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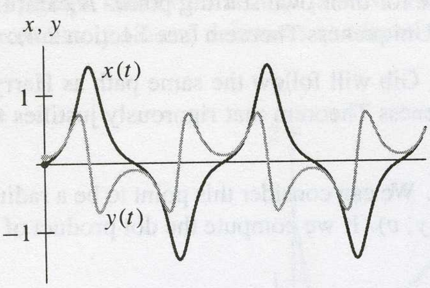

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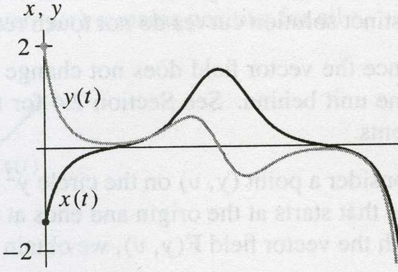

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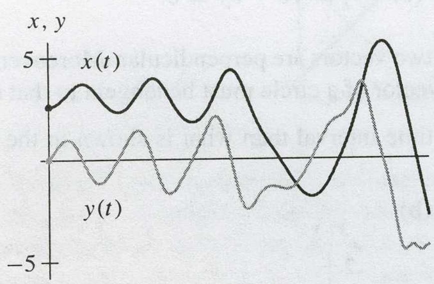


EXERCISES FOR SECTION 2.3

1. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 2e^t$$

and

$$2x + 2y = 4e^t - 2e^t = 2e^t.$$

To check that $dy/dt = x + 3y$, we compute both

$$\frac{dy}{dt} = -e^t,$$

and

$$x + 3y = 2e^t - 3e^t = -e^t.$$

Both equations are satisfied for all t. Hence $(x(t), y(t))$ is a solution.

2. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 6e^{2t} + e^t$$

and

$$2x + 2y = 6e^{2t} + 2e^t - 2e^t + 2e^{4t} = 6e^{2t} + 2e^{4t}.$$

Since the results of these two calculations do not agree, the first equation in the system is not satisfied, and $(x(t), y(t))$ is not a solution.

3. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 2e^t - 4e^{4t}$$

and

$$2x + 2y = 4e^t - 2e^{4t} - 2e^t + 2e^{4t} = 2e^t.$$

Since the results of these two calculations do not agree, the first equation in the system is not satisfied, and $(x(t), y(t))$ is not a solution.

4. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 4e^t + 4e^{4t}$$

and

$$2x + 2y = 8e^t + 2e^{4t} - 4e^t + 2e^{4t} = 4e^t + 4e^t.$$

To check that $dy/dt = x + 3y$, we compute both

$$\frac{dy}{dt} = -2e^t + 4e^{4t},$$

and

$$x + 3y = 4e^t + e^{4t} - 6e^t + 3e^{4t} = -2e^t + 4e^{4t}.$$

Both equations are satisfied for all t. Hence $(x(t), y(t))$ is a solution.

5. The second equation in the system is $dy/dt = -y$, and from Section 1.1, we know that $y(t)$ must be a function of the form $y_0 e^{-t}$, where y_0 is the initial value.

6. Yes. You can always show that a given function is a solution by verifying the equations directly (as in Exercises 1–4).

To check that $dx/dt = 2x + y$, we compute both

$$\frac{dx}{dt} = 8e^{2t} + e^{-t}$$

and

$$2x + y = 8e^{2t} - 2e^{-t} + 3e^{-t} = 8e^{2t} + e^t.$$

To check that $dy/dt = -y$, we compute both

$$\frac{dy}{dt} = -3e^{-t},$$

and

$$-y = -3e^{-t}.$$

Both equations are satisfied for all t. Hence $(x(t), y(t))$ is a solution.

7. From the second equation, we know that $y(t) = k_1 e^{-t}$ for some constant k_1. Using this observation, the first equation in the system can be rewritten as

$$\frac{dx}{dt} = 2x + k_1 e^{-t}.$$

This equation is a first-order linear equation, and we can derive the general solution using integrating factors from Section 1.8 or using the Extended Linearity Principle from Appendix A.

