t^2) = \mathbf{p}(t) = 1 + 4t + 7t^2.$$

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{aligned} c_1 \quad + \quad \quad \quad c_3 &= 1 \\ c_2 + 2c_3 &= 4 \\ c_1 + c_2 + c_3 &= 7 \end{aligned}$$

26. By definition, **w** is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p$ if and only if there exist scalars $c_1,\ldots,c_p$ such that

$$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p \tag{7}$$

Since the coordinate mapping is linear,

$$[\mathbf{w}]_B = c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_B \tag{8}$$

Conversely, (8) implies (7) because the coordinate mapping is one-to-one. Thus **w** is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p$ if and only if $[\mathbf{w}]_B$ is a linear combination of $[\mathbf{u}]_1,\ldots,[\mathbf{u}]_p$.

Note: Students need to be urged to *write* not just to compute in Exercises 27–34. The language in the *Study Guide* solution of Exercise 31 provides a model for the students. In Exercise 32, students may have difficulty distinguishing between the two isomorphic vector spaces, sometimes giving a vector in $\mathbb{R}^3$ as an answer for part (b).

27. The coordinate mapping produces the coordinate vectors (1, 0, 0, 1), (3, 1, –2, 0), and (0, –1, 3, –1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent.

28. The coordinate mapping produces the coordinate vectors (1, 0, –2, –3), (0, 1, 0, 1), and (1, 3, –2, 0) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

29. The coordinate mapping produces the coordinate vectors (1, –2, 1, 0), (–2, 0, 0, 1), and (–8, 12, –6, 1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & -2 & -8 \\ -2 & 0 & 12 \\ 1 & 0 & -6 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

30. The coordinate mapping produces the coordinate vectors (1, –3, 3, –1), (4, –12, 9, 0), and (0, 0, 3, –4) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing:

$$\begin{bmatrix} 1 & 4 & 0 \\ -3 & -12 & 0 \\ 3 & 9 & 3 \\ -1 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

31. In each part, place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form.

a. $\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Since there is not a pivot in each row, the original four column vectors do not span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials does not span $\mathbb{P}_2$.

b. $\begin{bmatrix} 0 & 1 & -3 & 2 \\ 5 & -8 & 4 & -3 \\ 1 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 2 & -6 & -3 \\ 0 & 0 & 0 & 7/2 \end{bmatrix}$

Since there is a pivot in each row, the original four column vectors span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials spans $\mathbb{P}_2$.

32. a. Place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form: $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The resulting matrix is invertible since it row equivalent to I_3. The original three column vectors form a basis for $\mathbb{R}^3$ by the Invertible Matrix Theorem. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials form a basis for $\mathbb{P}_2$.

b. Since $[\mathbf{q}]_B = (-3, 1, 2)$, $\mathbf{q} = -3\mathbf{p}_1 + \mathbf{p}_2 + 2\mathbf{p}_3$. One might do the algebra in $\mathbb{P}_2$ or choose to compute $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -8 \end{bmatrix}$. This combination of the columns of the matrix corresponds to the same combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. So $\mathbf{q}(t) = 1 + 3t - 8t^2$.

33. The coordinate mapping produces the coordinate vectors (3, 7, 0, 0), (5, 1, 0, –2), (0, 1, –2, 0) and (1, 16, –6, 2) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix and row reducing

$$\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$