For this equation the integrating factor is $\mu = e^{-2t}$. If we begin with the equation

$$\frac{dx}{dt} - 2x = k_1 e^{-t}$$

and multiply both sides by μ, we obtain

$$e^{-2t}\left(\frac{dx}{dt} - 2x\right) = e^{-2t}\left(k_1 e^{-t}\right),$$

which reduces to

$$\frac{d}{dt}\left(e^{-2t}x\right) = k_1 e^{-3t}.$$

Integrating both sides, we have

$$e^{-2t}x = -\frac{k_1}{3}e^{-3t} + k_2,$$

where k_2 is a constant of integration. Multiplying both sides by e^{2t}, we obtain

$$x(t) = -\frac{k_1}{3}e^{-t} + k_2 e^{2t}.$$

8. **(a)** No. Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3}e^{-t}, k_1 e^{-t}\right)$$

the function $y(t) = 3e^{-t}$ implies that $k_1 = 3$. But this choice of k_1 implies that the coefficient of e^{-t} in the formula for $x(t)$ is -1 rather than $+1$.

(b) To determine that $\mathbf{Y}(t)$ is not a solution without reference to the general solution, we check the equation $dx/dt = 2x + y$. We compute both

$$\frac{dx}{dt} = -e^{-t}$$

and

$$2x + y = 2e^{-t} + 3e^{-t}.$$

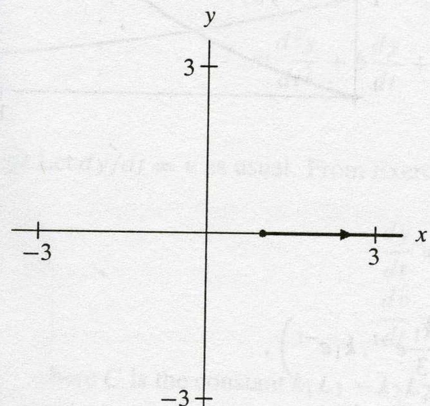
Since these two functions are not equal, $\mathbf{Y}(t)$ is not a solution.

9. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3}e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = 0$, and therefore $k_2 = 1$. We obtain $\mathbf{Y}(t) = (x(t), y(t)) = (e^{2t}, 0)$.

(b)



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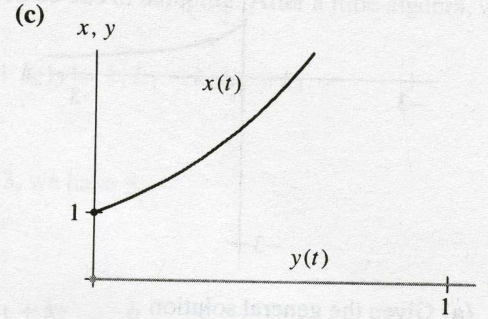
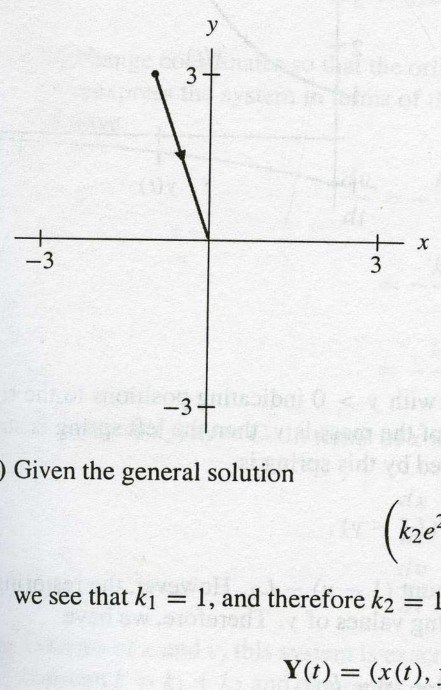


10. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3}e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = 3$, and therefore $k_2 = 0$. We obtain $\mathbf{Y}(t) = (x(t), y(t)) = (-e^{-t}, 3e^{-t})$.

(b)



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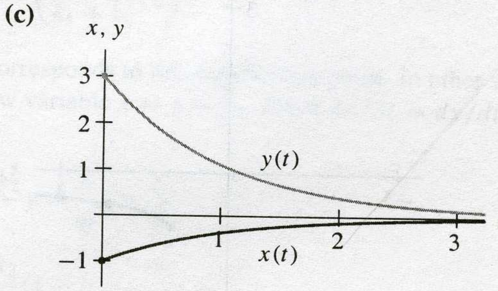


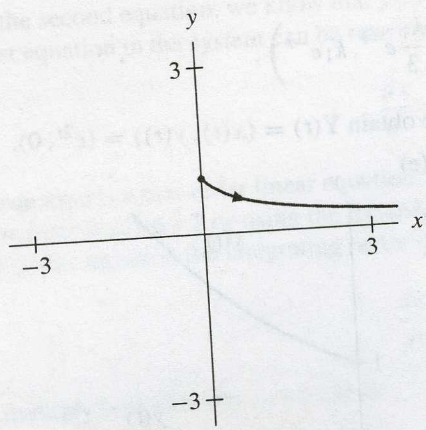
11. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3}e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = 1$, and therefore $k_2 = 1/3$. We obtain

$$\mathbf{Y}(t) = (x(t), y(t)) = \left(\tfrac{1}{3}e^{2t} - \tfrac{1}{3}e^{-t}, e^{-t}\right).$$

(b)



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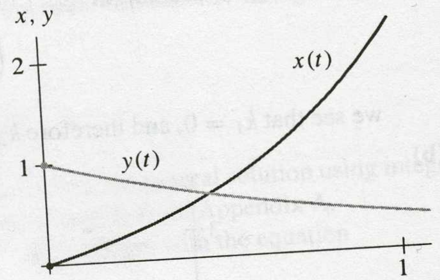


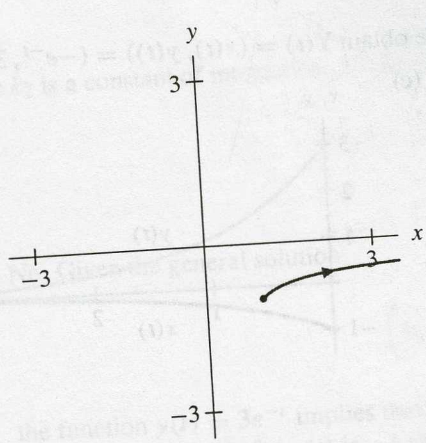
12. (a) Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3} e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = -1$, and therefore $k_2 = 2/3$. We obtain

$$\mathbf{Y}(t) = (x(t), y(t)) = \left(\tfrac{2}{3}e^{2t} + \tfrac{1}{3}e^{-t}, -e^{-t}\right).$$

(b)



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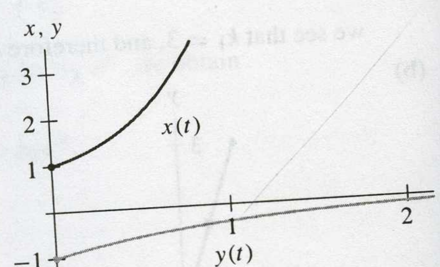


13. We choose the left wall to be the position $y = 0$ with $y > 0$ indicating positions to the right. Each spring exerts a force on the mass. If the position of the mass is y, then the left spring is stretched by the amount $y - L_1$. Therefore, the force F_1 exerted by this spring is

$$F_1 = k_1 (L_1 - y).$$

Similarly, the right spring is stretched by the amount $(1 - y) - L_2$. However, the restoring force F_2 of the right spring acts in the direction of increasing values of y. Therefore, we have

$$F_2 = k_2 ((1 - y) - L_2).$$

Using Newton's second law, we have

$$m\frac{d^2y}{dt^2} = k_1(L_1 - y) + k_2((1 - y) - L_2) - b\frac{dy}{dt},$$

where the term involving dy/dt represents the force due to damping. After a little algebra, we obtain

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + (k_1 + k_2)y = k_1L_1 - k_2L_2 + k_2.$$

14. **(a)** Let $dy/dt = v$ as usual. From Exercise 13, we have

$$\begin{aligned}\frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k_1 + k_2}{m}y - \frac{b}{m}v + \frac{C}{m},\end{aligned}$$

where C is the constant $k_1L_1 - k_2L_2 + k_2$.

(b) To find the equilibrium points, we set $dy/dt = 0$ and obtain $v = 0$. Setting $dv/dt = 0$ with $v = 0$, we obtain

$$(k_1 + k_2)y = C.$$

Therefore, this system has one equilibrium point,

$$(y_0, v_0) = \left(\frac{C}{k_1 + k_2}, 0\right).$$

(c) We change coordinates so that the origin corresponds to this equilibrium point. In other words, we reexpress the system in terms of the new variable $x = y - y_0$. Since $dx/dt = dy/dt = v$, we have

$$\begin{aligned}\frac{dv}{dt} &= -\frac{k_1 + k_2}{m}y - \frac{b}{m}v + \frac{C}{m} \\ &= -\frac{k_1 + k_2}{m}(x + y_0) - \frac{b}{m}v + \frac{C}{m} \\ &= -\frac{k_1 + k_2}{m}x - \frac{C}{m} - \frac{b}{m}v + \frac{C}{m},\end{aligned}$$

since $(k_1 + k_2)y_0 = C$. In terms of x and v, we have

$$\begin{aligned}\frac{dx}{dt} &= v \\ \frac{dv}{dt} &= -\frac{k_1 + k_2}{m}x - \frac{b}{m}v.\end{aligned}$$

(d) In terms of x and v, this system is exactly the same as a damped harmonic oscillator with spring constant $k = k_1 + k_2$ and damping coefficient b.

15. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

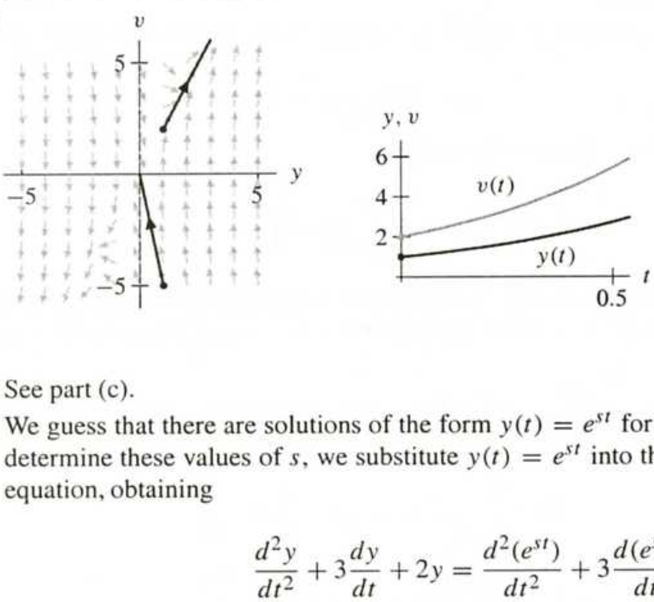
$$\begin{aligned}\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y &= \frac{d^2(e^{st})}{dt^2} + 3\frac{d(e^{st})}{dt} - 10(e^{st}) \\ &= s^2e^{st} + 3se^{st} - 10e^{st} \\ &= (s^2 + 3s - 10)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 3s - 10 = 0.$$

This equation is satisfied only if $s = -5$ or $s = 2$. We obtain two solutions, $y_1(t) = e^{-5t}$ and $y_2(t) = e^{2t}$, of this equation.

(c)



16. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

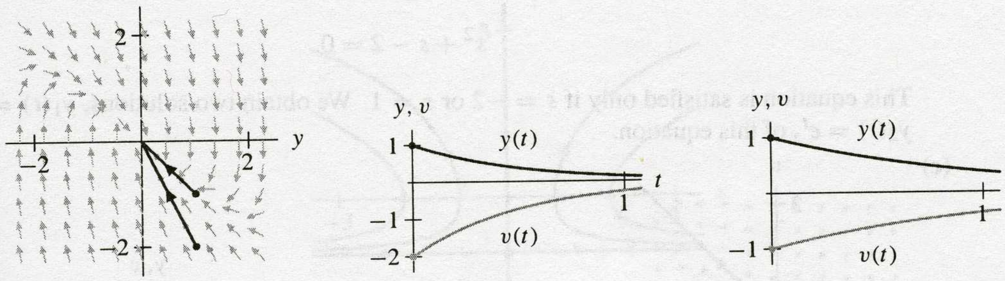
$$\begin{aligned}\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y &= \frac{d^2(e^{st})}{dt^2} + 3\frac{d(e^{st})}{dt} + 2(e^{st}) \\ &= s^2e^{st} + 3se^{st} + 2e^{st} \\ &= (s^2 + 3s + 2)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 3s + 2 = 0.$$

This equation is satisfied only if $s = -2$ or $s = -1$. We obtain two solutions, $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-t}$, of this equation.

(c)



17. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

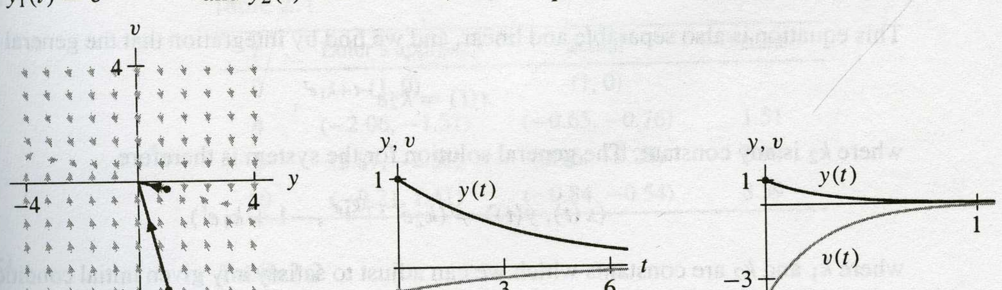
$$\begin{aligned}\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y &= \frac{d^2(e^{st})}{dt^2} + 4\frac{d(e^{st})}{dt} + e^{st} \\ &= s^2e^{st} + 4se^{st} + e^{st} \\ &= (s^2 + 4s + 1)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 4s + 1 = 0.$$

Applying the quadratic formula, we obtain the roots $s = -2 \pm \sqrt{3}$ and the two solutions, $y_1(t) = e^{(-2-\sqrt{3})t}$ and $y_2(t) = e^{(-2+\sqrt{3})t}$, of this equation.

(c)



18. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

$$\begin{aligned}\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y &= \frac{d^2(e^{st})}{dt^2} + \frac{d(e^{st})}{dt} - 2(e^{st}) \\ &= s^2e^{st} + se^{st} - 2e^{st} \\ &= (s^2 + s - 2)e^{st}\end{aligned}$$

EXERCISES FOR SECTION 3.1

1. Since $a > 0$, Paul's making a profit ($x > 0$) has a beneficial effect on Paul's profits in the future because the ax term makes a positive contribution to dx/dt. However, since $b < 0$, Bob's making a profit ($y > 0$) hinders Paul's ability to make profit because the by term contributes negatively to dx/dt. Roughly speaking, business is good for Paul if his store is profitable and Bob's is not. In fact, since $dx/dt = x - y$, Paul's profits will increase whenever his store is more profitable than Bob's.

Even though $dx/dt = dy/dt = x - y$ for this choice of parameters, the interpretation of the equation is exactly the opposite from Bob's point of view. Since $d < 0$, Bob's future profits are hurt whenever he is profitable because $dy < 0$. But Bob's profits are helped whenever Paul is profitable since $cx > 0$. Once again, since $dy/dt = x - y$, Bob's profits will increase whenever Paul's store is more profitable than his.

Finally, note that both x and y change by identical amounts since dx/dt and dy/dt are always equal.

2. Since $a = 2$, Paul's making a profit ($x > 0$) has a beneficial effect on Paul's future profits because the ax term makes a positive contribution to dx/dt. However, since $b = -1$, Bob's making a profit ($y > 0$) hinders Paul's ability to make profit because the by term contributes negatively to dx/dt. In some sense, Paul's profitability has twice the impact on his profits as does Bob's profitability. For example, Paul's profits will increase whenever his profits are at least one-half of Bob's profits since $dx/dt = 2x - y$.

Since $c = d = 0$, $dy/dt = 0$. Consequently, Bob's profits are not affected by the profitability of either store, and hence his profits are constant in this model.

3. Since $a = 1$ and $b = 0$, we have $dx/dt = x$. Hence, if Paul is making a profit ($x > 0$), then those profits will increase since dx/dt is positive. However, Bob's profits have no effect on Paul's profits. (Note that $dx/dt = x$ is the standard exponential growth model.)

Since $c = 2$ and $d = 1$, profits from both stores have a positive effect on Bob's profits. In some sense, Paul's profits have twice the impact of Bob's profits on dy/dt.

4. Since $a = -1$ and $b = 2$, Paul's making a profit has a negative affect on his future profits. However, if Bob makes a profit, then Paul's profits benefit. Moreover, Bob's profitability has twice the impact as does Paul's. In fact, since $dx/dt = -x + 2y$, Paul's profits will increase if $-x + 2y > 0$ or, in other words, if Bob's profits are at least one-half of Paul's profits.

Since $c = 2$ and $d = -1$, Bob is in the same situation as Paul. His profits contribute negatively to dy/dt since $d = -1$. However, Paul's profitability has twice the positive affect.

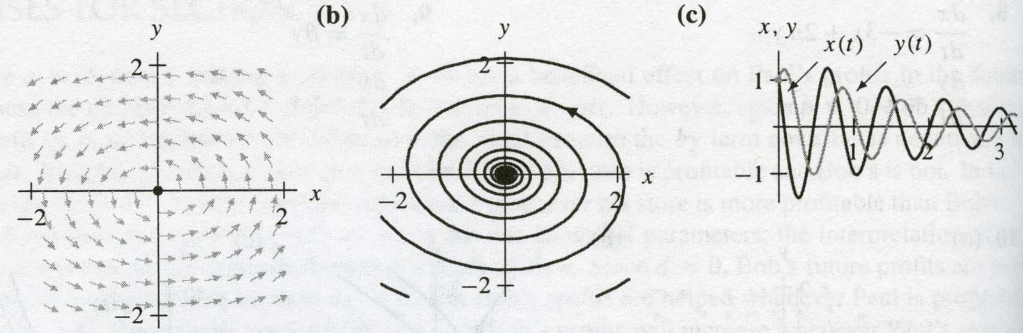
Note that this model is symmetric in the sense that both Paul and Bob perceive each others profits in the same way. This symmetry comes from the fact that $a = d$ and $b = c$.

5. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\mathbf{Y}$

6. $\mathbf{Y} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 3 \\ -0.3 & 3\pi \end{pmatrix}\mathbf{Y}$

7. $\mathbf{Y} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & -2 & -7 \\ -2 & 0 & 6 \\ 0 & 7.3 & 2 \end{pmatrix}\mathbf{Y}$

13. (a) (b) (c)



14. (a) If $a = 0$, then $\det \mathbf{A} = ad - bc = bc$. Thus both b and c are nonzero if $\det \mathbf{A} \neq 0$.

(b) Equilibrium points (x_0, y_0) are solutions of the simultaneous system of linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a = 0$, the first equation reduces to $by_0 = 0$, and since $b \neq 0$, $y_0 = 0$. In this case, the second equation reduces to $cx_0 = 0$, so $x_0 = 0$ as well. Therefore, $(x_0, y_0) = (0, 0)$ is the only equilibrium point for the system.

15. The vector field at a point (x_0, y_0) is (ax_0+by_0, cx_0+dy_0), so in order for a point to be an equilibrium point, it must be a solution to the system of simultaneous linear equations

$$\begin{cases} ax_0 + by_0 = 0 \\ cx_0 + dy_0 = 0. \end{cases}$$

If $a \neq 0$, we know that the first equation is satisfied if and only if

$$x_0 = -\frac{b}{a} y_0.$$

Now we see that any point that lies on this line $x_0 = (-b/a)y_0$ also satisfies the second linear equation $cx_0 + dy_0 = 0$. In fact, if we substitute a point of this form into the second component of the vector field, we have

$$\begin{aligned} cx_0 + dy_0 &= c\left(-\frac{b}{a}\right) y_0 + dy_0 \\ &= \left(-\frac{bc}{a} + d\right) y_0 \\ &= \left(\frac{ad - bc}{a}\right) y_0 \\ &= \frac{\det \mathbf{A}}{a} y_0 \\ &= 0, \end{aligned}$$

since we are assuming that $\det \mathbf{A} = 0$. Hence, the line $x_0 = (-b/a)y_0$ consists entirely of equilibrium points.

If $a = 0$ and $b \neq 0$, then the determinant condition $\det \mathbf{A} = ad - bc = 0$ implies that $c = 0$. Consequently, the vector field at the point (x_0, y_0) is (by_0, dy_0). Since $b \neq 0$, we see that we get equilibrium points if and only if $y_0 = 0$. In other words, the set of equilibrium points is exactly the x-axis.

Finally, if $a = b = 0$, then the vector field at the point (x_0, y_0) is $(0, cx_0 + dy_0)$. In this case, we see that a point (x_0, y_0) is an equilibrium point if and only if $cx_0 + dy_0 = 0$. Since at least one of c or d is nonzero, the set of points (x_0, y_0) that satisfy $cx_0 + dy_0 = 0$ is precisely a line through the origin.

16. **(a)** Let $v = dy/dt$. Then $dv/dt = d^2y/dt^2 = -qy - p(dy/dt) = -qy - pv$. Thus we obtain the system

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

In matrix form, this system is written as

$$\begin{pmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}.$$

(b) The determinant of this matrix is q. Hence, if $q \neq 0$, we know that the only equilibrium point is the origin.

(c) If y is constant, then $v = dy/dt$ is identically zero. Hence, $dv/dt = 0$.
Also, the system reduces to

$$\begin{pmatrix} \frac{dy}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} y \\ 0 \end{pmatrix},$$

which implies that $dv/dt = -qy$.
Combining these two observations, we obtain $dv/dt = -qy = 0$, and if $q \neq 0$, then $y = 0$.

17. The first-order system corresponding to this equation is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv.$$

(a) If $q = 0$, then the system becomes

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -pv,$$